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Groups with Minimal Conditions Related to Finiteness Properties on Conjugacy Classes.

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1. Introduction.

In a series of papers Belyaev [2], Belyaev-Sesekin [3] and Bruno-Phillips [4] (see also [11; section 8]) have obtained results on minimal non FC-groups and classified them when they have a non-trivial finite factor. Furthermore, in [5], Bruno-Phillips have generalized these results studying minimal non finite-by-nilpotent groups; this generalization is based on the well-known fact that a finite-by-abelian group, that is, a group with finite derived group, is a particular type of an FC-group (see [10; 4.35]). In any case these minimals are special cases of Cernikov groups (see [5; section 7] and [11; 8.11 and 8.13]).

Lately, in [8] and [9], the authors have extended the above problems in a direction of research that, roughly speaking, consists in replacing the term « finite group » by « Cernikov group » and so « FC-group » by « CC-group », although it should be remarked that the results we have found have a rather different nature than those of Bruno-Phillips because our theorems give subgroup characterizations of the conditions under consideration.

The aim of the present paper is to combine these ideas considering stronger conditions on the conjugacy classes of a group, namely the

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condition of being an FL-group or a CL-group (the terminology will be explain below). We shall first prove the Cernikov case and then deduce the finite case from it in the following way.

THEOREM 1. *Let $\mathfrak{X}$ be one of the following classes of groups* (i) *the theoretical union of the class of CL-groups and that of Cernikov-by-abelian groups*; (ii) *the class of CL-groups. Then a locally graded group G is an $\mathfrak{X}$-group if and only if every proper subgroup of G is an $\mathfrak{X}$-group.*

THEOREM 2. *Let $\mathfrak{Y}$ be one of the following classes of groups*: (i) *the theoretical union of the class of FL-groups and that of finite-by-abelian groups*; (ii) *the class of FL-groups. Then*

(1) *A locally graded minimal non $\mathfrak{Y}$-group is a non-perfect minimal non FC-group.*

(2) *A locally graded group G satisfies the minimal condition for non $\mathfrak{Y}$-subgroups if and only if G is either a $\mathfrak{Y}$-group or a Cernikov group.*

We recall some facts about a locally graded minimal non FC-group G (see [11]); such G is countable and locally finite. If G is perfect, then it is $\mathcal{F}$-perfect by [11; 8.9]; otherwise G is a Cernikov group and its structure is given by the results quoted above. Thus the structure of the groups in (1) of Theorem 2 is known.

Finally a comment on minimals; if $\mathfrak{C}$ is a class of groups, by *a minimal non $\mathfrak{C}$-group* we shall mean a group G which is not a $\mathfrak{C}$-group but in which each proper subgroup is a $\mathfrak{C}$-group. Attempts to classify these groups, for $\mathfrak{C}$ adequate, are certain to fail because of the existence of the Tarski groups, infinite nonabelian groups in which each proper subgroup is finite ([7]); indeed, these monsters show that our results are false without some sort of restriction on the groups under consideration. This difficulty has been avoided by the imposition of the local gradedness condition: a group G is said to be *locally graded* if every non-trivial finitely generated subgroup of G contains a proper subgroup of finite index.

2. Notation.

Throughout our group-theoretic notation is standard and is taken from [10]. For the reader's convenience we quickly review the definitions and some elementary facts to be used in the sequel. A group G is said to be an *FC-group* (a *CC-group*) if $G/C_G(x^G)$ is a finite group (a Cernikov group) for all x in G; Polovickii's theorems of characterization of these groups assure that G is an FC-group if and only if x^G is finite-by-cyclic for every x in G ([10; Corollary 3 to 4.32]) and that if G is a CC-group, then $[G, x]$ is Cernikov and so x^G is Cernikov-by-cyclic for every x in G ([10; 4.36]). More properties of FC-groups can be found in [11] and of CC-groups in [1] and [8].

Let G be a group and let m be a positive integer or infinity; then the subgroup G_m generated by all elements of G with order m is called the *m-layer of G*. A group each of whose layers is a Cernikov group is called a *CL-group* and a group whose layers are all finite is called an *FL-group*. The classification of CL- and FL-groups, again due to Polovickii, is collected in [10; 4.42 and 4.43]. Using the well-known Sunkov-Kegel-Wehrfritz theorem [6; 5.8], the more convenient way of dealing with these groups is as follows: *A group G is a CL-group (an FL-group) if and only if G is a periodic CC-group (a periodic FC-group) in which every Sylow subgroup is a Cernikov group.* Clearly both classes are closed under subgroups and taking homomorphic images. They are not closed under extensions but satisfy a certain form of weak extensibility, which can be stated as follows:

(*) *Let G be a periodic CC-group (a periodic FC-group) with a normal subgroup N such that G/N is Cernikov (finite). Then G is a CL-group (an FL-group) if and only if N is a CL-group (an FL-group).*

FL-groups or finite-by-abelian groups are FC-groups and clearly CL-groups are CC-groups. In general Cernikov-by-abelian groups need not to be CC-groups; for example, the semidirect product of a Prüfer p-group P by the group generated by an automorphism α of infinite order of P given by $x^\alpha = x^r$, x in P, where $(p, r) = 1$. However a periodic Cernikov-by-abelian group, that is, a periodic group G with Cernikov derived group, is a CC-group by [10; 4.36], which will be frequently used in what follows. Finally note that each notion defined in the finite case trivially implies its corresponding in the Cernikov case.

3. Proof of the Theorems.

PROOF OF THEOREM 1. Suppose that the case (i) has been proved but the case (ii) is false. Let G be a counterexample, that is, a locally graded minimal non CL-group. By (i) G' is Cernikov and, since G is clearly periodic, it follows that G is a CC-group. Then there exists a Sylow subgroup P of G that does not satisfy Min so that, by minimality, $G = P$ and every proper subgroup of G satisfies Min. Therefore G satisfies Min, a contradiction which shows (ii).

