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Extension of Certain Classes of Generating Functions.

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Abstract - In this paper a new class of generating functions for the arbitrary sequence of functions is derived with the aid of an expansion formula. This provides extension to various known classes of generating functions. Examples are cited to illustrate the applications, and a further generalization of the main result is also considered.

1. Introduction.

Two useful consequences of the Lagrange's expansion theorem are the expansion formulae [7, p. 355]:

$$\sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n} t^n = (1+w)^{\alpha+1}(1-\beta w)^{-1}, \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha + (\beta+1)n} \binom{\alpha + (\beta+1)n}{n} t^n = (1+w)^{\alpha}, \tag{1.2}$$

where α and β are arbitrary numbers, and w is a function defined implicitly in terms of t given by

$$w = t(1+w)^{\beta+1}, \quad w(0) = 0. \tag{1.3}$$

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Indeed, it has been pointed out by Srivastava and Panda [5, p. 181], that (1.1) and (1.2) are equivalent in the sense that either one implies the other. An interesting unification of the equivalent expansions (1.1) and (1.2) is the one given by Gould [1, p. 196, Eq. (6.1)]:

$$(1.4) \qquad \sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n = \\ = (1+w)^{\alpha} \sum_{n=0}^{\infty} f_n^{(\alpha,\beta,\gamma)}(w/1+w) ,$$

where the arbitrary parameters α, β and γ are independent of n; w is given by (1.3), and

$$(1.5) \qquad f_n^{(\alpha,\beta,\gamma)}(z) = (-1)^n \binom{\alpha-\gamma}{n} \binom{n+\gamma/(\beta+1)}{n}^{-1} z^n .$$

A number of results on generating functions originating from the pair of equivalent relations (1.1) and (1.2) or their generalized form (1.4), have appeared in literature. A fairly great deal of account of such results is incorporated in a recent monograph on the subject by Srivastava and Manocha [7]. One may also refer to the papers of Srivastava and Raina [6] and Raina ([2], [3]) (*) which judiciously uses the familiar Fox's H-function in their results on generating functions.

In this paper we derive a new class of generating functions by invoking a certain expansion formula (2.1), below.

This provides extension to many known generating-function relationships. The main result is further generalized and applications are given indicating the relevance with known results.

2. Generating functions.

Before stating the main result, we require the following result:

LEMMA. *If α, β, γ, δ and μ are arbitrary parameters (independent*

(*) A minor correction in [3] may be noted. In eq. (2.1) on p. 151 (and elsewhere also) of [3], the variable « x » should be changed to « $-x$ ».

of n), then

$$\sum_{n=0}^{\infty} \frac{\gamma(\delta+\mu n)}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n = \tag{2.1}$$

$$= (1+w)^{\alpha} \left[\frac{\gamma\mu(1+w)}{(1+\beta)(1-\beta w)} + \left(\delta - \frac{\gamma\mu}{1+\beta}\right) \sum_{n=0}^{\infty} f_n^{(\alpha,\beta,\gamma)}(w/1+w) \right],$$

where w is given by (1.3), *and the function $f_n(-)$ is given by* (1.5).

Proof. The formula (2.1) follows straightforwardly, if we write

$$\text{L.H.S.} = \frac{\gamma\mu}{1+\beta} \sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} t^n +$$

$$+ \left(\delta - \frac{\gamma\mu}{1+\beta}\right) \sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n,$$

and then invoke (1.1) and (1.4).

Remark 1. For $\gamma = \alpha$, and noticing from (1.5) that

$$\sum_{n=0}^{\infty} f_n^{(\alpha,\beta,\alpha)}(z) = 1, \tag{2.2}$$

then (2.1) evidently corresponds to the expansion formula recorded, for instance, in [7, p. 385, Problem 4].

