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Pointwise Estimates for Minimizers of Some Non-uniformly Degenerate Functionals.

VITTORIO SCORNAZZANI (*)

RIASSUNTO - In questo articolo si prova l'hölderianità e una disuguaglianza di Harnack per i minimi di funzionali che degenerano in maniera non uniforme. Si adattano le tecniche di Giaquinta-Giusti e di Di Benedetto-Trudinger introducendo su R^n una metrica « naturale », diversa da quella euclidea, per i funzionali considerati che tiene conto della non uniforme degenerazione dei funzionali stessi.

Introduction.

In this paper we prove the Hölder regularity and the Harnack inequality for the minimizers of following functional:

$$\mathsf{F}(u, \Omega) = \int_{\Omega} F(x, Du)\, dx\,, \qquad D = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \tag{1}$$

where Ω is an open subset of $\mathbb{R}^n$. We assume that the following hypotheses are satisfied by function F: F is a Caratheodory function, and there exist two constants $M > 0$ and $m > 1$ such that:

$$M^{-1} w(x)\left(\sum_{j=1}^{n} \lambda_j^2 P_j^2\right)^{m/2} \leqslant F(x, P) \leqslant M w(x)\left(\sum_{j=1}^{n} \lambda_j^2 P_j^2\right)^{m/2}. \tag{2}$$

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Here w is a nonnegative weight in the sense of Muckenhaupt (see later on) and λ_j, $j = 1, \ldots, n$ are nonnegative functions defined on $\mathbf{R}^n$ satisfying the same hypotheses of [FL1] and [FL2]. In the case $\lambda_j = 1$ and $w = 1$, Giaquinta and Giusti ([GG1], [GG2]) proved the Hölder regularity of minimizers of $\boldsymbol{F}$ while Di Benedetto and Trudinger ([BT]) proved the Harnack inequality for nonnegative minimizers of $\boldsymbol{F}$. Afterwards Modica ([M]) extended these results to the case $\lambda_j = 1$ and w is a weight that satisfies the following assumptions: $w \geqslant 0$ and there exist $p > 1$, $c_w \equiv c(w, p) \geqslant 1$ such that

$$\left(\frac{1}{|S_R|}\int_{S_R} w\,dx\right)\left(\frac{1}{|S_R|}\int_{S_R} w^{-1/(p-1)}\,dx\right)^{p-1} \leqslant c_w \tag{3}$$

for any euclidean ball S_R of radius R. Here $|S_R|$ is Lebesgue measure of S_R. The results of the works listed above cannot apply to nonuniformly degenerating functionals like, for example, to the following one

$$\boldsymbol{F}(u, \Omega) = \int_\Omega \left(|D_x u|^2 + |x|^{2\sigma}|D_y u|^2\right)^{m/2} dx\,dy\,, \qquad (x, y) \in \mathbf{R}^p \times \mathbf{R}^q, \quad \sigma > 0\,.$$

On the other hand it seems to be natural to handle such a functional by equipping $\mathbf{R}^n$ by a metric d constructed in order to taking account of the special nonuniformly degeneration of F (and therefore of λ_j). A class of metrics of this kind has been studied by several authors, see, for example, [FL1], [FL2], [FL3], [NSW]. By using the metric d in the place of the euclidean metric one can adapt the technics of Giaquinta and Giusti, of Di Benedetto and Trudinger and of Modica to the study of functionals of the kind (1) (2). It is reasonable to require that the weight w satisfies the condition of Muckenhaupt with respect to the metric d; hence we substitute the hypothesis (3) with the following one: $w \geqslant 0$, there exist $p > 1$, and $c_w \equiv c(w, p) \geqslant 1$ such that

$$\left(\frac{1}{|B_R|}\int_{B_R} w\,dx\right)\left(\frac{1}{|B_R|}\int_{B_R} w^{-1/(p-1)}\,dx\right)^{p-1} \leqslant c_w \tag{4}$$

for any d-ball of radius R, $B_R \subset \Omega$.

As a conseguence, our results can be applied, for example, to the functional defined by:

$$F(x, y, D_x u, D_y u) = \left(|D_x u|^2 + |x|^{2\sigma} |D_y u|^2\right)^{m/2} \|(x, y\|^\alpha ,$$
$$(x, y) \in \mathbf{R}^p \times \mathbf{R}^q,\ \sigma > 0,\ \alpha \in \mathbf{R} .$$

In Section 1 we introduce the class $\mathfrak{DG}_m(\Omega, \lambda, w)$ which is the natural De Giorgi's class modelled on metric d and weight w. Then we prove that the functions in De Giorgi's class $\mathfrak{DG}_m(\Omega,\lambda, w)$ are Hölder continuous and then we prove that the minimizers are in $\mathfrak{DG}_m(\Omega, \lambda, w)$ and therefore they are Hölder continuous.

