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No rational nonsingular quartic curve $C_4 \subset \mathbb{P}^3$ can be set-theoretic complete intersection on a cubic surface

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No Rational Nonsingular Quartic Curve $\mathscr{C}_4 \subset \mathbf{P}^3$ can be Set-Theoretic Complete Intersection on a Cubic Surface.

P. C. Craighero - R. Gattazzo (*)

Riassunto - In questo lavoro si dimostra che in $\mathbf{P}^3$, spazio proiettivo di dimensione 3 sopra un campo k algebricamente chiuso di caratteristica zero, ogni curva $\mathcal{C}_4$, del quarto ordine, razionale e non singolare, non può essere sottoinsieme intersezione completa di alcuna coppia di superficie $(\mathcal{F}_3, \mathcal{F}_n)$, con $\mathcal{F}_3$ superficie cubica ed $\mathcal{F}_n$ superficie di ordine n, qualunque sia n.

Introduction.

Let $\mathbf{P}^3$ be the projective space of dimension 3 over an algebraically closed field of characteristic zero.

The first authors who analyzed the problem whether a nonsingular rational quartic curve $\mathscr{C}_4 \subset \mathbf{P}^3$ can be set-theoretic complete intersection (s.t.c.i.) of two surfaces were:

1) L. Godeaux and D. Gallarati (see Introduzione in $[S_2]$), who proved that, under some hypotheses of generality concerning the kind of the singularities that the two surfaces have on $\mathscr{C}_4$, $\mathscr{C}_4$ itself cannot be s.t.c.i. of a cubic surface $\mathscr{F}_3$ and a quartic surface $\mathscr{F}_4$;

2) P. C. Graighero in [C], who proved that $\mathscr{C}_4' = (\lambda^4, \lambda^3\mu, \lambda\mu^3, \mu^4)$ is not s.t.c.i. of any pair of surfaces $\mathscr{F}_3$ and $\mathscr{F}_4$;

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3) E. Stagnaro in [S_2], who proved that every nonsingular rational quartic curve $\mathscr{C}_4$, is not s.t.c.i. of any pair of surfaces $\mathscr{F}_3$ and $\mathscr{F}_4$;

4) P. C. Craighero and R. Gattazzo in [C-G], who proved that $\mathscr{C}'_4$ is not s.t.c.i. of any pair of surfaces both of degree 4.

In this paper the authors prove that $\mathscr{C}_4$ is not s.t.c.i. of any pair of surfaces $\mathscr{F}_3$ and $\mathscr{F}_n$, for every n. This is a generalization of the result n. 3) here above, and it is in the frame of a program of research in which other authors are presently engaged, see for example D. B. Jaffe, who concludes the Introduction of its paper [J] by saying: « We hope that the techniques of this article will contribute to the determination of which smooth curves are set-theoretic complete intersections on other types of surfaces in $\mathbf{P}^3$, for instance on arbitrary ruled surfaces, on arbitrary cubic surfaces, or on surfaces having only rational singularities ».

We remark that, in the case when $\mathscr{F}_3$ is a ruled surface (but not a cone), a new method is set up, which we think could be of interest in a more general context (see Remark 3, § 4).

Many authors begin to conjecture that $\mathscr{C}'_4$, and even $\mathscr{C}_4$, is not s.t.c.i. of any pair of surfaces in $\mathbf{P}^3$. The aim of this paper, as well as the papers listed above, is that of finding new methods and devices which can be finally used in proving this conjecture, or at least to state it for particular degrees, or for particular kinds, of the two surfaces, in the hope that the conjecture could be reduced to these cases. Actually somewhat of this sort happens here in reaching our result: indeed, the main Theorem in [S_2] is of help in proving our Prop. 1; the key Lemma 1 in [S_2] is used in proving Prop. 5, case 3); and finally the recent result of D. B. Jaffe about curves on cones in [J] is used in the case when $\mathscr{F}_3$ is a cone (see Prop. 2).

Notations.

In what follows $\mathbf{P}^3$ denotes the projective space of dimension 3 over an algebraically closed field of characteristic zero. By a cubic (cubic ruled) surface $\mathscr{F}_3$, we always mean a reduced and irreducible cubic (cubic ruled) surface in $\mathbf{P}^3$; by a curve $\mathscr{C}$ we mean a reduced and irreducible curve in $\mathbf{P}^3$. If $\mathscr{F}$ is a surface in $\mathbf{P}^3$ and $\mathscr{C}$ a curve, with $\mathscr{C} \subset \mathscr{F}$, we say that $\mathscr{C}$ is set-theoretic complete intersection (s.t.c.i.) on $\mathcal{F}$, or complete intersection (c.i.) on $\mathcal{F}$ if there exists a surface $\mathcal{G}$ such that $\mathcal{F} \cap \mathcal{G} = \mathcal{C}$, or respectively if $\mathcal{F} \cap \mathcal{G} = \mathcal{C}$ and $I(\mathcal{C}, \mathcal{F} \cap \mathcal{G}) = 1$. The expressions « non singular rational quartic curve » and « straight line(s) » will be in the sequel respectively shortened in « non s.r.q.c. » and « s.l. ». Given a s.l. r and a point $P \in r$, $[P, r]$ will denote the plane joining P and r; given two coplanar lines r, s $[r, s]$ will denote the plane joining r and s.

1. In this paragraph we examine the cubic surfaces $\mathcal{F}_3$, containing a non s.r.q.c. $\mathcal{C}_4$, that are either non singular in codimension 1, or cones, concluding (see Prop. 1, and Prop. 2) that on these surfaces $\mathcal{C}_4$ cannot be s.t.c.i.

LEMMA 1 (D. Gallarati). *Let $\mathcal{F}$ be an irreducible surface in $\mathbf{P}^3$, non singular in codimension 1, and let $\mathcal{G}$ be a surface in $\mathbf{P}^3$ such that*

$$\mathcal{F} \cdot \mathcal{G} = q_1 \mathcal{C}_1 + \dots + q_t \mathcal{C}_t ,$$

where $\mathcal{C}_i$ denotes a curve on $\mathcal{F}$, and $q_i = I(\mathcal{C}_i, \mathcal{F} \cap \mathcal{G})$, $i = 1, \dots, t$. If there exists a surface $\mathcal{H}'$ s.t. $\mathcal{F} \cdot \mathcal{H}' = s' \mathcal{C}_1$, $s' > 0$, then there exists a surface $\mathcal{H}$ and a positive integer $s \leq s'$, s.t.

$$\mathcal{F} \cdot \mathcal{H} = s(q_2 \mathcal{C}_2 + \dots + q_t \mathcal{C}_t) .$$

LEMMA 2 (E. Stagnaro). *Let $\mathcal{M}$ be an irreducible (hence reduced) monoid surface in $\mathbf{P}^3$, with the vertex at $O = (0, 0, 0)$, whose degree is $n \geq 2$, and which is non singular in codimension 1:*

$$\mathcal{M} = \{F = \alpha x_0 + \beta = 0\} , \qquad \alpha, \beta \in k[X_1, X_2, X_3] .$$

Let us suppose that we have either

$$(*) \qquad \alpha = \alpha_0^d , \qquad \text{with } d > 0 , \ \alpha_0 \text{ irreducible} ,$$

or

$$(**) \qquad \alpha = \alpha_1^{d_1} \alpha_2^{d_2} , \qquad d_i > 0 \ (i = 1, 2)$$

with α_i $(i = 1, 2)$ indipendent linear polynomials. Let $r \subset \{\alpha = \beta = 0\}$ be a s.l. on $\mathcal{M}$ through O, and let $\mathcal{G} = \{G = 0\}$ be a surface of degree m

such that

$$\mathscr{M}\cdot\mathscr{G} = mn\imath .$$

Under these assumptions we have the following:

— *in the first case* (*), *within a constant of* k, *we have that*

(#) $G = \alpha_0^q + \delta F$, *with suitable* $q > 0$, *and* $\delta \in k[X_0, X_1, X_2, X_3]$;

— *in the second case* (**) *we have three possibilities:*

(##) 1) *either* $G = \alpha_i^q + \delta_i F$
(*with* $i = 1$ *or* $i = 2$, *and within a constant of* k);

2) *or* $\{\alpha_1\} \cap \mathscr{M} = \imath \cup \imath'$,
where $\imath'$ *is a s.l. different from* $\imath$; *moreover we have*
$\{\alpha_2 = 0\} \cap \mathscr{M} = \imath'$, *and* $I(\imath', \{\alpha_1 = 0\} \cap \mathscr{M}) = 1$;

3) *or* $\{\alpha_2\} \cap \mathscr{M} = \imath \cup \imath''$,
where $\imath''$ *is a s.l. different from* $\imath$; *moreover we have*
$\{\alpha_1 = 0\} \cap \mathscr{M} = \imath''$, *and* $I(\imath'', \{\alpha_2 = 0\} \cap \mathscr{M}) = 1$.

