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**Derivations and multilinear polynomials**

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# Derivations and Multilinear Polynomials.

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Let $R$ be a ring and $f = f(x_1, x_2, \ldots, x_n)$ a multilinear homogeneous polynomial in $n$ noncommuting variables.

We recall (see [7]) that $R$ is an $f$-radical extension of a subring $S$ if, for every $r_1, r_2, \ldots, r_n \in R$, there is an integer $m = m(r_1, r_2, \ldots, r_n) \geqslant 1$ such that $f(r_1, \ldots, r_n)^m \in S$.

When $R$ is $f$-radical over its center $Z(R)$ we say that $f$ is power central valued.

Rings with a power central valued polynomial have been studied in [10]. Results on $f$-radical extensions of rings have been obtained in [1] and [7] also.

Let now $d$ be a nonzero derivation on $R$; in this paper we will study the case in which there exists a polynomial $f(x_1, \ldots, x_n)$ such that $d(f(r_1, \ldots, r_n)^m) = 0$ for all $r_i \in R$ with $m = m(r_1, \ldots, r_n) \geqslant 1$. This is equivalent to say that $R$ is $f$-radical over $S = \{x \in R \colon d(x) = 0\}$.

Notice that when $f = x_1$ and $R$ is a prime ring with no nonzero nil ideals then, by [6], the above condition forces $R$ to be commutative. Moreover, if $d$ is an inner derivation on $R$, a prime ring with no nonzero nil right ideals, then in [4] it was proved that $f$ is power central valued and $R$ satisfies the standard identity of degree $n+2$, $S_{n+2}(x_1, \ldots, x_{n+2})$ provided an additional technical hypothesis also holds.

This is related to the following open question: « Let $D$ be a division ring and $f$ a polynomial power central valued in $D$, then is $D$ finite dimensional over its center? » (see [10]).

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In [4] and [10] it is proved that if $R$ is a prime ring with no nonzero nil right ideals and $f$ is power central valued in $R$, then $R$ satisfies a polynomial identity; the proof in [4] and [10] that $R$ is P.I. holds under the assumption that $f$ is not an identity for $p \times p$ matrices in char. $p > 0$. Hence, to apply this results in our paper we assume this extra hypothesis:

(A) *If* $\operatorname{char} R = p \neq 0$ *then* $f$ *is not an identity for* $p \times p$ *matrices in characteristic* $p$.

The main result of this paper is the following.

THEOREM 1. *Let $R$ be a prime ring,* $\operatorname{char} R \neq 2$, *with no nonzero nil right ideals and let* $f(x_1, \ldots, x_n)$ *be a multilinear homogeneous polynomial. Suppose that* $d$ *is a nonzero derivation on* $R$ *such that, for every* $r_1, \ldots, r_n \in R$, *there exists* $m \in N$, $m = m(r_1, \ldots, r_n)$ *with*

$$d(f(r_1, \ldots, r_n)^m) = 0 .$$

*If hypothesis* (A) *holds, then* $f(x_1, \ldots, x_n)$ *is power central valued and* $R$ *satisfies* $S_{n+2}(x_1, \ldots, x_{n+2})$.

*Moreover if* $f(x_1, \ldots, x_n)$ *is not a polynomial identity for* $R$ *and* $d(Z(R)) \neq 0$ *then* $Z(R)$ *is infinite of characteristic* $p \neq 0$.

As a consequence we will prove the following result of independent interest on Lie ideals (see [3]).

THEOREM 2. *Let $R$ be a prime ring with no nonzero nil right ideals,* $\operatorname{char} R \neq 2$, *and let* $U$ *be a noncentral Lie ideal of* $R$.

*Suppose that* $d$ *is a nonzero derivation on* $R$ *such that for every* $u \in U$ *there is* $m = m(u) \geqslant 1$ *with* $d(u^m) = 0$. *Then* $R$ *satisfies* $S_4(x_1, \ldots, x_4)$.

Throughout this paper we will use the following notation:

1) $R$ will always be an associative algebra over $C$, where $C$ is a commutative ring with 1.