In Section 2 we prove the Harnack inequality by using a Krylov-Safanov ([KS]) covering type Lemma, that we prove by using fundamentally the homogeneity of the space $(\mathbf{R}^n, d, w(x)\,dx)$.

Now we list explicitely the hypotheses on the function λ_i. We suppose:

i) $\lambda_j \geqslant 0,\ \lambda_1 \equiv 1,\ \lambda_j(x) = \lambda_j(x_1, \ldots, x_{j-1})\ \forall x \in \mathbf{R}^n,\ j = 1, \ldots, n$.

ii) Put $\Pi = \left\{x \in \mathbf{R}^n \colon \prod_{k=1}^n x_k = 0\right\}$, then

$$\lambda_j \in C(\mathbf{R}^n) \cap C^1(\mathbf{R}^n - \Pi) , \qquad 0 < \lambda_j(x) \leqslant \Lambda \qquad \forall x \in \mathbf{R}^n - \Pi,\ j = 1, \ldots, n$$
$$\lambda_j(x_1, \ldots, x_i, \ldots, x_{j-1}) = \lambda_j(x_1, \ldots, - x_i, \ldots, x_{j-1})$$
$$j = 2, \ldots, n ; \qquad i = 1, \ldots, j-1 .$$

iii) There exists a family of nonnegative numbers ϱ_{ij} such that

$$0 \leqslant x_i \frac{\partial}{\partial x_i} (\lambda_j(x)) \leqslant \varrho_{ij} \lambda_j(x) , \qquad j = 2, \ldots, n ,\ i = 1, \ldots, j-1 .$$

These hypotheses allow to construct a «natural» metric d for the functional, associated with the fields $X_j = \lambda_j(\partial/\partial x_j)$, $j = 1, \ldots, n$ (see [FL1], [FL2], [FL3], [NSW]) in the following way.

A continuous curve $\gamma \subseteq \Omega$ is X-admissible if:

i) γ is piecevise C^1;

ii) each piece C^1 of γ is an integral curve of one of the vector fields $\pm X_j$, $j = 1, \ldots, n$.

If $\gamma: [0, T] \to \Omega$, we put $1(\gamma) = T$. The hypotheses on λ allow to prove that for any $x, y \in \Omega$ there exists an X-admissible curve γ joing x and y. Hence we can define $d(x, y) = \inf\{1(\gamma): \gamma$ is X-admissible and joins x and $y\}$. For every compact subset K of Ω there exists $c = c(K)$ and $\varepsilon > 0$ such that

$$c^{-1}|x - y| \leqslant d(x, y) \leqslant c|x - y|^{\varepsilon} \tag{5}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbf{R}^n$ (see [FL2], Prop. 2.9). In the following we shall write

$$w(E) = \int_E w\,dx$$

and, for $k \in \mathbf{R}$

$$B(k, R) = \{x \in B_R / u(x) > k\}\,.$$

It is well know that $|E| = 0$ iff $w(E) = 0$ (see, for example, Lemma 4 in [C]) and that for the measure $w(x)\,dx$ the following doubling property holds: there exists a constant $\beta = \beta(p, c(w, p), d)$ such that

$$w(B_{2R}) \leqslant \beta w(B_R) \tag{6}$$

for every d-ball of radius R (see [FS], Lemma 2.10).

Given $u \in \mathrm{Lip}(\Omega)$, we denote

$$D_\lambda u = \left(\lambda_1 \frac{\partial u}{\partial x_1}, \ldots, \lambda_n \frac{\partial u}{\partial x_n}\right) \quad \text{and} \quad |D_\lambda u| = \left(\sum_{j=1}^{n} \lambda_j^2 \left(\frac{\partial u}{\partial x_j}\right)^2\right)^{\frac{1}{2}},$$

besides we denote by $W_m^1(\Omega)$ the closure of the space $\mathrm{Lip}(\Omega)$ with respect to the norm

$$\|u\,;\, W_m^1(\Omega)\| = \left(\int_\Omega |u|^m w\,dx\right)^{1/m} + \left(\int_\Omega |D_\lambda u|^m w\,dx\right)^{1/m}.$$