For the proofs of the above two Lemmas see [S_1], pp. 139-143.

LEMMA 3 (L. Robbiano). *Let* $\mathscr{F}$ *be a (reduced and irreducible) surface in* $\mathbf{P}^3$; *let* $\mathscr{C}$ *be a curve on* $\mathscr{F}$, *and let* $\mathscr{F}$ *have no singular point on* $\mathscr{C}$. *Then, if* $\mathscr{C}$ *is s.t.c.i. on* $\mathscr{F}$, $\mathscr{C}$ *os even a c.i. on* $\mathscr{F}$.

PROOF. See [R].

LEMMA 4. *Let* $\mathscr{F}_3$ *be a cubic surface nonsingular in codimension* 1 *and not a cone. If* $\imath \subset \mathscr{F}_3$ *is a s.l. which is s.t.c.i. on* $\mathscr{F}_3$, *then there exists a surface* $\mathscr{H}$ *such that*

$$\mathscr{F}_3 \cdot \mathscr{H} = 6\imath .$$

PROOF. From Lemma 3 it follows that $\mathscr{F}_3$ has necessarily a singular point on $\imath$: being not a cone, $\mathscr{F}_3$ must be a monoid surface with a double point D on $\imath$. Then we can apply to $\mathscr{F}_3$ and $\imath$ Lemma 2 (of course within a linear isomorphism we can suppose $D = O$). Then the following cases can happen:

a) $\alpha = \alpha_0^d$, with α_0 irreducible of degree 2, and $d = 1$: we have in this case that

$$\mathscr{F}_3 \cdot \{\alpha_0 = 0\} = 6\imath .$$

b) $\alpha = \alpha_0^d$, with α_0 of degree 1, and $d = 2$: we have in this case that

$$\mathscr{F}_3 \cdot \{\alpha_0 = 0\} = 3\imath ,$$

and, assuming $\mathscr{H} = \{\alpha_0^2 = 0\}$, we have

$$\mathscr{F}_3 \cdot \mathscr{H} = 6\imath .$$

c) $\alpha = \alpha_1 \alpha_2$, with α_1, α_2 independent linear polynomials: we have in this case four possibilities

c_1) $\mathscr{F}_3 \cdot \{\alpha_1 = 0\} = 3\imath$;

c_2) $\mathscr{F}_3 \cdot \{\alpha_2 = 0\} = 3\imath$;

c_3) $\mathscr{F}_3 \cdot \{\alpha_1 = 0\} = 2\imath + \imath'$ and $\mathscr{F}_3 \cdot \{\alpha_2 = 0\} = 3\imath'$;

c_4) $\mathscr{F}_3 \cdot \{\alpha_2 = 0\} = 2\imath + \imath''$ and $\mathscr{F}_3 \cdot \{\alpha_1 = 0\} = 3\imath''$.

Now, if c_1) or c_2) happens, we take $\mathscr{H} = \{\alpha_1^2 = 0\}$ or $\mathscr{G} = \{\alpha_2^2 = 0\}$: in both cases we get

$$\mathscr{F}_3 \cdot \mathscr{H} = 6\imath .$$

If c_3) or c_4) happens, then, applying Lemma 1 to $\mathscr{M}$ $(= \mathscr{F})$, $\{\alpha_1 = 0\}$ $(= \mathscr{G})$ and $\imath$ $(= \mathscr{C}_1)$, or to $\mathscr{M}$ $(= \mathscr{F})$, $\{\alpha_2 = 0\}$ $(= \mathscr{G})$, and $\imath$ $(= \mathscr{C}_1)$ respectively, we can find a surface $\mathscr{H}$ such that

$$\mathscr{F} \cdot \mathscr{H} = 6\imath .$$

PROPOSITION 1. *In* $\mathbf{P}^3$, *let* $\mathscr{F}_3$ *be a cubic surface containing a non s.r.q.c.* $\mathscr{C}_4$, *and let* $\mathscr{F}_3$ *be neither singular in codimension* 1, *nor a cone. Then* $\mathscr{C}_4$ *cannot be s.t.c.i. on* $\mathscr{F}_3$.

PROOF. Let us suppose the contrary, that is that there exists a surface $\mathscr{G}$, necessarily of degree $4m$, s.t.

$$\mathscr{F}_3 \cdot \mathscr{G} = 3m\mathscr{C}_4 . \qquad (\#)$$

Let $\mathscr{Q}$ be the (unique) quadric containing $\mathscr{C}_4$. It is well known that $\mathscr{Q}$ is not a cone (see e.g. [H], Ch. VI, Ex. 6.1, p. 355), and that, of the two scrolls of s.l. on $\mathscr{Q}$, one consists of unisecants of $\mathscr{C}_4$, and the others of trisecants of $\mathscr{C}_4$, so $\mathscr{C}_4$ cannot be s.t.c.i. on $\mathscr{Q}$, being of type $(1, 3)$, whereas a curve which is s.t.c.i. on $\mathscr{Q}$ must be of type (d, d) (see e.g. [H], Ch. II, § 6, pp. 135-136). Let Γ_2 be the algebraic cycle of order 2, which is the residual intersection, with respect to $\mathscr{C}_4$, of $\mathscr{Q}$ and $\mathscr{F}_3$:

$$\mathscr{F}_3 \cdot \mathscr{Q} = \mathscr{C}_4 + \Gamma_2 .$$

Γ_2 can be neither an irreducible conic, nor a pair of distinct and coplanar s.l., because Γ_2 would be c.i. on $\mathscr{Q}$ with the plane containing Γ_2, so that, by Lemma 1, applied to $\mathscr{Q}$ $(= \mathscr{F})$, $\mathscr{F}_3$ $(= \mathscr{G})$ and Γ_2 $(= \mathscr{C}_1)$, $\mathscr{C}_4$ would even be c.i. on $\mathscr{Q}$: absurd.

Γ_2 cannot consist of two skew s.l., because, by Lemma 1, applied to $\mathscr{F}_3$ $(= \mathscr{F})$, $\mathscr{Q}$ $(= \mathscr{G})$ and $\mathscr{C}_4$ $(= \mathscr{C}_1)$, ($\mathscr{C}_4$ is by (#) s.t.c.i. on $\mathscr{F}_3$) even Γ_2 would be s.t.c.i. on $\mathscr{F}_3$, which is notoriously impossible, being a disconnected variety (see e.g. [K], Ch. VI, § 4, Ex. 4.4, p. 199).

So it remains for Γ_2 only the possibility of being a pair of coincident s.l., that is

$$\mathscr{F}_3 \cdot \mathscr{Q} = \mathscr{C}_4 + 2r . \tag{\#\#}$$

Now, always by Lemma 1, applied to $\mathscr{F}_3$ $(= \mathscr{F})$, $\mathscr{Q}$ $(= \mathscr{G})$, and $\mathscr{C}_4$ $(= \mathscr{C}_1)$, r is s.t.c.i. on $\mathscr{F}_3$. By Lemma 3, r must pass through a singular point D of $\mathscr{F}_3$: since $\mathscr{F}_3$ is not a cone, D must be a double point of $\mathscr{F}_3$, which can be considered a monoid surface, non singular in codimension 1 by assumption, of vertex D: from Lemma 1, applied to $\mathscr{F}_3$ $(= \mathscr{F})$, $\mathscr{Q}$ $(= \mathscr{G})$, $\mathscr{C}_4$ $(= \mathscr{C}_1)$, we get that r is s.t.c.i. on $\mathscr{F}_3$; then, from Lemma 4 we can say that there exists a surface $\mathscr{H}$ s.t.

$$\mathscr{F}_3 \cdot \mathscr{H} = 6r . \tag{\#\#\#}$$

Now, from (##) it follows that one can find a surface $\mathscr{G}$ (actually one can take for $\mathscr{G}$ the non reduced $3\mathscr{Q}$) s.t.

$$\mathscr{F}_3 \cdot \mathscr{G} = 3\mathscr{C}_4 + 6r . \tag{\#\#\#\#}$$

Finally from (###), (####), and Lemma 1, applied to $\mathscr{F}_3$ $(= \mathscr{F})$, $\mathscr{G}$, and $\mathscr{C}_4$ $(= r)$, it follows that there exists a surface $\mathscr{H}'$ (necessarily of degree 4) s.t. $\mathscr{F}_3 \cdot \mathscr{H} = 3\mathscr{C}_4$, which is absurd, according to [S_2].