Thus it remains to show (i). As above we suppose that the result is false and take G to be a counterexample. We claim that G must be periodic. If not, we note that whenever G/L is a non-trivial periodic image of G, then L' is a Cernikov group because L cannot be periodic. If G is finitely generated, then G has a proper normal subgroup N of finite index so that N' is Cernikov. Clearly G/N' satisfies Max and it follows immediately that every proper subgroup of G/N' is finite-by-abelian. By [9; Theorem 1] (or [5; 4.1]) G/N' is finite-by-abelian, a contradiction. Therefore every finitely generated subgroup H of G is proper and, in any case, H' is Cernikov. Thus G' is locally finite. If T is the periodic part of G, then $G' \leqslant T$ and G/T is a non-trivial torsion-free abelian group. Choose a proper subgroup M of G containing T such that G/M is Cernikov; the election of a such M is clear if G/T is not $\mathcal{F}$-perfect and it is a consequence of the well-known structure of divisible torsion-free abelian groups otherwise. As we remarked above M' must be Cernikov and so T' is Cernikov. Therefore every CL-subgroup of G, being contained in T, has Cernikov derived group. Hence every proper subgroup of G is Cernikov-by-abelian and so is G by [9; Theorem 1], a contradiction which shows that G must be periodic as claimed. As a consequence every proper subgroup of G is a CC-group.

If G itself is a CC-group, then G cannot be $\mathcal{F}$-perfect because a periodic $\mathcal{F}$-perfect CC-group is abelian by [8; 1.2]. Hence G has a proper normal subgroup K of finite index and, since K cannot be a CL-group by the property (*), it follows that K' is Cernikov. On the other hand G/K' is an abelian-by finite CC-group and so it is Cernikov-by-abelian by [8; 1.1] and [10; 4.23]. Therefore G' is Cernikov, a contradiction.

Thus G is a minimal non CC-group. By the result of [8], G is countable, locally finite, perfect, $\mathcal{F}$-perfect and every non-trivial image of G is again a minimal non CC-group having the same properties mentioned above. $G/\zeta(G)$ has trivial center by Grün's Lemma ([10; part 1, p. 48]) so that replacing G by $G/\zeta(G)$ we may assume that $\zeta(G) = 1$. Since G is perfect and $\mathcal{F}$-perfect, any Cernikov normal subgroup of G is central in G by [10; 3.29] and so trivial. Therefore proper and non-trivial normal subgroups of G cannot have Cernikov layers and so they all are abelian.

Let J be a non-trivial simple image of G. By the result of [6; 1.D], for every prime p, J has a Sylow p-subgroup that contains a conjugate of every finite p-subgroup of J. Clearly a proper p-subgroup of J is soluble-by-finite so that it has finite derived length. Hence J cannot be enormous (terminology of [6; p. 122]) and this contradicts [8; 4.2]. Therefore G has no non-trivial simple images so that G can be expressed as the union of a chain of proper normal subgroups. As we showed in the previous paragraph every proper normal subgroup of G is abelian. Therefore G is abelian, a contradiction which shows our theorem.

The proof of the second part of the next Theorem needs an auxiliary result.

LEMMA. *Let G be a periodic CC-group in which every proper subgroup is either an FC-group or a Cernikov group. Then G is either an FC-group or a Cernikov group.*

PROOF. Again we assume that the result is false and take G to be a counterexample. Pick x in G such that $|G:C_G(x)|$ is infinite. Since x^G is Cernikov, G/x^G cannot be Cernikov and so there exists a proper subgroup H of G containing x^G such that H/x^G is not Cernikov. Thus H is an FC-group and then x^G is an abelian-by-finite FC-group. Therefore x^G is central-by-finite and then it is known that $(x^G)'$ is finite (see [10], for example). Hence x^G is a finite-by-abelian Cernikov group generated by elements of order $|x|$. This implies x^G finite, a contradiction.

PROOF OF THEOREM 2. (1) Let G be a locally graded minimal non $\mathfrak{Y}$-group; clearly every proper subgroup of G is an FC-group and, in any case, G is either a CL-group or a Cernikov-by-abelian group by Theorem 1. We claim that it suffices to show that G is not an FC-group, which is clear if G is a CL-group. For, G will be then a minimal non FC-group. If G were perfect, then it would be $\mathfrak{F}$-perfect and all this would give G abelian, reasoning directly in case of G being Cernikov-by-abelian or using [8; 1.2] in case of G being a CL-group, so that this assumption cannot occur and our claim follows.

Therefore it remains to show that G is not an FC-group in case (i) provided G is a Cernikov-by-abelian group. Assume that G is an FC-group. Since G is not abelian, it has a proper normal subgroup N of finite index. Obviously N cannot be an FL-group by the propperty (*) so that N' is finite. However G/N' is finite-by-abelian and therefore G is finite-by-abelian, a contradiction.

(2) We assume that there exists a locally graded group G satisfying the minimal condition for non $\mathcal{Y}$-subgroups which is neither a $\mathcal{Y}$-group nor a Cernikov group. Then G has a subgroup H which is minimal with respect to being neither a $\mathcal{Y}$-group nor a Cernikov group. In case (ii) H is a CL-group by Theorem 1 and so an FC-group by the Lemma, which is a contradiction. In case (i), as above, H cannot be a CL-group so that H' is Cernikov. As we did in the proof of Theorem 1, arguing with the periodic part of H, it is easy to show that H is periodic and so a CC-group. Then the Lemma again gives that H is in fact an FC-group and, as in the last part of the proof of the above (1), this yields to a contradiction, and the proof of Theorem 2 is now complete.

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