DEFINITION 1. We call $u \in W_m^1(\Omega)$ a minimizer for $F(\cdot, \Omega)$ if for all $\Phi \in W_m^1(\Omega)$ with $\operatorname{supp}\Phi \subset \Omega$

$$F(u, \operatorname{supp}\Phi) \leqslant F(u + \Phi, \operatorname{supp}\Phi)\,.$$

DEFINITION 2. A function u is said to belong to De Giorgi's class, $\mathfrak{DG}_m(\Omega, \lambda, w) \equiv \mathfrak{DG}_m(\Omega)$ if:

i) $u \in W^1_m(\Omega)$;

ii) for any d-ball $B_R \subset \Omega$ and for any $\varrho > 0$, $\varrho < R$, $k \in \mathbf{R}$

$$\int_{B(k,\varrho)} |D_\lambda u|^m w\, dx = \frac{C}{(R-\varrho)^m} \int_{B(k,R)} |u-k|^m w\, dx \quad (^1) \tag{8}$$

where B_ϱ is the d-ball concentric to B_R and radius ϱ.

REMARK. In the case $\lambda_j \equiv 1$, the class $\mathfrak{DG}_m(\Omega)$ coincide with the usual De Giorgi's class (see [DG], [GG1], [M]).

§ 1. Here we prove that the functions in $\mathfrak{DG}_m(\Omega)$ are Hölder continuous (Th. 5) and that the minimizers are in $\mathfrak{DG}_m(\Omega)$. To this aim we make use of the following embedding Theorem whose proof is similar to the one of Th. 4.1 of [FS]:

THEOREM 3. Let w be a weight such that satisfies (4) with $p \leqslant m$, let B_R be the d-ball with centre $\bar{x}$ and radius R and $u \in W^1_m(\Omega)$ such that for a $\beta > 0$ we have $w(\{x \in B_R / u(x) = 0\}) \geqslant \beta w(B_R)$. Then exist $l > 1$ and $C > 0$ such that:

$$\left(\int_{B_R} |u|^{lm} w\, dx\right)^{1/lm} \leqslant C R w(B_R)^{(1-l)/lm} \left(\int_{B_R} |D_\lambda u|^m w\, dx\right)^{1/m}. \tag{9}$$

Here C depends on $\bar{x}$ and β and on constants of (4), l depends on m, ϱ_{ij}, and constants of (4).

PROOF. Put $E = \{x \in B_R / u(x) = 0\}$ and $u_R = (w(B_R))^{-1}\int_{B_R} uw\, dx$. Then, making use of Hölder inequality:

$$|u_R| \leqslant (w(E))^{-1}\int_E |u_R - u| w\, dx \leqslant \beta^{-1}\left((w(B_R))^{-1}\int_{B_R} |u_R - u|^{lm} w\, dx\right)^{1/lm}$$

(1) In the following we denote by C a positive constant which is not always the same at each occurrence.

by that, we have:

$$\left((w(B_R))^{-1}\int_{B_R}|u|^{lm} w\,dx\right)^{1/lm} \leqslant C\left((w(B_R))^{-1}\int_{B_R}|u_R - u)^{lm} w\,dx\right)^{1/lm} \leqslant$$
$$\leqslant CR\left((w(B_R))^{-1}\int_{B_R}|D_\lambda u|^m w\,dx\right)^{1/m}$$

by Th. 4.5 of [FS].

By (9) it is possible to prove that, if u and $-u \in \mathfrak{D}\mathcal{G}_m(\Omega)$, then u is locally essentially bounded. In fact, we have:

THEOREM 4. Let $u \in \mathfrak{D}\mathcal{G}_m(\Omega)$, $k \in R$, $\sigma \in]0, 1[$, $0 < R < R_0$: Then for any d-ball $B_R \subset \Omega$ we have:

$$\sup_{B_{\sigma R}} u \leqslant k + \frac{C}{(1-\sigma)^{l\Theta/(l-1)}} \cdot \tag{10}$$
$$\cdot \left(w(B_R)\right)^{-\Theta/m}\Big(w\big(B(k, R)\big)\Big)^{(\Theta-1)/m}\left(\int_{B(k,R)}|u - k|^m w\,dx\right)^{1/m}$$

where $\Theta = \Theta(l) > 1$ and l is the constant in (9), and $C = C(\bar{x}) > 0$.