PROPOSITION 2. *Let $\mathcal{F}_3$ a cubic surface in $\mathbf{P}^3$, which is a cone containing a non s.r.q.c. $\mathcal{C}_4$. Then $\mathcal{C}_4$ cannot be s.t.c.i. on $\mathcal{F}_3$.*

PROOF. Let us suppose the contrary, that is that there exists a surface $\mathcal{G}$ s.t.

$$\mathcal{F}_3 \cdot \mathcal{G} = 3m\mathcal{C}_4 .$$

$\mathcal{F}_3$ cannot be singular in codimension one by Th. 4, 3, first case, in [J]. $\mathcal{C}_4$ must pass through the vertex V of $\mathcal{F}_3$, otherwise it would be, by Lemma 3, c.i. on $\mathcal{F}_3$, which is absurd. Moreover the generic s.l. of $\mathcal{F}_3$ meets $\mathcal{C}_4$ in V and in just one further point: indeed, if it met $\mathcal{C}_4$ in at least two further distinct points, it would be a trisecant of $\mathcal{C}_4$, and it would belong to $\mathcal{Q}$, so $\mathcal{Q} \subset \mathcal{F}_3$: absurd again. This means that the projection of $\mathcal{C}_4$ from V to a generic plane section $\mathcal{C}_3$ of $\mathcal{F}_3$ with a plane not passing through V, which presently is an elliptic cubic curve, is a birational correspondence, and this would imply that

$$1 = \text{genus of } \mathcal{C}_3 = \text{genus of } \mathcal{C}_4 = 0 :$$

absurd. Q.E.D.

2. In this paragraph we examine the cubic ruled surfaces of general type, those having two base skew s.l., one double, and one simple. Proposition 3 states that, even in this case, $\mathcal{C}_4$ cannot be s.t.c.i. on such a surface.

LEMMA 5. *Let $\mathcal{C}_4$ be a non s.r.q.c. in $\mathbf{P}^3$, let $\mathcal{F}_3$ be a cubic ruled surface of general type containing $\mathcal{C}_4$, and let d and s be respectively its double and simple base s.l.. Then s is a trisecant of $\mathcal{C}_4$ (may be tangent at a point A and secant at a point B, $B \neq A$, or even tangent to $\mathcal{C}_4$ at a possible flex), whereas s is a chord of $\mathcal{C}_4$ (may be tangent at a point which is not a flex).*

PROOF. Since d is a double s.l. on $\mathcal{F}_3$, every s.l. meeting s, $\mathcal{C}_4$, and d, necessarily belongs to $\mathcal{F}_3$, so that $\mathcal{F}_3$ is the ruled surface of the s.l. meeting the three curves s, d, $\mathcal{C}_4$. Setting $\mathcal{L}_1 = s$, $\mathcal{L}_2 = \mathcal{C}_4$, $\mathcal{L}_3 = d$, we have that the degrees of these curves are respectively

$$d_1 = 1 , \quad d_2 = 4 , \quad d_3 = 1 ,$$

whereas the multiplicities of $\mathscr{L}_1 = \mathscr{s}$ and $\mathscr{L}_3 = d$ for $\mathscr{F}_3$ are respectively 1 and 2. Applying to $\mathscr{F}_3$ G. Salmon's Theorem (see [G], p. 377) we get

$$(*) \qquad 2 = d_1 d_2 - r_{12} = 1\cdot 4 - r_{12}\,, \qquad 1 = d_2 d_3 - r_{23} = 4\cdot 1 - r_{23}$$

where r_{ij} is the number of (distinct or coincident) points common to $\mathscr{L}_i$ and $\mathscr{L}_j$, $1 \leq i,\ j \leq 3$, $i \neq j$. From $(*)$ we get $r_{12} = 4 - 2 = 2$ and $r_{23} = 4 - 1 = 3$; whence the Lemma.

LEMMA 6. *Let $\mathscr{C}_4$, $\mathscr{F}_3$, d, $\mathscr{s}$ be as in Lemma 5. Let $\mathscr{s}$ be a chord of $\mathscr{C}_4$, intersecting transversally $\mathscr{C}_4$ just at two distinct points U and V. Let t_U and t_V be the two tangent s.l. to $\mathscr{C}_4$ at U and V respectively. Let us set $\Pi_U = [\mathscr{s}, t_U]$, $\Pi_V = [\mathscr{s}, t_V]$, and let g_U and g_V the two rulings of $\mathscr{F}_3$ passing through U and V respectively. Then we have*

$$g_U \subset \Pi_U \quad \text{and} \quad g_V \subset \Pi_V\,.$$

PROOF. $\mathscr{C}_4$ has surely a stationary point P, for which there exists a plane Π through P such that $\Pi\cdot\mathscr{C}_4 = 4P$. Assume $\mathbf{A}^3$ to be $\mathbf{P}^3 - \Pi$, $P = Z_\infty$, and the tangent to $\mathscr{C}_4$ at P as the s.l. $Z_\infty Y_\infty$. Then, with a suitable choice of $O = (0, 0, 0)$, of the unit point $U = (1, 1, 1)$, and of the parameter t, we can suppose that $t = \infty$ corresponds to Z_∞, and that $\mathscr{C}_4^{(a)} = \mathscr{C}_4 \cap \mathbf{A}^3$ admits one of the following two parametric representations:

$$(**) \qquad (at + t^2, bt + t^3, ct + t^4)\,; \qquad (t, dt^2 + t^3, et^2 + t^4)$$

in the first case Z_∞ being not a flex of $\mathscr{C}_4$, in the second being a flex of $\mathscr{C}_4$. Working with these representations of $\mathscr{C}_4$, one gets easily the result: we avoid putting down explicitely the quite elementary computations involved, together with the obvious adaptations of the proof when U, or V is Z_∞.

REMARK 1. Using the representations $(**)$ of a rational nonsingular quartic $\mathscr{C}_4^{(a)}$ in $\mathbf{A}^3 \subset \mathbf{P}^3$, one can easily recognize that $\mathscr{C}_4^{(a)}$ has at least one affine stationary point, so in any case $\mathscr{C}_4$ has at least two distinct stationary points.

LEMMA 7. *Let $\mathscr{C}_4$, $\mathscr{F}_3$, d, $\mathscr{s}$ be as in Lemma 5. Let $\mathscr{s}$ be a tangent to $\mathscr{C}_4$ at one of its points U, which is not a flex for $\mathscr{C}_4$. Let Π_U be the* *obsculating plane to $\mathscr{C}_4$ at U, and let g_U be the ruling of $\mathscr{F}$ through U. Then we have*

$$g_U \subset \Pi_U .$$

PROOF. One proceeds in a way quite similar to that followed in proving Lemma 6.

