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Existence of almost-periodic ultra-weak solutions to the equation $u'(t) = a(t)Au(t) + f(t)$ in Hilbert spaces

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Existence of Almost-Periodic Ultra-Weak Solutions to the Equation $u'(t) = a(t)\,\mathrm{Au}(t) + f(t)$ in Hilbert Spaces.

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ABSTRACT - In this work we consider the non-homogeneous first order differential equation: $du/dt - a(t)Au(t) = f(t)$ in a separable Hilbert space H, under a few assumptions about the complex-valued almost-periodic function $a(t)$ and the linear operator A in H. We establish a sufficient condition, ensuring, for almost-periodic $f(t)$, $\mathbb{R} \to H$, the existence (and uniqueness) of an almost-periodic ultra-weak solution of the above equation.

Introduction.

This paper is intended as a continuation to [3], [4]. As in [3], let H be a Hilbert space over $\mathbf{C}$ and then A be a linear hermitian operator with dense domain, $D(A) \subset H \to H$. We assume also the existence of a complete sequence $(e_k)_1^\infty$ in H, the e_k's being orthonormal eigen-vectors of A corresponding to (real) nonzero eigen-values λ_k.

Next, let $a(t)$ be a complex-valued continuous almost-periodic

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(Bohr) function such that

$$(0.1) \qquad a^* = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \mathrm{Re}\, a(\sigma)\, d\sigma\right)$$

belongs to $\mathbf{R}/\{0\}$.

Finally, let $f(t)$ be a H-valued (Bochner) almost-periodic function. We shall indicate a sufficient condition involving the above considered entities in order that an almost-periodic solution of the equation in the title exists, at least in the ultra-weak sense. (The unicity would then follow from [3]).

The « simplest » case when $H = \mathbf{C}$ and A is the identity operator, that is the scalar ordinary differential equation $u'(t) = a(t)\,u(t) + f(t)$ is explained as Theorem 6.6 in [1]. We had to extend the arguments used there so as to apply them in the general case of operator differential equations.

1. Let us define the complex-valued almost-periodic functions $f_j(t)$ given by the scalar product $(f(t), e_j)_H$, $\forall j = 1, 2, \dots$.

Then let us consider (as in [1]) the following functions:

$$(1.1) \qquad u_j(t) = \int_{-\infty}^{t} \exp\left(\int_s^t \lambda_j\, a(u)\, du\right) f_j(s)\, ds\,, \qquad \text{for } \lambda_j a^* < 0\,,$$

$$(1.2) \qquad u_j(t) = -\int_t^{\infty} \exp\left(\int_s^t \lambda_j\, a(u)\, du\right) f_j(s)\, ds\,, \qquad \text{for } \lambda_j a^* > 0\,.$$

These functions are solutions on the real line of the ordinary differential equations

$$(1.3) \qquad u_j'(t) = \lambda_j\, a(t)\, u_j(t) + f_j(t)$$

and they are almost-periodic (this follows from Favard's theorem, see [1]). For our purposes some very careful estimates of (1.1) and (1.2) are quite essential; we shall give them in the following lines.

i) Assume $\lambda_j > 0$ and $a^* < 0$; then

$$(1.4) \qquad \left|\exp\left(\int_s^t \lambda_j\, a(u)\, du\right)\right| = \exp\left(\int_s^t \lambda_j (\mathrm{Re}\, a)(u)\, du\right) = \exp\left(\lambda_j \int_{t-\sigma}^{t} (\mathrm{Re}\, a)(u)\, du\right)$$

where we denote: $\sigma = t - s$.

On the other hand we know that:

$$\lim_{\sigma\to\infty}\frac{1}{\sigma}\int\limits_{t-\sigma}^{t}(\mathrm{Re}\,a)(u)\,du = a^* ,$$

uniformly with respect to $t\in\mathbf{R}$. Hence, given $\varepsilon = \frac{1}{2}|a^*|$, there is a positive number T_0 such that

$$(1.5)\qquad \frac{1}{\sigma}\int\limits_{t-\sigma}^{t}(\mathrm{Re}\,a)(u)\,du < \frac{1}{2}a^* , \qquad \text{for } \sigma > T_0 \text{ and } \forall t\in\mathbf{R}\,.$$

(Note that T_0 is independent of $j\in\mathbf{N}$).

