

# RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 81 (1989), p. 31-35

[http://www.numdam.org/item?id=RSMUP\\_1989\\_\\_81\\_\\_31\\_0](http://www.numdam.org/item?id=RSMUP_1989__81__31_0)

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## A Note on Fractional Integration.

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**ABSTRACT** - We give a new proof of the Hardy-Littlewood-Sobolev fractional integration theorem, based on a pointwise estimate of the rearrangement of the fractional integral.

The main purpose of this note is to give an alternative proof of the Hardy-Littlewood-Sobolev fractional integration theorem. Our proof is simple, of great generality (it applies to any locally compact unimodular group with  $\sigma$ -finite Haar measure) and does not depend on interpolation results or covering properties of the space. It generalizes some ideas of Hedberg (see [2]). The key step is a pointwise estimate of a variant of the nonincreasing rearrangement of the convolution  $f * K_\alpha$ , where  $K_\alpha$  is a function in  $L(n/(n-\alpha), \infty)$  (the weak- $L^{n/(n-\alpha)}$ ). Hedberg gives a pointwise estimate of  $f * K_\alpha$  in terms of the Hardy-Littlewood maximal function of  $f$ . Our estimate is in terms of a sort of nonincreasing rearrangement of  $f$ . This is advantageous when dealing with problems connected with Lorentz spaces. The pointwise estimate of  $f * K_\alpha$  we obtain allows us to generalize to kernels in  $L(n/(n-\alpha), \infty)$  the other inequalities contained in Hedberg's paper.

We recall that a function  $f$  is in the Lorentz space  $L(n/(n-\alpha), \infty)$  if  $|\{y: |f(y)| \geq \lambda\}| \leq A \lambda^{-n/(n-\alpha)}$ .

Given a locally integrable function  $f$ , we introduce a sort of re-

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arrangement of  $f$  (see [3]) in the following way

$$f^{**}(t) = \sup_{|E| \geq t} |E|^{-1} \int_E |f(y)| dy \quad t > 0.$$

It is well known that if  $p > 1$ ,  $\left(\int_0^\infty ((f^{**}(t))^p) dt\right)^{1/p}$  is comparable with the  $L^p$  norm of  $f$ .

We refer to Hunt's paper ([3]) for more about Lorentz spaces (see also [1]).

**LEMMA 1.** *Suppose that  $K_\alpha \in L(n/(n-\alpha), \infty)$  and  $f \in L^p(\mathbb{R}^n)$ ,  $1 < p < n/\alpha$ . Then, for any  $\delta > 0$*

$$(I_\alpha(f))^{**}(t) \leq A \{ \delta^\alpha f^{**}(t) + \delta^{\alpha-n/p} \|f\|_p \}.$$

**PROOF.** Set  $F_\delta = \{y : K_\alpha(x-y) \geq \delta^{\alpha-n}\}$ . Then

$$|I_\alpha(f)(x)| \leq \left( \int_{F_\delta^c} + \int_{F_\delta} \right) |f(y) K_\alpha(x-y)| dy = I_1(x) + I_2(x).$$

By Holder's inequality

$$I_2(x) \leq \|f\|_p \left( \int_{F_\delta^c} |K_\alpha(x-y)|^{p'} dy \right)^{1/p'}.$$

Since  $F_\delta^c = \bigcup_j B_{\delta,j}$ , where  $B_{\delta,j} = \{y : 2^{-j} \delta^{\alpha-n} \leq K_\alpha(x-y) < 2^{-j+1} \delta^{\alpha-n}\}$  and  $K_\alpha \in L(n/(n-\alpha), \infty)$  we have

$$\begin{aligned} \int_{F_\delta^c} |K_\alpha(x-y)|^{p'} dy &= \sum_{j=1}^\infty \int_{B_{\delta,j}} |K_\alpha(x-y)|^{p'} dy \leq \\ &\leq A \sum_{j=1}^\infty 2^{jn/(n-\alpha)} \delta^n (2^{-j} \delta^{\alpha-n})^{p'} \leq A \delta^{n+(\alpha-n)p'} \end{aligned}$$

whence  $I_2(x) \leq A \|f\|_p \delta^{\alpha-n/p}$ .