LEMMA 8. *Let $\mathscr{C}_4$, $\mathscr{F}_3$, d, δ be as in Lemma 5. Let D a point of $d \cap \mathscr{C}_4$. Let us assume that on the two (distinct or coincident) rulings of $\mathscr{F}_3$ through D there is no point of $\mathscr{C}_4$ other than D. Then the plane $\Pi = [\delta, D]$ is tangent to $\mathscr{C}_4$ at D.*

PROOF. We distinguish two cases, according as $\mathscr{C}_4$ intersects δ at two distinct points U and V, or it is tangent to δ at a point U (which is not a flex of $\mathscr{C}_4$). In the first case, the plane $\Pi = [\delta, D]$ cannot be tangent to $\mathscr{C}_4$ neither at U, nor at V: indeed let it be tangent to $\mathscr{C}_4$ for example at U; then, by Lemma 6, the ruling g_U of $\mathscr{F}_3$ through U would lay on Π, and consequently we would have $g_U = UD$, against our assumption on the rulings of $\mathscr{F}_3$ through D. So

$$I(U, \mathscr{C}_4 \cap \Pi) = I(V, \mathscr{C}_4 \cap \Pi) = 1 .$$

Moreover there does not exist any point $E \in \Pi \cap \mathscr{C}_4$, with $E \notin \{U, V, D\}$: indeed, if it existed, then E would belong to one of the rulings of $\mathscr{F}_3$ through D, and this would lead again to a contradiction. So necessarily $I(D, \mathscr{C}_4 \cap \Pi) = 2$.

In the second case one argues in an analogous way, taking into account Lemma 7.

LEMMA 9. *Let $\mathscr{C}_4$, $\mathscr{F}_3$, d, δ be as in Lemma 5. Then there exists a point $D \in d \cap \mathscr{C}_4$ such that at least one of the two (distinct or coincident) rulings of $\mathscr{F}$ through D intersects on $\mathscr{C}_4$ a point $P \neq D$.*

PROOF. We shall distinguish two cases, according as d meets $\mathscr{C}_4$ at three distinct points D_1, D_2, D_3, or it is tangent to $\mathscr{C}_4$ at D_1, and meets $\mathscr{C}_4$ in a further point D_2 (possibly coincident with D_1, when D_1 is a flex of $\mathscr{C}_4$).

Case 1. $\mathscr{C}_4 \cap d = \{D_1, D_2, D_3\}$, $D_i \neq D_j$, $i \neq j$. By Lemma 8, if for every $i = 1, 2, 3$, through D_i there does not pass any ruling containing a point of $\mathscr{C}_4$ different from D_i, each of the three distinct planes $\Pi_i = [\mathscr{s}, D_i]$, $i = 1, 2, 3$, would be tangent respectively at D_i, $i = 1, 2, 3$, to $\mathscr{C}_4$, which is not possible, because the number of planes through $\mathscr{s}$ which are tangent to $\mathscr{C}_4$ at points not belonging to $\mathscr{s}$ is at most 2, as it is easy to verify. This proofs the Lemma in the first case.

Case 2. Let be either $\mathscr{C}_4 \cap \mathscr{d} = \{D_1, D_2\}$, with $\mathscr{d}$ tangent to $\mathscr{C}_4$ at D_1, and $D_2 \neq D_1$, or $\mathscr{C}_4 \cap \mathscr{d} = \{D_1\}$, with $\mathscr{d}$ inflexional tangent to $\mathscr{C}_4$ at its flex D_1. In both these subcases, $\Pi = [\mathscr{s}, D_1]$ is not tangent to $\mathscr{C}_4$ at D_1, because it would contain the tangent to $\mathscr{C}_4$ at D_1, which is $\mathscr{d}$, and $\mathscr{d}$ and $\mathscr{s}$ would be coplanar. Then, by Lemma 8, there must exist a ruling of $\mathscr{F}_3$ through D_1 meeting $\mathscr{C}_4$ at D_1 and at a further point $P \neq D_1$. Whence the Lemma.

PROPOSITION 3. *In* $\mathbf{P}^3$, *for every non s.r.q.c.* $\mathscr{C}_4$, *and for every cubic ruled surface* $\mathscr{F}_3$ *of general type containing* $\mathscr{C}_4$, *we have that* $\mathscr{C}_4$ *cannot be s.t.c.i. on* $\mathscr{F}_3$.

PROOF. Let $\mathscr{d}$ and $\mathscr{s}$ be respectively the double and simple base s.l. of $\mathscr{F}_3$. Let $\mathscr{g}_1$ be a ruling of $\mathscr{F}_3$ (see Lemma 9) which, beside a point $D_1 \in \mathscr{C}_4 \cap \mathscr{d}$, contains a further point P_1 ($\notin \mathscr{d}$) of $\mathscr{C}_4$. Now, by Remark 1, $\mathscr{C}_4$ has at least two distinct stationary points, so that there exists surely one of them P which is different from P_1, and let us choose as plane at infinity Π_∞ the osculating plane to $\mathscr{C}_4$ at P.

With this $\mathscr{C}_4^{(a)} = \mathscr{C}_4 \cap \mathbf{A}^3$ admits a parametric biregular representation of the type

$$\mathscr{C}_4^{(a)}\colon\ (a(t),\ b(t),\ c(t))$$

with $a(t)$, $b(t)$, $c(t)$ polynomials in $k[t]$, of degree $\leqq 4$. Let t_1 be the value of the parameter corresponding to $P_1 = P(t_1)$. $\mathscr{F}_3^{(a)} = \mathscr{F}_3 \cap \mathbf{A}^3$ admits a parametric representation of the form

$$(*) \qquad \mathscr{F}_3^{(a)}\colon\ (a(t) + l(t)u,\ b(t) + m(t)u,\ c(t) + n(t)u)$$

where one can suppose that $l(t)$, $m(t)$, $n(t)$ are three coprime polynomials in $k[t]$ (in $(*)$ $\mathscr{C}_4^{(a)}$ plays the role of base curve). For every given $t \in k$, such that $P(t) = (a(t), b(t), c(t)) \notin \mathscr{d}$

$$(**) \qquad (a(t) + l(t)u,\ b(t) + m(t)u,\ c(t) + n(t)u)$$

is a parametric representation (in the parameter u) for the affine part $g(t)^{(a)}$ of the unique ruling $g(t)$ of $\mathscr{F}_3$ through $P(t)$. $(**)$, for $t = t_1$, represents a s.l. passing through $P_1 = P(t_1)$ and contained in $\mathscr{F}_3^{(a)}$, which actually is the affine part $g_1^{(a)}$ of the ruling g_1 of $\mathscr{F}_3$ through P_1.

Now let us suppose that there exists a surface $\mathscr{G}$ such that $\mathscr{F}_3 \cdot \mathscr{G} = 3m\mathscr{C}_4$. $\mathscr{G}$ has necessarily degree $4m$. Being d a trisecant of $\mathscr{C}_4$ (possibly in three points variously coincident), the generic ruling of $\mathscr{F}$ is a s.l. intersecting transversally $\mathscr{C}_4$ in just one point: this means that, for generic t, $g(t)$ intersects $\mathscr{C}_4$ just in $P(t)$, which in $(**)$ is obtained for the value $u = 0$ of the parameter u. Now we have, for every such t,

$$\{P(t)\} = g(t) \cap \mathscr{C}_4 = g(t) \cap \mathscr{F}_3 \cap \mathscr{G} = g(t) \cap \mathscr{G}$$

which means that $g(t)$ intersects $\mathscr{G}$ at $P(t)$ with $I(P(t), g(t) \cap \mathscr{G}) = 4m = \deg(\mathscr{G})$.

This implies that, if

$$\mathscr{G}^{(a)} = \{G(X, Y, Z) = 0\} ,$$

in $k[t, u]$ we have the following

$$(***) \qquad G\big(a(t) + l(t)u,\, b(t) + m(t)u,\, c(t) + n(t)u\big) = [A(t)]u^{4m} ,$$

with suitable $A(t) \in k[t]$. Putting in $(***)$ $t = t_1$, we get

$$G\big(a(t_1) + l(t_1)u,\, b(t_1) + m(t_1)u,\, c(t_1) + n(t_1)u\big) = A(t_1)u^{4m} .$$

Now, if it were $A(t_1) \neq 0$, this would mean that the ruling g_1 intersects $\mathscr{G}$ at $P_1 = P(t_1)$ with multiplicity $4m$, and this would prevent g_1 from having on it another point of $\mathscr{C}_4 \subset \mathscr{G}$: contradiction, because we know that $D_1 \in g_1$. So it is $A(t_1) = 0$; but then we have $g_1 \subset \mathscr{G}$, hence $g_1 \subset \mathscr{G} \cap \mathscr{F}_3 = \mathscr{C}_4$: absurd. Q.E.D.