Consequently, we see that

$$(1.6)\qquad \exp\Big(\lambda_j\int\limits_{t-\sigma}^{t}(\mathrm{Re}\,a)(u)\,du\Big)\leqslant \exp\left(\tfrac{1}{2}\lambda_j a^*\sigma\right) , \qquad \sigma > T_0,\ t\in\mathbf{R}$$

Consider now the integral:

$$\int\limits_{0}^{\infty}\exp\Big(\lambda_j\int\limits_{t-\sigma}^{t}(\mathrm{Re}\,a)(u)\,du\Big)f_j(t-\sigma)\,d\sigma = I_j$$

and write: $I_j = I_{j,1} + I_{j,2}$ where

$$(1.7)\qquad I_{j,1} = \int\limits_{0}^{T_0}\exp\Big(\lambda_j\int\limits_{t-\sigma}^{t}(\mathrm{Re}\,a)(u)\,du\Big)f_j(t-\sigma)\,d\sigma ,$$

$$(1.8)\qquad I_{j,2} = \int\limits_{T_0}^{\infty}\exp\Big(\lambda_j\int\limits_{t-\sigma}^{t}(\mathrm{Re}\,a)(u)\,du\Big)f_j(t-\sigma)\,d\sigma .$$

We see the estimate (for $\lambda_j > 0,\ \sigma > 0,\ 0 < \sigma < T_0,\ t\in\mathbf{R}$).

$$(1.9)\qquad \lambda_j\int\limits_{t-\sigma}^{t}(\mathrm{Re}\,a)(u)\,du \leqslant \lambda_j\|\mathrm{Re}\,a\|_\infty\cdot\sigma \leqslant \lambda_j\|a\|_\infty T_0$$

hence

$$I_{j,1} \leqslant T_0 \exp\left(\lambda_j \|a\|_\infty T_0\right) \|f_j\|_\infty \tag{1.10}$$

where $\|\varphi\|_\infty = \sup_{\xi \in \mathbf{R}} |\varphi(\xi)|$ in (1.9) and (1.10) above.

Next, we have, using (1.6)

$$I_{j,2} \leqslant \left(\int_{T_0}^{\infty} \exp\left(\frac{1}{2}\lambda_j a^* \sigma\right) d\sigma\right) \|f_j\|_\infty = \frac{2}{|a^*|\lambda_j} \exp\left(\frac{1}{2}\lambda_j a^* T_0\right) \|f_j\|_\infty . \tag{1.11}$$

Thus, altogether, we obtain

$$\begin{aligned} \left|\int_0^\infty \exp\left(\lambda_j \int_{t-\sigma}^{t} (\mathrm{Re}\, a)(u)\, du\right) f_j(t-\sigma)\, d\sigma\right| &= |I_{j,1} + I_{j,2}| \leqslant \\ &\leqslant \|f_j\|_\infty \left(T_0 \exp\left(\lambda_j \|a\|_\infty T_0\right) + \frac{2}{|a^*|\lambda_j} \exp\left(\frac{1}{2}\lambda_j a^* T_0\right)\right) \end{aligned} \tag{1.12}$$

which is the desired estimate for $u_j(t)$ in (1.1) in the case $\lambda_j > 0$, $a^* < 0$.

ii) Consider now the case where $a^* < 0$ and $\lambda_j < 0$, and estimate the expression (1.2) which becomes (with the substitution $s = \sigma + t$)

$$-\int_0^\infty \exp\left(-\int_t^{t+\sigma} \lambda_j a(u)\, du\right) f_j(\sigma + t)\, d\sigma . \tag{1.13}$$

Again we see that $\left|\exp\left(-\int_t^{t+\sigma} \lambda_j a(u)\, du\right)\right| = \exp\left(|\lambda_j| \int_t^{t+\sigma} (\mathrm{Re}\, a)(u)\, du\right)$ and

$$\int_t^{t+\sigma} (\mathrm{Re}\, a)(u)\, du < \tfrac{1}{2} a^* \sigma , \qquad \text{for } \sigma \geqslant T_0,\ t \in \mathbf{R} . \tag{1.14}$$