Given an arbitrary bounded sequence $\{C_n\}$, $n \geqslant 0$, let the functions $F_n(z)$ and $g(z)$ be defined by

$$F_n(z) = \sum_{k=0}^{[n/m]} \binom{\alpha+(\beta+1)n}{n-mk} C_k z^k, \tag{2.3}$$

and

$$g(z) = \sum_{k=0}^{\infty} C_k z^k, \qquad C_k \neq 0, \tag{2.4}$$

where α and β are arbitrary parameters, and m is any positive integer. We now propose to establish a new class of generating functions which is contained in the following:

THEOREM 1. *If the functions $F_n(z)$ and $g(z)$ are defined by* (2.3) *and* (2.4), *respectively, then for arbitrary values γ, δ and μ (independent of n),*

$$(2.5)\qquad \sum_{k=0}^{\infty} \frac{\delta+\mu n}{\gamma+(\beta+1)n} F_n(z)t^n =$$

$$= (1+w)^{\alpha}\left[\frac{\mu(1+w)}{(1+\beta)(1-\beta w)} g(zw^m) + \left(\frac{\delta}{\gamma} - \frac{\mu}{1+\beta}\right)\cdot \right.$$

$$\left. \cdot \sum_{n,k=0}^{\infty} \frac{(\gamma/(\beta+1))_{mk}}{(1+n+\gamma/(\beta+1))_{mk}} C_k(zw^m)^k f_n^{(\alpha,\beta,\gamma)}(w/1+w)\right],$$

where w and $f_n(-)$ are given, respectively, by (1.3) *and* (1.5), *provided that both the sides of* (2.5) *exist.*

PROOF: In view of (2.3), (2.5) gives

$$\text{L.H.S.} = \sum_{n=0}^{\infty} \frac{\delta+\mu n}{\gamma+(\beta+1)n}\left\{\sum_{k=0}^{[n/m]} \binom{\alpha+(\beta-1)n}{n-mk} C_k z^k\right\} t^n =$$

$$= \sum_{k=0}^{\infty} C_k \frac{(zt^m)^k}{\gamma+(\beta+1)mk}\left\{\sum_{n=0}^{\infty} \frac{(\gamma+(\beta+1)mk)(\delta+\mu mk+\mu n)}{(\gamma+(\beta+1)mk+(\beta+1)n)}\cdot\right.$$

$$\left.\cdot\binom{\alpha+(\beta+1)mk+(\beta+1)n}{n} t^n\right\}.$$

Now appealing to the expansion formula (2.1) to evaluate the inner sum by replacing δ by $\delta+\mu mk$, γ by $\gamma+(\beta+1)mk$ and α by $\alpha+(\beta+1)mk$, respectively; the desired right hand side of (2.5) is readily obtained in conjuction with (2.4).

REMARK 2. If $\delta=\gamma$, $\mu=0$ and $z=1$, then Theorem 1 reduces to the Zeitlin's generating function [8, p. 410, Theorem 3].

On the other hand, if $\gamma=\alpha$, then Theorem 1 by virtue of (2.2) is seen to correspond to Srivastava's result mentioned in [7, p. 391].

3. Applications.

The specialization of the various parameters and suitable setting of the arbitrary sequence would lead us to a wide range of known and new generating functions.

To illustrate, let us set

$$C_n = (-1)^{mn}\left\{\prod_{i=1}^{p}(a_i)_n\right\}\left\{n!\prod_{i=1}^{q}(b_i)_n\right\}^{-1}, \qquad n\geqslant 0, \tag{3.1}$$

then (2.3) and (2.4) give

$$F_n(z) = \binom{\alpha+(\beta+1)n}{n}\,{}_{p+m}F_{q+m}\left[\begin{matrix}\Delta(m;-n) & , (a);_p \\ \Delta(m;1+\alpha+\beta n), & (b_q);\end{matrix}\; z\right], \tag{3.2}$$

and

$$g(z) = {}_pF_q\left[\begin{matrix}(a_p); \\ (b_q);\end{matrix}\; z(-1)^m\right], \tag{3.3}$$

where $\Delta(m;x)$ condenses the array of parameters defined by

$$\Delta(m;x) = \prod_{j=1}^{m}(x+j-1)/m, \qquad m\geqslant 1. \tag{3.4}$$

The result which emerges by the substitution of (3.1) and (3.2) in Theorem 1, is the result