Set  $E_\delta \cup_j F_{\delta,j}$ , where

$$F_{\delta,j} = \{y: 2^j \delta^{\alpha-n} < K_\alpha(x-y) \leq 2^{j+1} \delta^{\alpha-n}\}.$$

Moreover set

$$\tilde{F}_{\delta,j} = \{y: 2^j \delta^{\alpha-n} < K_\alpha(y) \leq 2^{j+1} \delta^{\alpha-n}\}.$$

Then

$$\begin{aligned} |E|^{-1} \int_E I_1(x) dx &= |E|^{-1} \int_E \int_{F_\delta} |f(y) K_\alpha(x-y)| dy dx = \\ &= \sum_{j=1}^{\infty} 2^j \delta^{\alpha-n} \int_{\tilde{F}_{\delta,j}} |E|^{-1} \int_E |f(x-y)| dy dx = \sum_{j=1}^{\infty} 2^j \delta^{\alpha-n} \int_{\tilde{F}_{\delta,j}} \left( |E|^{-1} \int_E |f(y)| dy \right) dx \leq \\ &\leq A f^{**}(t) \sum_{j=1}^{\infty} 2^j \delta^{\alpha-n} (\sum_{i=1}^j \delta^{\alpha-n})^{-n/(n-\alpha)} = A \delta^\alpha f^{**}(t). \end{aligned}$$

Therefore

$$|E|^{-1} \int_E I_\alpha(f)(x) dx \leq |E|^{-1} \int_E (I_1(x) + I_2(x)) dx \leq A (\|f\|_p \delta^{\alpha-n/p} + \delta^\alpha f^{**}(t))$$

from which the lemma follows easily.

**THEOREM 2.** *Let  $0 < \alpha < n$ ,  $1 < p$ ,  $q < \infty$ ,  $1/q = 1/p - \alpha/n$ . Then*

$$\|I_\alpha(f)\|_q \leq A \|f\|_p.$$

**PROOF.** Set  $\delta = (f^{**}(t)/\|f\|_p)^{-p/n}$  and integrate both sides of the inequality in Lemma 1 to get the result.

**REMARKS.** (a) The proof of Muckenhoupt and Stein (see [4]) uses the Marcinkiewicz interpolation theorem, but gives also an endpoint result (the case  $p = 1$ ). Hedberg's proof relies on the Hardy-Littlewood maximal function theorem, which depends on a covering property of the ambient space. In general cases this property is not available or it is technically very difficult to prove.

(b) In the compact case slight modifications are in order; we omit the details.

(c) Our proof gives interesting results even when dealing with non convolution operators. Applications to such situations will be given elsewhere.

We can also prove the following generalization of Theorem 4 in [2]. We remark that this result for general kernels which belong to  $L(n/(n-\alpha), \infty)$  is new.

**THEOREM 3.** *If  $f \geq 0$  is measurable, then*

$$\|I_{\alpha\theta}(f^s)\|_r \leq \|f\|_p^{s-\theta} \|I_\alpha(f)\|_q^\theta$$

whenever  $0 < \alpha < n$ ,  $0 < \theta < 1$ ,  $1 < p$ ,  $q < \infty$ ,  $\theta < s < \theta + (1-\theta)p$  and  $1/r = (s-\theta)/p + \theta/q$ .

**PROOF.** For simplicity we give only the proof in the case  $s = 1$ . For different  $s$  the proof follows the lines of the proof of Theorem 4 in [2], although our calculations are more involved.

It is easy to verify that  $K_{\alpha\theta} \in L(n/(n-\alpha), \infty)$  implies  $K_\alpha \equiv \equiv K_{\alpha\theta}^{(n-\alpha)/(n-\alpha\theta)} \in L(n/(n-\alpha), \infty)$ . Set  $F_\delta$  as in the proof of Lemma 1. Then

$$\int_{F_\delta^c} |f(y)K_{\alpha\theta}(x-y)| dy = \int_{F_\delta^c} |f(y)K_\alpha(x-y)(K_{\alpha\theta}(x-y))^\beta| dy \leq \delta^{\alpha(1-\theta)} I_\alpha(f)(x),$$

where  $\beta = 1 - (n-\alpha)/(n-\alpha\theta)$ , while

$$\int_{F_\delta} |f(y)K_{\alpha\theta}(x-y)| dy = \int_{\{y: K_{\alpha\theta}(x-y) \geq \delta^{\alpha\theta-n}\}} |f(y)K(x-y)| dy.$$

Lemma 1 now implies:

$$[I_{\alpha\theta}(f)]^{**}(t) \leq \{\delta^{\alpha\theta} f^{**}(t) + \delta^{\alpha(1-\theta)} [I_\alpha(f)]^{**}(t)\}.$$

Choosing  $\delta^\alpha = ([I_\alpha(f)]^{**}(t)/f^{**}(t))$  and using Holder's inequality we get the result.

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Manoscritto pervenuto in redazione il 16 gennaio 1988.