It follows that the function $u_j(t)$ given by (1.2) is estimated by

$$\|f_j\|_\infty \left(T_0 \exp\left(|\lambda_j|\, \|a\|_\infty T_0\right) + \frac{2}{|a^* \lambda_j|} \exp\left(\frac{1}{2} |\lambda_j| a^* T_0\right)\right) . \tag{1.15}$$

iii) Take now $a^* > 0$ and $\lambda_j > 0$; again we estimate the integral (1.13). We see that

$$(1.16)\qquad \frac{a^*}{2} \leqslant \frac{1}{\sigma}\int_t^{t+\sigma} (\mathrm{Re}\, a)(u)\, du \quad \text{and} \quad -\lambda_j \int_t^{t+\sigma} (\mathrm{Re}\, a)(u)\, du \leqslant -\frac{1}{2}\sigma a^* \lambda_j ,$$

$$\text{for } \sigma \geqslant T_0 ,\ t \in \mathbf{R} .$$

Decomposing (1.13) as in i) above, we get the integrals from 0 to T_0 and from T_0 to $+\infty$. The first is estimated by

$$\|f_j\|_\infty \cdot T_0 \exp(\lambda_j \|a\|_\infty T_0)$$

while the second by

$$\|f_j\|_\infty \frac{2}{a^* \lambda_j} \exp\left(-\frac{1}{2} a^* \lambda_j T_0\right),$$

so that we get an estimate by

$$(1.17)\qquad \|f_j\|_\infty \left(T_0 \exp(\lambda_j \|a\|_\infty T_0) + \frac{2}{a^* \lambda_j} \exp\left(-\frac{1}{2} a^* \lambda_j T_0\right)\right).$$

iv) In the last case where $a^* > 0$ and $\lambda_j < 0$ we must estimate the integral (1.1) which is also

$$\int_0^\infty \exp\left(\int_{t-\sigma}^t \lambda_j a(u)\, du\right) f_j(t-\sigma)\, d\sigma .$$

As above we see that $\frac{1}{2}\sigma a^* \lambda_j \geqslant \lambda_j \int_{t-\sigma}^t (\mathrm{Re}\, a)(u)\, du$ for $\sigma \geqslant T_0$, $t \in \mathbf{R}$ and we get for (1.1) an upper bound equal to

$$(1.18)\qquad \|f_j\|_\infty \left(T_0 \exp(\lambda_j \|a\|_\infty T_0) + \frac{2}{a^* |\lambda_j|} \exp\left(\frac{1}{2} a^* \lambda_j T_0\right)\right).$$

Let us define, for any $j = 1, 2, 3, \ldots$, the numbers

$$(1.19)\qquad \omega_j = \left(T_0 \exp(|\lambda_j|\, \|a\|_\infty T_0) + \frac{2}{|a^* \lambda_j|} \exp\left(-\frac{1}{2} T_0 |a^* \lambda_j|\right)\right).$$

We can see then that

$$(1.20)\qquad |u_j(t)| \leqslant \omega_j \|f_j\|_\infty, \qquad \forall t \in \mathbf{R},\ j = 1, 2, \dots .$$

2. Let us make now the

MAIN HYPOTHESIS. *The numerical series* $\sum_{j=1}^{\infty} \omega_j^2 \|f_j\|_\infty^2$ *is convergent.* We state the following:

THEOREM. *Assuming the* M-H *to be true, the vector-series in* H: $\sum_{j=1}^{\infty} u_j(t)\, e_j$ *is uniformly convergent on the real line and is the unique almost-periodic ultra-weak solution of the abstract differential equation*

$$(2.1)\qquad u'(t) = a(t)\, A u(t) + f(t)$$

over the whole real line.