2) $f(x_1, \ldots, x_n)$ will denote a multilinear homogeneous polynomial in $n$ non commuting variables, and we will assume that

$$f(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n + \sum \alpha_\pi x_{\pi(1)} x_{\pi(2)} \ldots x_{\pi(n)}$$

where $\alpha_T \in C$ and $1 \neq \pi \in S_n$ the symmetric group on $\{1, \ldots, n\}$.

3) $f(x_1, \ldots, x_n)$ will often be abbreviated as $f$ or $f(x_i)$.

4) $Z(R)$ will always denote the center of $R$.

5) $d$ will be a nonzero derivation on $R$ which is $C$-linear. (i.e. for $c \in C$, $r \in R$, $d(cr) = cd(r)$).

6) $S = \{x \in R\colon d(x) = 0\}$.

Finally, in all that follows, unless stated otherwise, we will assume that *$R$ is a prime ring*, $\operatorname{char} R \neq 2$, *and $R$ is $f$-radical over $S$. Furthermore we will assume that hypothesis* (A) *holds.*

We now can begin a series of reductions necessary to prove our result.

Lemma 1. *If $R$ is a division ring then $f(x_1, \ldots, x_n)$ is power central valued.*

Proof. Let $0 \neq x \in S = \{r \in R\colon d(r) = 0\}$, then we have

$$0 = d(1) = d(xx^{-1}) = d(x)x^{-1} + xd(x^{-1}) = xd(x^{-1})$$

which implies $d(x^{-1}) = 0$, i.e. $x^{-1} \in S$, so that $S$ is a proper subdivision ring of $R$. Then, by [7, Theorem 1], $f$ is power central valued.

For the next lemma we need to recall the following:

Definition 1. We say that $a \in T(R)$ if for all $r_1, \ldots, r_n$ in $R$ there exists an integer $m = m(a, r_1, \ldots, r_n) \geqslant 1$ such that $af(r_1, \ldots, r_n)^m = f(r_1, \ldots, r_n)^m a$ (see [4]).

Definition 2. Let $x$ be a quasi regular element of $R$, i.e. there exists $x' \in R$ such that $x + x' + xx' = x + x' + x'x = 0$.

Notice that if $R$ has a unit element 1 then $1 + x$ is invertible and $(1 + x)^{-1} = 1 + x'$.

Let $\varphi_x\colon R \to R$ be the map defined by

$$\varphi_x(r) = r + xr + rx' + xrx'.$$

$\varphi_x$ is an automorphism of $R$, we write $\varphi_x(r) = (1 + x)r(1 + x)^{-1}$ and we say that $a = 1 + x$ is formally invertible.

We also write $r(1 + x)$ for $r + rx$ and $(1 + x)r$ for $r + xr$.

Lemma 2. *If $a \in R$ is invertible, or formally invertible, then there exists $z \in T(R)$ depending on $a$ such that $d(a) = az$.*

PROOF. If $r_1, \ldots, r_n \in R$ let $m \geqslant 1$ be such that $f(r_i)^m$ and $f(ar_i a^{-1})^m$ are in $S$. Thus $d\big(af(r_i)^m a^{-1} a\big) = d(a) f(r_i)^m$ and also $d\big(af(r_i)^m a^{-1} a\big) = af(r_i)^m a^{-1} d(a)$.

Therefore $a^{-1} d(a) = z \in T(R)$ and so $d(a) = az$.

LEMMA 3. *If $T(R) = Z(R)$ and $J(R)$, the Jacobson radical of $R$, is non zero then $R$ is commutative.*

PROOF. If $x \in J(R)$ then $1 + x$ is formally invertible. By Lemma 2 $d(x) = d(1 + x) = z + zx$ for some $z \in T(R) = Z(R)$, and so $d(x)$ commutes with $x$; that is $d(x)x = xd(x)$ for all $x \in J(R)$. Since $R$ is prime, by [6, Lemma], $R$ is commutative.

LEMMA 4. *Suppose that $T(R) = Z(R)$. If $t \in R$ is such that $t^2 = 0$ then $d(t) = 0$.*

PROOF. Since $1 + t$ is formally invertible, by Lemma 2, one has $d(t) = d(1 + t) = z + zt$ for some $z \in T(R) = Z(R)$.