3. In this paragraph we examine the case in which $\mathscr{F}_3$ is a cubic ruled surface of Cayley's type containing a non s.r.q.c. $\mathscr{C}_4$. The main statements are Prop. 4 and Prop. 5.

Let us remember that $\mathscr{F}_3$ has a base double s.l. d; that there exists just one plane Π^* through d which osculates $\mathscr{F}_3$ along d, that is s.t.

$\Pi^* \cdot \mathscr{F}_3 = 3d$; and that there exists just one point $D \in d$ s.t. the only s.l. of $\mathscr{F}_3$ through D is d (D is the uniplanar double point of $\mathscr{F}_3$, with Π^* as principal plane).

REMARK 2. The proofs of the statements contained in this Remark are not given explicitely, being either well known, or of not difficult verification. Let $\mathscr{C}_4$ be a non s.r.q.c. in $\mathbf{P}^3$, and let $\imath$ be a s.l. which is an ordinary chord of $\mathscr{C}_4$, intersecting it transversally just at two distinct points A and B. Let P be a generic point of $\mathscr{C}_4$: The plane $\Pi = [\imath, P]$ intersects on $\mathscr{C}_4$, beside A, B and P, a further point P'.

1) The correspondence

$$\Omega: \mathscr{C}_4 \to \mathscr{C}_4$$

such that $\Omega(P) = P'$ is birational and biregular: since $\mathscr{C}_4 \simeq \mathbf{P}^1$, Ω is a bilinear involution.

2) Let P_1 and P_2 be the two (necessarily distinct) fixed points of Ω. Moreover let $\mathscr{H}$ be the ruled surface whose generic ruling is the s.l. $[P, \Omega(P)]$ (such surface has obviously as base curve both $\mathscr{C}_4$ and $\imath$). Then the two tangents g_1 and g_2 to $\mathscr{C}_4$, respectively at P_1 and P_2, belong to $\mathscr{H}$.

3) Let t_A and t_B be the tangents to $\mathscr{C}_4$ at A and B, respectively. Suppose that $\Pi_A = [t_A, \imath]$ is osculating $\mathscr{C}_4$ at A: then $\Omega(A) = A$, and so, by 2), t_A belongs to $\mathscr{H}$ (similarly $\Omega(B) = B$, and $t_B \subset \mathscr{H}$, if $\Pi_B = [t_B, \imath]$ is osculating $\mathscr{C}_4$ at B).

4) If t_A and t_B, (see 3)), are not coplanar, and neither Π_A, nor Π_B (see 3)) is osculating $\mathscr{C}_4$ respectively at A or at B, then every plane Π through $\imath$ intersects on $\mathscr{C}_4$ at least one point $P \neq A, B$.

LEMMA 10. *Let $\mathscr{C}_4$ be a non s.r.q.c. $\mathscr{C}_4 \subset \mathbf{P}^3$. Let $\mathscr{F}_3$ be a cubic ruled surface of Cayley's type containing $\mathscr{C}_4$. Let d be the base s.l. of $\mathscr{F}$, and suppose that d is an ordinary chord of $\mathscr{C}_4$ intersecting it transversally just at A and B, $A \neq B$. Then necessarly the two s.l. t_A and t_B, which are tangent to $\mathscr{C}_4$ respectively at A and B, must be coplanar.*

PROOF. Suppose that t_A and t_B were not coplanar. In our situation, the ruled surface $\mathscr{H}$ of Remark 2 is just $\mathscr{F}_3$, with $\imath = d$. Let Π be any plane through d, and let us prove that Π cannot be osculating $\mathscr{F}_3$ along d, that is it cannot be $\Pi \cdot \mathscr{F}_3 = 3d$: this shall be a contradiction, so t_A and t_B must be coplanar. Let us choose $\Pi = \Pi_A$. First suppose $I(A, \mathscr{C}_4 \cap \Pi_A) = 2$; then $I(B, \mathscr{C}_4 \cap \Pi_A) = 1$ (it cannot be 2, otherwise t_A and t_B would be both contained in Π_A, that is they would be coplanar); so it exists $P \in \mathscr{C}_4 \cap \Pi_A$, $P \notin d$, and this means that it cannot be $\Pi_A \cdot \mathscr{F}_3 = 3d$.

Now suppose $I(A, \mathscr{C}_4 \cap \Pi_A) = 3$; then, by Remark 2, 3), it follows that $t_A \subset \mathscr{F}_3$, and again it cannot be $\Pi_A \cdot \mathscr{F}_3 = 3d$. Same argument when $\Pi = \Pi_B$. Finally if $\Pi \neq \Pi_A$, Π_B, then $I(A, \mathscr{C}_A \cap \Pi) = I(B, \mathscr{C}_4 \cap \Pi) = 1$, so it exists at least one point $P \in \Pi \cap \mathscr{C}_4$, and $P \notin d$, and this again prevents Π from osculating $\mathscr{F}_3$ along d.

LEMMA 11. *Let $\mathscr{C}_4$ be a non s.r.q.c. in $\mathbf{P}^3$. Let A be a point of $\mathscr{C}_4$ which is not a flex of $\mathscr{C}_4$ itself, and d the tangent to $\mathscr{C}_4$ at A. Let us suppose that $\mathscr{C}_4 \cap d = \{A\}$. Let B be any point of d different from A. Then through B there passes at least one s.l., different from d, which either intersects $\mathscr{C}_4$ at two distinct points, or it is tangent to $\mathscr{C}_4$ at a suitable point* (*different from A*).

PROOF. Let γ be a plane transversal to d, and let $\varphi_B\colon \mathscr{C}_4 \to \gamma$ be the projection of $\mathscr{C}_4$ from B to γ, and $\mathscr{C}_4'$ the image of $\mathscr{C}_4$ under φ_B; it is easy to see that $\mathscr{C}_4'$ is a quartic curve of γ and $\varphi_B\colon \mathscr{C}_4 \to \mathscr{C}_4'$ is a birational (and regular) correspondence, so that $\mathscr{C}_4'$ is a rational quartic curve of γ. Let us set $A' = \gamma_B(A)$: it is quite simple also to see that A' is an ordinary cusp of $\mathscr{C}_4'$, which implies that $\mathscr{C}_4'$ itself has at least one further multiple point D, $D \neq A'$. Now, if through B there did not pass any chord or tangent to $\mathscr{C}_4$ different from d, every point of $\mathscr{C}_4'$ different from A' would be of course a simple point of $\mathscr{C}_4'$: absurd.

LEMMA 12. *In $\mathbf{P}^3$ there does not exist any cubic ruled surface $\mathscr{F}_3$ of Cayley's type containing a non s.r.q.c. $\mathscr{C}_4$, and having as its double base s.l. d a tangent to $\mathscr{C}_4$ at a point A, which is not a flex of $\mathscr{C}_4$, and such that $\mathscr{C}_4 \cap d = \{A\}$.*

PROOF. Let us suppose that such a surface $\mathscr{F}_3$ exists and deduce a contradiction. Let P be a generic point of $\mathscr{C}_4$, and let us considere $\Pi_P = [d, P]$. We have $\Pi_P \cdot \mathscr{C}_4 = 2A + P + P'$. As in Remark 2, the correspondence

$$\Omega\colon \mathscr{C}_4 \to \mathscr{C}_4$$

such that $\Omega(P) = P'$ is a linear involution, and let $\mathscr{H}$ be the ruled surface whose generic ruling is $[P, \Omega(P)]$. Of course it is $\mathcal{H} = \mathcal{F}_3$, and, as in Remark 2, 2) on can see that, if $\Omega(P_1) = P_1$, then the tangent to $\mathcal{C}_4$ at P_1 belongs to $\mathcal{H} = \mathcal{F}_3$. Now, as it is well known, through A there passes a trisecant t of $\mathcal{C}_4$ (intersecting it at three non necessarily distinct points) which presently must be different from d, so that t is transversal to $\mathcal{C}_4$ at A, and must then meet $\mathcal{C}_4$ at two (distinct or coincident) points different from A. t then belongs to $\mathcal{H} = \mathcal{F}_3$. This means that through A there passes a ruling of $\mathcal{F}_3$ different from d.