We first prove two Lemmas in which we use the following notation like those used by Giaquinta ([G])

$$u(h, \varrho) := \int_{B(h,\varrho)} |u - k|^m w dx\,,$$
$$b(h, \varrho) := w(B(h, \varrho))\,, \qquad \Phi(h, \varrho) := u^{l\Theta/(l-1)}(h, \varrho)\, b(h, \varrho)$$

where $\Theta = \left(1 + (1 + 4(l-1)/l)^{\frac{1}{2}}\right)/2$.

LEMMA 4.1. Let $0 < \varrho < R < R_0$ and $h > k$. For any $u \in \mathfrak{D}\mathcal{G}_m(\Omega)$ we have:

$$\Phi(h, \varrho) = \frac{CR^{ml\Theta/(l-1)}\left(w(B_R)\right)^{-\Theta}}{(R-\varrho)^{ml\Theta/(l-1)}\,|h - k|^m}\,\Phi^{\Theta}(k, R)\,. \tag{11}$$

PROOF. Let B_R be a d-ball contained in Ω. Then there exists $\eta \in C_0^\infty(B_{(R+\varrho)/2})$, $\eta = 1$ on B_ϱ, $0 \leqslant \eta \leqslant 1$, and such that $|D_\lambda \eta| \leqslant C/(R - \varrho)$ (see [FL2], proof of Lemma 4.2). Let $u \in \mathfrak{D}\mathcal{G}_m(\Omega)$. In particular

$u \in W^1_m(\Omega)$, so by Corollary 5.2 in [FS], it results $\eta \max(u-k, 0) \in \in w^1_m(\Omega)$. From inequality (9) and doubling property for the measure $w(x)\,dx$, we have:

$$\left(\int_{B((R+\varrho)/2)} |\eta \max(u-k, 0)|^{lm} w\,dx\right)^{1/lm} \leqslant$$
$$\leqslant CR(w(B_R))^{(1-l)/lm}\left(\int_{B((R+\varrho)/2)} |D_\lambda(\eta \max(u-k, 0))|^m w\,dx\right)^{1/m}$$

for some $l>1$. By Hölder inequality, we get:

$$\int_{B_\varrho} |\max(u-k, 0)|^m w\,dx \leqslant \left(\int_{B_\varrho} |\max(u-k, 0|^{lm} w\,dx\right)^{1/l} \big(w(B(k, \varrho))\big)^{(l-1)/l}.$$

Hence, because $\varrho < (R+\varrho)/2 < R$ and $u \in \mathfrak{DG}_m(\Omega)$:

$$\int_{B(k,\varrho)} |u-k|^m w\,dx \leqslant CR^m\big(w(B_R)\big)^{(1-l)/l}\cdot$$
$$\cdot \int_{B(k,(R+\varrho)/2)} |D_\lambda(\eta \max(u-k, 0))|^m w\,dx \cdot \big(w(B(k, \varrho))\big)^{(l-1)/l} \leqslant$$
$$\leqslant \frac{CR^m(w(B_R))^{(1-l)/l}}{(R-\varrho)^m} \int_{B(k,R)} |u-k|^m w\,dx \cdot \big(w(B(k, \varrho))\big)^{(l-1)/l}.$$

Now, if $h>k$, and $0<\varrho<R$, we have

$$|h-k|^m w(B(h, \varrho)) \leqslant \int_{B(h,\varrho)} |u-k|^m w\,dx \leqslant \int_{B(k,R)} |u-k|^m w\,dx$$

hence, because $u(h, \varrho) \leqslant u(k, \varrho)$, using our notation we get

$$\begin{cases} u(h, \varrho) \leqslant \dfrac{CR^m(w(B_R))^{(1-l)/l}}{(R-\varrho)^m}\, u(k, R)\, b(k, R)^{(l-1)/l}, \\[2ex] b(h, \varrho) = \dfrac{1}{|h-k|^m}\, u(k, R)\,. \end{cases}$$

For any positive numbers ξ and ζ we find

$$u^{\xi}(h, \varrho)b^{\zeta}(h, \varrho) \leqslant \frac{CR^{m\xi}(w(B_R))^{(1-l)\xi/l}}{(R-\varrho)^{m\xi}|h-k|^{m\zeta}} u^{\xi+\zeta}(k, R)b(k, R)^{(l-1)\xi/l}.$$

Now we choose ξ and ζ in such way that for some $\Theta > 0$, we have $\xi + \zeta = \Theta\xi$, $(l-1)\xi/l = \Theta\zeta$, (then Θ must be the positive solution of $\Theta^2 - \Theta - (l-1)/l = 0$ i.e. $\Theta = \big(1 + (1 + 4(l-1)/l)^{\frac{1}{2}}\big)/2 > 1$. We can choose $\zeta = 1$ and $\xi = l\Theta/(l-1)$ and consequently the Lemma follows.