But $0 = d(t^2) = td(t) + d(t)t = 2zt$. Since $\operatorname{char} R \neq 2$ $zt = 0$. Moreover since $z \in Z(R)$, either $z = 0$ or $z$ is not a zero divisor in $R$. In any case $d(t) = 0$.

LEMMA 5. *Let $R$ be without nonzero nil right ideals. If there exists a non trivial idempotent $e = e^2 \neq 0, 1$ in $R$ then $f$ is power central valued.*

PROOF. Let $A$ be the subring of R generated by the elements of square zero. $A$ is invariant under all automorphism of $R$. Since $R$ is a prime ring with nontrivial idempotents than, by [9, Theorem], $A$ contains a nonzero ideal $I$ of $R$. On the other hand by [4, Theorem] either $f$ is power central valued in $R$ or $T(R) = Z(R)$. In this last case, by Lemma 4, $d(x) = 0$ for all $x \in A$ and so $d(I) = 0$.

Now, since $0 = d(I) \supseteq d(IR) = Id(R)$, by the primeness of $R$ we obtain $d(R) = 0$ which is a contradiction.

In the next Lemma we examine the case when $R$ is primitive.

Lemma 6. *If $R$ is primitive then $f$ is power central valued.*

PROOF. Let $V$ be a faithful irreducible right $R$-module with endomorphism ring $D$, a division ring. By Lemma 1 and Lemma 5 we may assume that $V$ is infinite dimensional over $D$ and $R$ does not contain a non trivial idempotent. By [8] this says that $R$ does not have nonzero linear transformations of finite rank.

We will prove that these assumptions lead to a contradiction.

Now, (see [1, Lemma 7]), $C$ acts on $V$ and we may assume that both $R$ and $S = \{x \in R: d(x) = 0\}$ act densely on $V$ over $D$.

Let now $vr = 0$, for some $v \in V$ and $r \in R$, and suppose that $vd(r) \neq 0$.

Since $r$ has infinite rank there exist $w_1, \ldots, w_n \in \operatorname{Im} r$ such that $vd(r), w_1, \ldots, w_n$ are linearly independent, and let $v_1, \ldots, v_n \in V$ such that $w_i = v_i r$, $1 \leqslant i \leqslant n$.

Now by the Jacobson density theorem there exist $a_1, \ldots, a_n \in R$ such that $w_i a_i = v_{i+1}$ $(i = 1, \ldots, n \bmod n)$, $w_i a_j = 0$ otherwise, and $vd(r) a_1 = v_2$, $vd(r) a_i = 0$ $(i = 2, \ldots, n)$.

Notice that for all $r_1, \ldots, r_n \in R$ we have

$$d(f(r_1, \ldots, r_n)^m) = \sum_{p+q=m-1} f(r_i)^p d(f(r_i)) f(r_i)^q$$

and also, since $f(x_1, \ldots, x_n)$ is multilinear, we have:

$$d(f(r_i)) = \sum_{t=1}^{n} f(r_1, \ldots, d(r_t), \ldots, r_n) .$$

Let $m \geqslant 1$ be such that $d(f(ra_i)^m) = 0$, hence one has:

$$\begin{aligned} 0 = vd(f(ra_i)^m) &= \sum_{p+q=m-1} vf(ra_i)^p d(f(ra_i)) f(ra_i)^q = \\ &= vd(f(ra_i)) f(ra_i)^{m-1} = \sum_t vf(ra_1, \ldots, d(ra_t), \ldots, ra_n) f(ra_i)^{m-1} = \\ &= vf(d(r)a_1, ra_2, \ldots, ra_n) f(ra_i)^{m-1} = v_1 f(ra_i)^{m-1} = \ldots = v_1 , \end{aligned}$$

a contradiction.

Hence if $vr = 0$ then $vd(r) = 0$.

Let $0 \neq v \in V$ and suppose that $vr$ and $vd(r)$ are linearly dependent for all $r \in R$. Let $x, y \in R$ be such that $vx$ and $vy$ are linearly independent, then

$$vd(x) = \lambda_x vx , \quad vd(y) = \lambda_y vy \quad \text{and} \quad vd(x+y) = \lambda_{x+y} v(x+y) ,$$

where $\lambda_x$, $\lambda_y$ and $\lambda_{x+y}$ are in $D$.