On the other hand, by Lemma 11, we have that through any point B, $B \neq A$, of d there passes a ruling of $\mathcal{F}_3$ different from d. This is inconsistent with $\mathcal{F}_3$ being a cubic ruled surface of Cayley's type, for which there exists (exactly) one point D of d such that the only s.l. of $\mathcal{F}_3$ through D is d.

PROPOSITION 4. *In* $\mathbf{P}^3$ *let* $\mathcal{C}_4$ *be a non s.r.q.c., and let* $\mathcal{F}_3$ *be a cubic ruled surface* $\mathcal{F}_3$ *of Cayley's type containing* $\mathcal{C}_4$, *and having its double base s.l.* d *which is a simple chord of* $\mathcal{C}_4$, *intersecting it transversally just at two distinct points* A *and* B. *Then* $\mathcal{C}_4$ *cannot be s.c.t.i. on* $\mathcal{F}_3$.

PROOF. From Lemma 10 we know that the two s.l. t_A, t_B, which are tangent to $\mathcal{C}_4$ respectively at A and B, are coplanar. Through A there passes a (unique) trisecant $\imath_A$ of $\mathcal{C}_4$, meeting $\mathcal{C}_4$ either at three distinct points, or at two distinct points being at (only) one of them tangent to $\mathcal{C}_4$, or finally tangent to $\mathcal{C}_4$ at one of its (possible) flexes. Of course $\imath_A \neq t_A$, because, on the contrary, the plane $\Pi = [t_A, t_B]$ would cut on $\mathcal{C}_4$ at least three (distinct or variously coincident) points laying on t_A, and in at least two (coincident) points at B, which would imply that $\mathcal{C}_4 \subset \Pi$: absurd. Same argument for the trisecant $\imath_B$ of $\mathcal{C}_4$ through B. It is also clear that $\imath_A$, and $\imath_B$, meet transversally $\mathcal{C}_4$ respectively at A and at B. This implies that $\imath_A$ and $\imath_B$ both meet $\mathcal{C}_A$, beside at A and at B respectively, at two further (distinct or coincident) points.

Now the surface $\mathcal{H}$ introduced in Remark 2 presently coincides with our $\mathcal{F}_3$, and $\imath_A$ and $\imath_B$ belong to $\mathcal{F}_3$. Let us choose an affine $\mathbf{A}^3$ in such a way that the improper plane Π_∞ is hyperosculating $\mathcal{C}_4$ at one of its (at least two and distinct, see Remark 1) stationary points P_∞; moreover we can even suppose that one, let it be $\imath_A$, of the two trisecants $\imath_A$ and $\imath_B$ is such that $\imath_A \cap \mathcal{C}_4 \subset \mathbf{A}^3$. By this $\mathcal{C}_4^{(a)} = \mathcal{C}_4 \cap \mathbf{A}^3$ is a polynomial curve $\mathcal{C}_4^{(a)} = (a(t), b(t), c(t))$ with $a(t)$, $b(t)$, $c(t) \in k[t]$, and of degree ≤ 4. Even $\mathcal{F}_3^{(a)} = \mathcal{F}_3 \cap \mathbf{A}^3$ can be represented parametrically in the form

$$(*) \qquad (a(t) + l(t)u,\ b(t) + m(t)u,\ c(t) + n(t)u)\ , \qquad (t,\ u \text{ parameters})\ ,$$

where $l(t)$, $m(t)$, $n(t) \in k[t]$, and can be supposed coprime. For every given t s.t. $P(t) = (a(t), b(t), c(t)) \notin d$, the equations $(*)$ represent the affine part $g(t)^{(a)}$ of the unique ruling $g(t)$ of $\mathscr{F}_3$ through $P(t)$. Let t_1 be a value of the parameters for which $P(t_1) \in r_A \cap \mathscr{C}_4$, but $P(t_1) \neq A$. The equation $(*)$, for $t = t_1$, represents the affine part of a s.l. through $P(t_1)$ contained in $\mathscr{F}_3$: since through $P(t_1)$ there exists only one ruling of $\mathscr{F}_3$, and through $P(t_1)$ there passes $r_A \subset \mathscr{F}_3$, then $g(t_1) = r_A$.

Now let us suppose that there exists a surface $\mathscr{G}$ such that $\mathscr{F} \cdot \mathscr{G} = 3m\mathscr{C}_4$. $\mathscr{G}$ has necessarily degree $4m$. Given a generic point $P(t)$ of $\mathscr{C}_4^{(a)}$, let be $P(A(t)/B(t)) = \Omega(P(t))$, Ω being the linear involution introduced in Remark 2, and $A(t)$, $B(t)$ are indipendent linear polynomials. Let $G(X, Y, Z) = 0$ the equation of $\mathscr{G}^{(a)} = \mathscr{G} \cap \mathbf{A}^3$. For the generic ruling $g(t)$ of $\mathscr{F}_3$ we have

$$g(t) \cap \mathscr{G} = g(t) \cap \mathscr{F}_3 \cap \mathscr{G} = g(t) \cap \mathscr{C}_4 = \{P(t),\ P(\omega(t))\}\ ,$$

where $\omega(t) = A(t)/B(t)$. Introducing $(*)$ in $G(X, Y, Z)$, we get in $k[t, u]$ the following

$$(**) \qquad G(a(t) + l(t)u,\ b(t) + m(t)u,\ c(t) + n(t)) = \\ = u^r[Q_r(t) + \ldots + Q_{4m}(t)u^{4m-r}]\ ,$$

with suitable

$$Q_i(t) \in k[t]\ ,\quad i = r, \ldots, 4m - r\ , \qquad \text{and} \qquad r = I(P(t),\ \mathscr{G} \cap g(t)) > 0\ ,$$

because for $u = 0$ in $(*)$ we get $P(t)$.

On the other hand the equation

$$Q_r(t) + \ldots + Q_{4m}(t)u^{4m-r} = 0$$

has a unique root in u, because $g(t)$ intersects $\mathscr{C}_4$ just at $P(t)$ and $P(\omega(t))$. This implies

$$Q_r(t) + \ldots + Q_{4m}(t)u^{4m-r} = Q_{4m}(t)[u - C(t)/D(t)]^{4m-r}$$

where $C(t)$ and $D(t)$ are suitable coprime elements in $k[t]$.

Let us prove that $D(t_1) \neq 0$. Suppose the contrary. Now, of course we have

$$(**)\qquad \begin{cases} x(\omega(t)) = a(t) + l(t)\,C(t)/D(t)\,, \\ y(\omega(t)) = b(t) + m(t)\,C(t)/D(t)\,, \\ z(\omega(t)) = c(t) + n(t)\,C(t)/D(t)\,, \end{cases}$$

and, by multiplying by $D(t)$, we get

$$(***)\qquad \begin{cases} D(t)x(\omega(t)) = D(t)\,a(t) + l(t)\,C(t)\,, \\ D(t)y(\omega(t)) = D(t)\,b(t) + m(t)\,C(t)\,, \\ D(t)\,z(\omega(t)) = D(t)\,c(t) + n(t)\,C(t)\,. \end{cases}$$

By our assumption that $P_\infty \notin \imath_A$, we have that $\omega(t_1)$ is defined at t_1, (that is $B(t_1) \neq 0$) because $\Omega(P(t_1))$ is a point which belongs to $\imath_A$, and to the infinite value of the parameter there corresponds on $\mathscr{C}_4$ the point P_∞. Putting in $(***)$ $t = t_1$, we get then

$$l(t_1)\,C(t_1) = 0\,, \qquad m(t_1)\,C(t_1) = 0\,, \qquad n(t_1)\,C(t_1) = 0$$

whence $C(t_1) = 0$, being $l(t)$, $m(t)$, $n(t)$ coprime: absurd. So $D(t_1) \neq 0$.