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\delta+\mu n}{\gamma+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n}\cdot \\ \cdot{}_{p+m}F_{q+m}\left[\begin{matrix}\Delta(m;-n) & , (a_p); \\ \Delta(m;1+\alpha+\beta n), & (b_q);\end{matrix}\; z\right]t^n = \\ = (1+w)^{\alpha}\left[\frac{\mu(1+w)}{(1+\beta)(1-\beta w)}\,{}_pF_q\left[\begin{matrix}(a_p); \\ (b_q);\end{matrix}\; z(-w)^m\right] + \right.\\ + \left(\frac{\delta}{\gamma}-\frac{\mu}{1+\beta}\right)\sum_{n=0}^{\infty} f_n^{(\alpha,\beta,\gamma)}(w/1+w)\cdot \\ \left.\cdot{}_{p+m}F_{q+m}\left[\begin{matrix}\Delta(m;\gamma/(\beta+1)) & , (a_p); \\ \Delta(m;1+n+\gamma/(\beta+1)), & (b_q);\end{matrix}\; z(-w)^m\right]\right],\end{aligned} \tag{3.5}$$

where w is given by (1.3) and $f_n(-)$ by (1.5).

If $\delta = \alpha$, and keeping in mind the relation (2.2), then restructurring the parameters in the hypergeometric function and effecting elementary simplification in the process, we find that (3.5) leads to the same result as listed in [7, p. 391]. If we put $p = q = 0$, and choose

$m = 1$, then (3.5) gives the result

$$(3.6)\qquad \sum_{n=0}^{\infty} \frac{\delta + \mu n}{\gamma + (\beta + 1)n} L_n^{(\alpha+\beta n)}(z)\, t^n =$$

$$= (1 + w)^\alpha \left[\frac{\mu(1 + w)\exp(-wz)}{(1 + \beta)(1 - \beta w)} + \left(\frac{\delta}{\gamma} - \frac{\mu}{1 + \beta}\right) \sum_{n=0}^{\infty} f_n^{(\alpha,\beta,\gamma)}(w/1 + w) \cdot {}_1F_1\left[\begin{matrix}\gamma/(\beta + 1) & ; \\ 1 + n + \gamma/(\beta + 1); & \end{matrix} -wz\right]\right],$$

where w and $f_n(-)$ (as before) are given, respectively, by (1.3) and (1.5).

For $\delta = \gamma$ and $\mu = 0$, (3.6) is substantially the same result as given in [8, p. 409, Eq. (3.4)].

4. A generalization of Theorem 1.

A generalization of Theorem 1 can be contemplated if we consider the Srivastava-Buschman polynomial system [4] defined by

$$(4.1)\qquad S_{n,m}^{(\alpha,\beta)}(h; z) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}(\alpha + (\beta + 1)n + 1)_{mk}}{(\alpha + \beta n + 1)_{(h+m)k}} A_k z^k,$$

where α, β and h are arbitrary parameters (real or complex), and m is any positive integer.

By proceeding on the same lines as indicated in the proof of Theorem 1, and applying (2.1), we arrive at the following generalization of Theorem 1.

Theorem 2. *If a polynomial system is defined by* (4.1), *then for arbitrary constants* δ *and* μ,

$$(4.2)\qquad \sum_{n=0}^{\infty} \frac{\delta + \mu n}{\gamma + (\beta + 1)n} \binom{\alpha + (\beta + 1)n}{n} S_{n,m}^{(\alpha,\beta)}(h; z)\, t^n =$$

$$= (1 + w)^\alpha \left[\frac{\mu(1 + w)}{(1 + \beta)(1 - \beta w)} g\big(z(-w)^m(1 + w)^h\big) + \left(\frac{\delta}{\gamma} - \frac{\mu}{1 + \beta}\right) \cdot \sum_{n,k=0}^{\infty} \frac{(\gamma/(\beta + 1))_{mk}}{(1 + n + \gamma/(\beta + 1))_{mk}} A_k \big(z(-w)^m(1 + w)^h\big)^k f_n^{(\alpha+hk,\beta,\gamma)}(w/1 + w)\right],$$

where w *and* $f_n(-)$ *are, respectively, given by* (1.3) *and* (1.5).

It is easy to notice that Theorem 2 corresponds to Theorem 1 in the special case when $h = 0$, and $A_k = (-1)^{mk} C_k$, $k \geqslant 0$. Also, it is worth noting that the main result in the Srivastava-Buschman paper [4, p. 361, Theorem 2] is a limiting case of our Theorem 2, when $\delta = \gamma$, $\mu = 0$, and $|\gamma| \to \infty$ (formally). Further, we conclude by remarking that Theorem 2 would widely be applicable to all those polynomials which arise from the system (4.1), including those listed in [4].

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