Therefore $\lambda_{x+y} vx + \lambda_{x+y} vy = \lambda_x vx + \lambda_y vy$, thus $\lambda_x = \lambda_y$.

As a result there exists $\lambda \in D$ such that $vd(x) = \lambda vx$ for all $x \in R$

with $vx \neq 0$. On the other hand, as we said above, $vr = 0$ implies $vd(r) = 0$, hence $vd(x) = \lambda vx$ for all $x \in R$.

However since $S$ acts densely on $V$ there is $x \in S$ such that $vx \neq 0$ and we obtain $0 = vd(x) = \lambda vx$, hence $\lambda = 0$. By this argument, if $vr$ and $vd(r)$ are linearly dependent for all $v \in$ and $r$ in $R$ then $Vd(R) = 0$ and so $d = 0$.

Therefore, we may assume that there exist $v \in V$ $r \in R$ such that $vr$ and $vd(r)$ are linearly independent; moreover, as above $r$ has infinite rank.

Let $w_1, \ldots, w_n \in \operatorname{Im} r$ be such that $vr, vd(r), w_1, \ldots, w_n$ are linearly independent, and let $v_1, v_2, \ldots, v_n \in V$ be such that $v_i r = w_i$ $(i = 1, \ldots, n)$.

By the density of $S$ on $V$ there exist $s_1, \ldots, s_n \in S$ such that $vrs_i = 0$ $(i \geqslant 1)$, $vd(r)s_1 = v_2$, $vd(r)s_i = 0$ $(i \geqslant 2)$, $w_i s_i = v_{i+1}$ $(i = 1, \ldots, n \bmod. n)$, $w_i s_j = 0$ for $i \neq j$.

Then we have:

$$vf(rs_1, \ldots, rs_n) = 0\,, \qquad vf(d(r)s_1, rs_2, \ldots, rs_n) = v_1\,,$$

$$vf(rs_1, \ldots, d(r)s_t, \ldots, rs_n) = 0 \ \ (t \neq 1)\,, \qquad v_1 f(rs_1, \ldots, rs_n) = v_1\,.$$

Let now $m \geqslant 1$ be such that $d(f(rs_1, \ldots, rs_n)^m) = 0$; hence we have

$$\begin{aligned} 0 = vd(f(rs_i)^m) &= \sum_{p+q=m-1} vf(rs_i)^p\, d(f(rs_i))\, f(rs_i)^q = \\ &= vd(f(rs_i))\, f(rs_i)^{m-1} = \sum_t vf(rs_1, \ldots, d(rs_t), \ldots, rs_n)\, f(rs_i)^{m-1} = \\ &= vf(d(r)s_1, rs_2, \ldots, rs_n)\, f(rs_i)^{m-1} = v_1 f(rs_i)^{m-1} = \ldots = v_1\,, \end{aligned}$$

a contradiction, and this proves the result.

Next we are going to examine the general case. First we will study a special kind of ideals invariant under the derivation.

Let $I$ be any ideal of $R$. We define

$$I' = \{x \in I \colon d^n(x) \in I \ \ \forall n \geqslant 1\}\,.$$

Then $I'$ is an ideal of $R$ invariant under $d$; in fact $I'$ is the largest subset of $I$ invariant under $d$. We have the following:

LEMMA 7. *Let $P$ be a primitive ideal such that* $\operatorname{char} R/P \neq 2$. *If* $f(x_1, \dots, x_n)$ *is not power central valued in* $R/P$ *then*

1) $\operatorname{char} R/P' \neq 2$;

2) $T(R/P') = Z(R/P')$;

3) $R/P'$ *is a prime ring.*

PROOF. To prove 1), let $x \in R$ be such that $2x \in P'$; hence $d^i(2x) \in P$, $\forall i \geq 0$, and so $2d^i(x) \in P$, $\forall i \geq 0$. Since $\operatorname{char} R/P \neq 2$ this implies that $d^i(x) \in P$ $\forall i \geq 0$, thus $x \in P'$.

This says that $R/P'$ is 2-torsion free.

We now prove 2). Let

$$A = \{x \in R : x + P' \in T(R/P')\} .$$

$A$ is a subring of $R$ invariant under $d$. In fact, for $x \in A$ and $r_1, \dots, r_n \in R$ there exists $m \geq 1$ such that $xf(r_i)^m - f(r_i)^m x$ is in $P'$ and we may assume that $d(f(r_i)^m) = 0$.