We get then

$$G(a(t_1) + l(t_1)u, b(t_1) + m(t_1)u, c(t_1) + n(t_1)) = \\ = u^{4r}[u - C(t_1)/D(t_1)]^{4m-r}Q_{4m}(t_1)\,.$$

Let us assume that $Q_{4m}(t_1) \neq 0$. Now we distinguish two cases, according as $C(t_1) \neq 0$ or $C(t_1) = 0$. In the first case $\imath_A$ meets $\mathscr{G}$, beside at A, at the two further distinct points, both different from A, $P(t_1)$ and $P(\omega(t_1))$, which correspond to the two values 0, and $C(t_1)/D(t_1)$ of the parameter u in the parametrization of $\mathscr{g}(t)^{(a)}$. From $(***)$ we have

$$I(P(t_1), \mathscr{G} \cap \imath_A) = r\,, \qquad I(P(\omega(t_1)), \mathscr{G} \cap \imath_A) = 4m - r\,.$$

But $\imath_A$ intersects $\mathscr{G}$ also at A, so that $\imath_A \subset \mathscr{G}$ because

$$I(P(t_1), \mathscr{G} \cap \imath_A) + I\big(P(\omega(t_1)), \mathscr{G} \cap \imath_A\big) + I(A, \mathscr{G} \cap \imath_A) \geqq$$
$$\geqq r + (4m - r) + 1 = 4m + 1 > \deg(\mathscr{G}) .$$

So $\imath_A \subset \mathscr{G} \cap \mathscr{F}_3 = \mathscr{C}_4$: absurd.

In the second case we have already $I(P(t_1), \mathscr{G} \cap \imath_A) = 4m$, and $P(t_1) \neq A$, hence the same conclusion.

So it must be $Q_{4m}(t_1) = 0$, and this implies again that $\imath_A \subset \mathscr{G}$, which leads once more to a contradiction with the hypothesis

$$\mathscr{G} \cap \mathscr{F}_3 = \mathscr{C}_4 . \qquad \text{Q.E.D.}$$

LEMMA 13. *In* $\mathbf{P}^3$ *let* $\mathscr{F}_3$ *be cubic ruled surface of Cayley's type containing a non s.r.q.c.* $\mathscr{C}_4$, *let* $\mathscr{Q}$ *be the unique (non singular) quadric containing* $\mathscr{C}_4$, *and let* $\mathscr{d}$ *be the double base s.l. of* $\mathscr{F}_3$. *Let us suppose* $\mathscr{d} \subset \mathscr{Q}$, *and let* P *be a point of* $\mathscr{C}_4 \cap \mathscr{d}$, *at which* $\mathscr{C}_4$ *and* $\mathscr{d}$ *meet each other transversally. Then the uniplanar double point* D *of* $\mathscr{F}_3$ *on* $\mathscr{d}$ *is different from* P.

PROOF. Within a linear isomorphism of $\mathbf{P}^3$ and with a suitable choice of an affine $\mathbf{A}^3 \subset \mathbf{P}^3$, we can suppose that

$$\mathscr{F}^{(a)} = \{X^3 - 2XYZ + Z^2 = 0\} ,$$

and by consequence we have $D = (0, 0, 0)$, $\mathscr{d}^{(a)} = \{X = Z = 0\}$. Since $\mathscr{d} \subset \mathscr{Q}$, we have

$$\mathscr{Q}^{(a)} = \{X(aX + bY + cZ + d) + Z(eY + fZ + g) = 0\} .$$

Let us suppose that $P = D = O = (0, 0)$. From direct calculation it follows that $\mathscr{C}_4$ passes through O only if $d = 0$. On the other hand it must be $b \neq 0$, otherwise no component of $\mathscr{F}_3 \cap \mathscr{Q}$ would be a quartic curve, as it is easy to see. Then from the parametric representation of $\mathscr{C}_4$ it is immediate to verify that $\mathscr{C}_4$ is tangent to $\mathscr{d}$ at O: contradiction. So $D \neq P$.

LEMMA 14. *In* $\mathbf{P}^3$ *let* $\mathscr{d}$ *be a trisecant of a non s.r.q.c.* $\mathscr{C}_4$, *meeting it at three distinct points* P_i $(i = 1, 2, 3)$, *and let* $\mathscr{F}_3$ *a cubic ruled surface of Cayley's type having* $\mathscr{d}$ *as its double base s.l.. Then on at* *least one of the rulings* g_i $(i = 1, 2, 3)$ *of* $\mathcal{F}_3$, *respectively through* P_i $(i = 1, 2, 3)$, *there is a point* $R \in \mathcal{C}_4$, *with* $R \notin d$.

PROOF. By Lemma 13 it is $D \neq P_i$ $(i = 1, 2, 3)$ so through each of the points P_i, there passes a ruling g_i, with $g_i \neq d$ $(i = 1, 2, 3)$. If on every g_i $(i = 1, 2, 3)$ there did not exist a point $R \in \mathcal{C}_4 \cap g_i$, $R \notin d$, then the three distinct planes $\alpha_i = [g_i, d]$, would not meet $\mathcal{C}_4$ outside d. On the other hand there are exactly three planes through d behaving like this, and they are the planes containing respectively the three tangents to $\mathcal{C}_4$ at P_i $(i = 1, 2, 3)$. For any other plane Π through d we have $I(P_i, \mathcal{C}_4 \cap \Pi) = 1$ $(i = 1, 2, 3)$, so that Π and $\mathcal{C}_4$ must meet each other at a further point $P \notin d$.

From all this it would follow that no plane through d can intersect on $\mathcal{F}_3$ only d: contradiction, because $\mathcal{F}_3$ is of Cayley's type.

LEMMA 15. *Let* $\mathcal{C}_4$ *be a non s.r.q.c. in* $\mathbf{P}^3$, *and let* d *be a tangent-chord of* $\mathcal{C}_4$, *tangent to it at* P_1, *and intersecting it transversally at* $P_2 \neq P_1$. *Let* $\mathcal{F}_3$ *be a cubic ruled surface of Cayley's type containing* $\mathcal{C}_4$ *and having* d *as its double base s.l.; moreover let us suppose that the uniplanar double point* D *of* $\mathcal{F}_3$ *on* d *is different from* P_1. *Then on at least one of the rulings* g_i, $(i = 1, 2)$ *of* $\mathcal{F}_3$ *respectively through* P_i $(i = 1, 2)$ *there is a point* $R \in \mathcal{C}_4$, *with* $R \notin d$.

PROOF. Through each of the P_i $(i = 1, 2)$ there passes a ruling g_i different from d: as for P_1, since $D \neq P_1$, and as for P_2, by Lemma 13. After that the argument is an easy adaptation of that used in proving Lemma 14.

PROPOSITION 5. *In* $\mathbf{P}^3$ *let* $\mathcal{C}_4$ *be a non s.r.q.c., and let* $\mathcal{F}_3$ *be a cubic ruled surface of Cayley's type containing* $\mathcal{C}_4$, *and such that its double base s.l.* d *is*:

1) *either a trisecant of* $\mathcal{C}_4$ *at three distinct points* P_i $(i = 1, 2, 3)$;
2) *or a tangent-chord of* $\mathcal{C}_4$, *tangent to* $\mathcal{C}_4$ *at* P_1, *and intersecting it transversally at* P_2;
3) *or, if* $\mathcal{C}_4$ *actually has a flex, a tangent to* $\mathcal{C}_4$ *at one of its flexes* P_0.

Then, in all these cases, $\mathcal{C}_4$ *cannot be s.t.c.i. on* $\mathcal{F}_3$.

PROOF. *Case* 1). By Lemma 14, through at least one of the points P_i $(i = 1, 2, 3)$ there passes a ruling g of $\mathcal{F}_3$ different from d, having on it (at least) two distinct points of $\mathcal{C}_4$. We can then argue as in the proof of Prop. 3, that is we can show that, if there existed a surface $\mathscr{G}$ such that $\mathscr{G} \cap \mathscr{F}_3 = \mathscr{C}_4$, (i.e. $\mathscr{G} \cdot \mathscr{F}_3 = 3m\mathscr{C}_4$), then g would be contained in $\mathscr{G}$, so that $g \subset \mathscr{G} \cap \mathscr{F}_3 = \mathscr{C}_4$: absurd.