Note that « ultra-weak » solution of the above equation means here, as in the paper [3], that the integral identity

$$(2.2)\qquad \int_{\mathbf{R}} \big(u(t), \varphi'(t) + \bar{a}(t) A^* \varphi(t)\big)_H\, dt = 0 \qquad \forall \varphi \in K_{A^*}(\mathbf{R})$$

holds. Here

$$K_{A^*}(\mathbf{R}) = \{\varphi(t), \mathbf{R} \to D(A^*), \varphi \in C_0^1(\mathbf{R}; H), A^* \varphi \in C(\mathbf{R}; H)\}$$

where $A^* \supset A$ is the adjoint operator to A, and $\bar{a}(t)$ is the complex conjugate of $a(t)$, $\forall t \in \mathbf{R}$.

The uniqueness of all almost-periodic solutions follows from [3]. Hence, it remains to prove existence, which is done essentially as in [4].

The uniform (over $\mathbf{R}$) convergence of the series $\sum_{1}^{\infty} u_j(t)\, e_j$ in H-norm follows from the relation

$$(2.3)\qquad \Big\| \sum_{N}^{N+p} u_j(t)\, e_j \Big\|_H^2 = \sum_{N}^{N+p} |u_j(t)|^2$$

from the Main Hypothesis and the estimates in section 1. Hence its sum $u(t)$ is H-almost-periodic.

Now, let $v_j(t) = u_j(t)\,e_j$; we have

$$v_j'(t) = u_j'(t)\,e_j = \lambda_j\,a(t)\,u_j(t)\,e_j + f_j(t)\,e_j = \lambda_j\,a(t)\,v_j(t) + f_j(t)\,e_j\,,$$

hence

$$v_j'(t) = a(t)\,Av_j(t) + f_j(t)\,e_j \tag{2.4}$$

holds, $\forall j = 1, 2, \ldots$, in H-sense. $\big($Due to $Av_j(t) = A\big(u_j(t)\,e_j\big) = u_j(t)\,Ae_j = u_j(t)\,\lambda_j\,e_j = \lambda_j\,v_j(t)\big)$.

Next, put $w_N = v_1(t) + \ldots + v_N(t)$. We see that the equality

$$w_N'(t) = a(t)\,Aw_N(t) + g_N(t) \tag{2.5}$$

in H-sense, over $\mathbb{R}$ holds, where

$$g_N(t) = \sum_1^N f_j(t)\,e_j\,.$$

Taking any $\varphi \in K_{A^*}(\mathbf{R})$ we obtain

$$\int_{\mathbf{R}} \big(w_N'(t) - a(t)\,Aw_N(t), \varphi(t)\big)_H\,dt = \int_{\mathbf{R}} \big(g_N(t), \varphi(t)\big)_H\,dt\,. \tag{2.6}$$

By partial integration we derive

$$\int_{\mathbf{R}} \big(w_N'(t), \varphi(t)\big)\,dt = -\int_{\mathbf{R}} \big(w_N(t), \varphi'(t)\big)\,dt\,. \tag{2.7}$$

Also

$$\int_{\mathbf{R}} \big(a(t)\,Aw_N(t), \varphi(t)\big)_H\,dt = \int_{\mathbf{R}} \big(Aw_N(t), \bar{a}(t)\,\varphi(t)\big)\,dt = \int_{\mathbf{R}} \big(w_N(t)\ \ \bar{a}A^*\varphi\big)(t)\,dt \tag{2.8}$$

Accordingly we see that for any natural N the identity

$$\int_{\mathbf{R}} \big(w_N(t), \varphi'(t) + \bar{a}(t)\,A^*\varphi(t)\big)_H\,dt = -\int_{\mathbf{R}} \big(g_N(t), \varphi(t)\big)_H\,dt \tag{2.9}$$

holds $\forall \varphi \in K_{A^*}(\mathbf{R})$.

Now, let us remember that $g_N(t) \to f(t) (N \to \infty)$, *uniformly* on **R**, due to the relative compactness of the trajectory of $f(t)$. Also, as previously seen,

$$w_N(t) \to u(t) = \sum_1^\infty u_j(t)\, e_j\,,$$

uniformly on **R**.

It follows that u is ultra-weak almost-periodic solution of $u'(t) = a(t) Au(t) + f(t)$. This proves the Theorem.

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