Since $P'$ is $d$-invariant we have:

$$P' \ni d\big(xf(r_i)^m - f(r_i)^m x\big) = d(x) f(r_i)^m - f(r_i)^m d(x)$$

and so $d(x)$ is also in $A$. Since $f$ is not power central valued in $R/P$, then by [4, Theorem] $T(R/P) = Z(R/P)$, hence, as $P' \subseteq P$, we have $x + P \in T(R/P) = Z(R/P)$ for all $x \in A$. This says that, for $x \in A$ and $y \in R$, $[x, y] = xy - yx \in P$.

Next we claim that $[x, y] \in P'$.

In fact, for $m \geq 1$, we have by Leibniz's formula

$$d^m(xy - yx) = d^m(xy) - d^m(yx) = \sum_i \binom{m}{i} [d^i(x), d^{m-i}(y)] .$$

Since $d^i(x) \in A$ one has, as above, that $[d^i(x), R] \subseteq P$, hence

$$d^m(xy - yx) \in P, \ \forall m \geq 1 .$$

This says that $xy - yx \in P'$ for $x \in A$ $y \in R$ and so $T(R/P') = Z(R/P')$.

To prove 3) we first show that $R/P'$ is a semiprime ring.

We remark that $R' = R/P'$ is a ring with induced derivation, defined by $d(x + P') = d(x) + P'$ and for all $r_1', \dots, r_n' \in R'$ there exists $m = m(r_i') \geq 1$ such that $d(f(r_i')^m) = 0 \in R'$; moreover if $d = 0$ then $P = P'$ and we are done. Hence we may assume that $d$ is nonzero *in* R$'$. Furthermore, as we said above, $R'$ is 2-torsion free and $T(R') = Z(R')$.

If $t \in R$ and $t^2 \equiv 0 \bmod P'$ then, since $(1+t) + P'$ is formally invertible, by the argument given in Lemma 4 it follows that $d(t) \equiv z + zt$ and $0 \equiv d(t^2) \equiv 2zt$ for some $z \in R$ such that $z + P' \in Z(R')$. Therefore, since $R'$ is 2-torsion free $zt \equiv 0$ and $d(t) \equiv z \bmod P'$.

Let $t \in R$ be such that $tRt \equiv 0 \pmod{P'}$ and $t^2 \equiv 0 \pmod{P'}$.

Then, for every $r \in R$, we have $d(t) + P' \in Z(R')$ and also $d(tr) + P' \in Z(R')$; which implies $(d(t)r + td(r)) + P' \in Z(R')$, and so

$$(d(t)^2 r + d(t)td(r)) + P' = d(t)^2 r + P' \in Z(R') .$$

Therefore, for $r, s \in R$, we have $d(t)^2(rs - sr) \equiv 0 \bmod P'$ and so $d(t)^2 R(rs - sr) \equiv 0 \bmod P'$ (recall that $d(t) + P' \in Z(R')$).

Let now

$$B = \{x \in R : xR(rs - sr) \equiv 0 \bmod P' \ \forall r, s \in R\} .$$

Notice that $B$ is invariant under $d$; in fact

$$0 \equiv d(xR(rs - sr)) \equiv d(x)R(rs - sr) + xd(R)(rs - sr) + \\ + xR(d(r)s - sd(r)) + xR(rd(s) - d(s)r) \equiv d(x)R(rs - sr) .$$

Moreover, since $R/P$ is noncommutative, there exists $r$, $s$ in $R$ such that $rs - sr \notin P$. But, for all $x \in B$, we have $xR(rs - sr) \subseteq P' \subseteq P$; since $R/P$ is primitive this implies $B \subseteq P$. Hence $B \subseteq P'$, the largest subset of $P$ $d$-invariant.

In other words we have proved that $tRt \equiv 0$ and $t^2 \equiv 0 \bmod P'$ implies $d(t)^2 \equiv 0$ and $d(t)Rd(t) \equiv 0 \bmod P'$.