Case 2). If the uniplanar double point D of $\mathscr{F}_3$ is different from P_1, then we can apply Lemma 15, and then we argue again as in case 1). On the other hand, if $D = P_1$, certainly there exists on $\mathscr{C}_4$ a stationary point $S \neq D = P_1$, (see Remark 1). Assuming the stationary plane of $\mathscr{C}_4$ at S as improper plane Π_∞, and with a suitable choice of an affine $\mathbf{A}^3 \subset \mathbf{P}^3$, we can represent biregularly $\mathscr{C}_4^{(a)} = \mathscr{C}_4 \cap \mathbf{A}^3$ in the form

$$\mathscr{C}_4^{(a)}\colon\ (a(t),\, b(t),\, c(t))$$

with $a(t)$, $b(t)$, $c(t)$ polynomials in $k[t]$, of degree $\leqq 4$. Let t_1 be the value of the parameter t corresponding to $D = P(t_1)$. $\mathscr{F}_3^{(a)}$ can be represented in the form

$$(*) \qquad (a(t) + l(t)u,\ b(t) + m(t)u,\ c(t) + n(t)u)\ , \qquad (t,\ u \text{ parameters})$$

where $l(t)$, $m(t)$, $n(t) \in k[t]$, and are coprime polynomials. For $t = t_1$ $(*)$ represents a s.l. of $\mathscr{F}_3$, passing through D, which of course must be d. We can then argue as in the previous cases 1) and 2), since on d there are two distinct points $P_1 = D$, and P_2.

Case 3). Let us suppose that P_0 is a flex of $\mathscr{C}_4$, and d is the tangent to $\mathscr{C}_4$ at P_0. Within a linear isomorphism of $\mathbf{P}^3$, and the choice of a suitable $\mathbf{A}^3 \subset \mathbf{P}^3$, $\mathscr{C}_4^{(a)}$ can be represented biregularly (see Proof of Lemma 6) in the form

$$(t,\ at^2 + t^3,\ bt^2 + t^4)\ , \qquad \text{with } a, b \in k\ .$$

With this, the (unique) quadric $\mathscr{Q}$ containing $\mathscr{C}_4$ has an affine equation

$$Q = XY + (a^2 + b)X^2 - aY - Z = 0\ .$$

As for the cubic ruled surface of Cayley's type $\mathscr{F}_3 \subset \mathscr{C}_4$, it is easy to verify that it must have as its osculating plane along d (presently $d = Y_\infty Z_\infty$, and $D = Z_\infty$) just the stationary plane Π to $\mathscr{C}_4$ at D: by this the equation of $\mathscr{F}_3^{(a)}$ is

$$F = (a - c)X^3 - (b + ac)X^2 - XY + cY + Z = 0\ , \qquad \text{with } a \neq c\ .$$

As for the cone $\mathscr{H}$ projecting $\mathscr{C}_4$ from $Z_\infty = D$, its affine part $\mathscr{H}^{(a)}$ has equation

$$H = Y - aX^2 - X^3 = hF - hQ = 0 , \qquad \text{where } h = 1/(a-c) .$$

Now let us suppose that there exists a surface $\mathscr{G}$ (necessarily of degree $4m$), such that $\mathscr{G}\cdot\mathscr{F}_3 = 3m\mathscr{C}_4$. Let be $G(X, Y, Z) = 0$ the affine equation of $\mathscr{G}$.

$\mathscr{F}_3$ can be considered as a monoid with vertex Z_∞, and to it we can apply Lemma 1, p. 54, in [S_2], (interchanging there of course X_0 with X_3): by this we get

$$(X_0^2)^{4m-3}G^* = X_0^b H^{*3m} + A^*F^* ,$$

where G^*, H^*, F^* are the homogeneizations of G, H, F. Dehomogenizing this equation, we get

$$G = H^{3m} + AF , \qquad \text{(where } A \text{ is the dehomogeneization of } A^*) .$$

From $H = hF - hQ$, we get

$$(**) \qquad G = h^{3m}Q^{3m} + A'F ,$$

where A' is a suitable polynomial of $k[X, Y, Z]$.

Let $G = G_{4m} + G_{4m-1} + \ldots$ the representation of G as sum of homogeneous polynomials in $k[X, Y, Z]$ of degree $4m, 4m-1, \ldots$; $\{G_{4m} = 0\}$ represents $\mathscr{G}_\infty = \Pi_\infty \cap \mathscr{G}$ on the improper plane Π_∞.

Since $\mathscr{C}_4 \cap \Pi = \{Z_\infty\}$, and $\mathscr{G} \cap \mathscr{F} = \mathscr{C}_4$, it follows that it must be

$$\mathscr{G}_\infty \cap d \subset (\mathscr{G} \cap \Pi_\infty) \cap (\Pi_\infty \cap \mathscr{F}_3) = \Pi_\infty \cap \mathscr{C}_4 = \{Z_\infty\} .$$

From this, and from $(**)$, we have

$$(\#) \qquad Y^{4m} + XD_{4m-1}(X, Y, Z) + G_{4m-1} + \ldots =$$
$$= h^{3m}\{X^{3m}[Y + (a^2+b)X]^{3m} + \ldots (-aY - Z)^{3m}\} +$$
$$+ A'(X, Y, Z)[a - cX^3 - (b + ac)X^2 - XY + cY + Z] .$$

with $D_{4m-1}(X, Y, Z)$ homogeneous polynomial in $k[X, Y, Z]$ of degree $4m-1$, and where the former dots stand for the sum of the terms of G of degree $< 4m-1$, whereas the latter ones stand for a term which is a multiple of X. Setting in (#) $X = 0$, we get the following equality in $k[X, Y]$

$$(\#\#) \qquad Y^{4m} + G_{4m-1}(0, Y, Z) = h^{3m}(-aY - Z)^{3m} + A'(0, Y, Z)(cY + Z)$$

which is seen obviously absurd by simply looking at the forms of maximum degree in both sides: on the left hand side it is Y^{4m}, on the right hand side, since $3m < 4m$, it is necessarily the form of maximum degree of $A'(0, Y, Z)(cY + Z)$: so the two forms cannot coincide, and (##) is absurd. Q.E.D.

4. From all what has been obtained up to now we can conclude with the following

THEOREM. *In* $\mathbf{P}^3$, *for any non s.r.q.c.* $\mathscr{C}_4$, *and for any cubic surface* $\mathscr{F}_3$ *containing* $\mathscr{C}_4$, *we have that* $\mathscr{C}_4$ *cannot be s.t.c.i. on* $\mathscr{F}_3$.

PROOF. If $\mathscr{F}_3$ is non singular in codimension 1, or a cone, this theorem respectively follows from Prop. 1, or Prop. 2.

If $\mathscr{F}_3$ is a cubic ruled surface of general type, the theorem follows from Prop. 3.

If $\mathscr{F}_3$ is a ruled surface of Cayley's type, let d be its base double s.l.. First of all we can see easily that d and $\mathscr{C}_4$ cannot be neither skew curves, nor curves intersecting transversally each other at just one point P: indeed the generic ruling $g \subset \mathscr{F}_3$ would meet $\mathscr{C}_4$ in 4 points in the first case, and in 3 points in the second one. This would imply $g \subset \mathscr{Q}$, where $\mathscr{Q}$ is the (unique) quadric containing $\mathscr{C}_4$, and we would have $\mathscr{Q} \subset \mathscr{F}_3$: absurd. From Lemma 10 and Lemma 12 then, there remains for d and $\mathscr{C}_4$ only the following configurations:

1) either d and $\mathscr{C}_4$ meet each other transversally at just two distinct points P_1 and P_2 (with the tangents to $\mathscr{C}_4$ at P_1 and P_2 coplanar);

2) or d and $\mathscr{C}_4$ meet each other at three points P_1, P_2 and P_3 (variously coincident in the sense said in the statement of Prop. 5).

But, in both cases, respectively Prop. 4 and Prop. 5 tell us that $\mathscr{C}_4$ cannot be s.t.c.i. on $\mathscr{F}_3$. Q.E.D.

REMARK 3. In proving Prop. 3 and Prop. 4 the following fact plays a key role: if q is the number of points at which the generic ruling of $\mathscr{F}_3$ meets $\mathscr{C}_4$, there exists some ruling g s.t. $\#(\mathscr{C}_4 \cap g) > q$.

Most likely a similar fact happens in more general situations, and could be of use in establishing that some curve $\mathscr{C}$ of degree ≥ 4 is not s.t.c.i. on a ruled surface $\mathscr{F}$ of degree ≥ 3.

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