Hence, by induction, we have $d^i(t)^2 \equiv 0$ and $d^i(t)Rd^i(t) \equiv 0 \bmod P'$; since $P' \subseteq P$ and $R/P$ is primitive this says that $d^i(t) \in P$, $\forall i \geqslant 0$, that is $t \in P'$ and $R' = R/P'$ is semiprime.

Finally, let $a, b \in R$ and suppose that $aRb \subseteq P'$.

Then, for any $x \in R$ we have $d(axb) \in P'$ and $ad(x)b \in P'$, so

$$(*) \qquad d(a)xb + axd(b) \in P' .$$

Now $R/P'$ is a semiprime ring, hence $aRb \subseteq P'$ forces $bRa \subseteq P'$. Multiplying $(*)$ on the left by $bR$ we obtain $bRd(a)xb \subseteq P'$ and consequently $d(a)xb$ is in $P'$. From (*) it follows that $axd(b)$ is also in $P'$.

We have proved that $d(a)Rb \subseteq P'$ and also $aRd(b) \subseteq P'$. At this stage an easy induction leads to $d^i(a)Rd^j(b) \subseteq P'$ $\forall i, j \geq 0$. Since $P' \subseteq P$ and $R/P$ is primitive, we conclude as above that either $a \in P'$ or $b \in P'$. This completes the proof.

Now we are ready to prove the main result of this paper.

PROOF OF THEOREM 1. As quoted above, since $R$ is a pıime ring with no nonzero nil right ideals and hypothesis (A) holds then either $f$ is power central valued or $T(R) = Z(R)$ (see [4]).

In the first case, by [4, Lemma 6], $R$ satisfies $S_{n+2}$. In the last case, if $J(R) \neq 0$ then by Lemma 3 $R$ is commutative.

Suppose now that $R$ is semisimple, so that $R$ is a subdirect product of primitive rings $R_\alpha$ of characteristic different from 2. Let $P_\alpha$ be a primitive ideal of $R$ such that $R_\alpha \cong R/P_\alpha$; we now partition these primitive ideals into four sets:

$$\mathscr{Q}_1 = \{P: d(R) \subseteq P\}$$

$$\mathscr{Q}_2 = \{P: d(P) \subseteq P \text{ but } d(R) \nsubseteq P\}$$

$$\mathscr{Q}_3 = \{P: d(P) \nsubseteq P \text{ and } f \text{ is power central valued in } R/P\}$$

$$\mathscr{Q}_4 = \{P: d(P) \nsubseteq P \text{ and } f \text{ is not power central valued in } R/P\}$$

in addition, let $I_i = \cap P$ for $P \in \mathscr{Q}_i$ $i = 1, \ldots, 4$.

Since $R$ is semisimple $I_1 I_2 I_3 I_4 \subseteq I_1 \cap I_2 \cap I_3 \cap I_4 = 0$.

Since $R$ is prime we must have that at least one among $I_1, I_2, I_3$ or $I_4$ is zero. However $I_1 \neq 0$, otherwise $d(R) \subseteq I_1 = 0$, a contradiction. If $I_2 = 0$ then $R$ is a subdirect product of primitive rings on which $d$ induces a nonzero derivation $d'$ satisfying all the hypotheses of Lemma 6. Then $f$ is power central valued on $R/P$, for each $P \in \mathscr{Q}_2$, and so by [4, Lemma 6] $R/P$ satisfies $S_{n+2}(x_1, \ldots, x_{n+2})$.

Therefore if $I_2 = 0$ then $R$ satisfies $S_{n+2}(x_1, \ldots, x_{n+2})$.

We also remark that if $P \in \mathscr{Q}_3$ then, as above, $R/P$ satisfies $S_{n+2}$. Hence, if $I_3 = 0$ then $R$ satisfies also this identity.

Finally we claim that $\mathscr{Q}_4 = \emptyset$.

Let $P \in \mathscr{Q}_4$, and let $P' = \{x \in P: d^i(x) \in P, \forall i \geq 1\}$. By Lemma 7 $R/P'$ is a prime ring, char. $R/P' \neq 2$ and $T(R/P') = Z(R/P')$. Moreover $d$ induces on $R' = R/P'$ a non zero derivation $d$ which also satisfies $d(f(r_1', \ldots, r_n')^m) = 0$ for all $r_i'$ in $R'$ for some $m = m(r_i', \ldots, r_n') \geq 1$.

We remark again that $f(x_1, \ldots, x_n)$ is nil valued on the nonzero ideal $P/P'$ of $R' = R/P'$. If $R'$ is with no nonzero nil right ideals then $f(x_1, \ldots, x_n)$ is a polynomial identity for $P/P'$ and so for $R'$ (see [5]).

Of course, this implies that $f(x_1, \ldots, x_n)$ is a polynomial identity for $R/P$, a contradiction since $P \in \mathcal{Q}_4$.

Therefore $R'$ has a nonzero nil right ideal and so $J(R') \neq 0$. But, in this case, by Lemma 3 $R'$ is commutative, and this is also a contradiction.

As a result $R$ satisfies the standard identity $S_{n+2}$ and $R'' = R_Z = \{rz^{-1} : r \in R,\ 0 \neq z \in Z(R)\}$ is a central simple algebra finite dimensional over $F$, the quotient field of $Z(R)$.

At it is well known, $d$ extends uniquely to a derivation on $R''$ (which we shall also denote by $d$) as follows:

$$d(rz^{-1}) = d(r)z^{-1} - rd(z)z^{-2} \qquad \forall r \in R,\ 0 \neq z \in Z(R)\,.$$

If $R$ does not satisfies $f$ then there exist $r_1, \ldots, r_n \in R$ such that $f(r_1, \ldots, r_n)$ is not nilpotent [5]. If $0 \neq z \in Z(R)$ there is an $m \geqslant 1$ such that $d(f(zr_1, r_2, \ldots, r_n)^m) = 0$ and $d(f(r_1, \ldots, r_n)^m) = 0$. Hence, we have $0 = d(f(zr_1, r_2, \ldots, r_n)^m) = d(z^m f(r_1, \ldots, r_n)^m) = d(z^m) f(r_1, \ldots, r_n)^m$ and so $d(z^m) = 0$.

As a result, if $s_i = r_i z_i^{-1} \in R''$ there is an $m = m(s_i) \geqslant 1$ such that

$$d(f(r_i)^m) = 0 \qquad \text{and} \qquad d(z^m) = 0\,,$$

where $z = z_1 \ldots z_n$, hence $d(f(s_1, \ldots, s_n)^m) = 0$.

Therefore by Lemma 6 $f(x_1, \ldots, x_n)$ is power central valued in $R''$ and we are done. Moreover, if $d(Z(R)) \neq 0$ and $f$ is not a polynomial identity for $R$ we obtain, as above, $d(z^m) = 0$ for all $z \in Z(R)$. Of course this implies that $Z(R)$ is infinite of characteristic $p \neq 0$. This completes the proof.

Of some independent interest is the special case when $f(x, y) = xy - yx$. In particolar, we do not need any extra assumptions regarding the behavior of $f$ on $p \times p$ matrices. We state this result as:

COROLLARY. *Let $R$ be a prime ring with no nonzero nil right ideals, $\operatorname{char} R \neq 2$. Let $d$ be a nonzero derivation on $R$ such that for every $x, y \in R$ there exists $m = m(x, y) \geqslant 1$ with $d((xy - yx)^m) = 0$. Then $R$ satisfies $S_4(x_1, \ldots, x_4)$.*

We conclude this paper with an easy application of this result to Lie ideals. This extend to arbitrary derivations a result of [3].

PROOF OF THEOREM 2. Since char $R \neq 2$ and $U$ is a non central Lie ideal of $R$, by [2, Lemma 1] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, I] \subseteq U$.

Let $I' = \{x \in I: d^i(x) \in I, \forall i \geqslant 1\}$, $I'$ is an ideal of $R$ invariant under $d$. Moreover, by hypothesis, for every $x, y \in I$ some power of $(xy - yx)$ lies in $I'$. Since $R$ has no nonzero nil right ideals and $R$ is not commutative we must have $I' \neq 0$. Then $I'$ is a prime ring with a nonzero derivation $d$ satisfying all the hypothesis of the Corollary, and so $I'$ satisfies $S_4(x_1, \dots, x_4)$. Since $R$ is prime, $R$ also satisfies $S_4(x_1, \dots, x_4)$.

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