LEMMA 4.2. For any $k \in \mathbf{R}$, $0 < R < R_0$, $\sigma \in]0, 1[$, it holds:

$$\Phi(k + d, \sigma R) = 0$$

where

$$d = \frac{2^{(l\Theta/(l-1)+1)\Theta/(\Theta-1)} C^{1/m}(w(B_R))^{-\Theta/m}}{(1-\sigma)^{l\Theta/(l-1)}} \Phi^{(\Theta-1)/m}(k, R)$$

and C is the constant that appears in Lemma 4.1.

PROOF. We set $h_n = k + d(1 - 2^{-n})$, $\varrho_n = \sigma R + (1-\sigma)R2^{-n}$; $n \in \mathbf{N}$ then $h_n \nearrow k + d$, $\varrho_n \searrow \sigma R$. For every $n \in \mathbf{N}$ we have

$$\Phi(h_n, \varrho_n) \leqslant \frac{\Phi(k, R)}{2^{m/(\Theta-1)(l\Theta/(l-1)+1)n}}.$$

This easily follows by Lemma 4.1 and by induction. Now it is enough take the limit $n \to +\infty$ to prove the Lemma.

PROOF OF THEOREM 4. By Lemma 4.2, we have $\Phi(k + d, \sigma R) = 0$, thus either $u(k + d, \sigma R) = 0$ or $b(k + d, \sigma R) = 0$. Hence, since $(\Theta - 1)\Theta l/(l-1) = 1$, the thesis follows.

This holds also for $-u$ and hence u is locally essentially bounded.

THEOREM 5. If u and $-u \in \mathfrak{D}\mathfrak{G}(\Omega)$, then u is Hölder continuous.

We first prove the following two Lemmas:

LEMMA 5.1. Let $h > k > k_0$, if $w(B(k_0, R)) \leqslant \gamma w(B_R)$, $0 < \gamma < 1$, then:

$$|h-k|^{(m+1)/2}\big(w(B(h, R))\big)^{1/l} \leqslant \tag{12}$$
$$\leqslant C(\gamma)R^{(m+1)/2}(w(B_R))^{(1-l)/l}\Bigg(\int_{B(k,R)} |D_\lambda u|^m w\, dx\Bigg)^{(m+1)/2m} \cdot$$
$$\cdot \big(w(B(k, R) - B(h, R))\big)^{(m-1)/2m}.$$

PROOF. Put $v = \min(u, h) - \min(u, k)$. Then:

$$w(\{x \in B_R/v(x) = 0\}) = w(B_R - B(k, R)) =$$
$$= w(\{x \in B_R/u(x) \leq k\}) \geq w(\{x \in B_R/u(x) \leq k_0\}) \geq (1-\gamma)\, w(B_R)\,.$$

So we can apply (9). There exists $l > 1$, such that

$$|h-k|^{(m+1)l/2} w(B(h, R)) = \int_{B(h,R)} |v|^{(m+1)l/2} w\, dx \leq$$
$$\leq \int_{B(k,R)} |v|^{(m+1)l/2} w\, dx \leq \int_{B_R} |v|^{(m+1)l/2} w\, dx \leq$$
$$\leq C(\gamma) R^{(m+1)l/2} (w(B_R))^{1-l} \left(\int_{B_R} |D_\lambda v|^{(m+1)/2} w\, dx \right)^l =$$
$$= C(\gamma) R^{(m+1)l/2} (w(B_R))^{1-l} \left(\int_{B(k,R)-B(h,R)} |D_\lambda u|^{(m+1)/2} w\, dx \right)^l .$$

By Hölder inequality (12) follows.

LEMMA 5.2. Let $u \in \mathfrak{D}\mathfrak{G}_m(\Omega)$; setting

$$M(2R) = \sup_{B_{2R}} u\,, \qquad m(2R) = \inf_{B_{2R}} u\,;$$

$$k = \tfrac{1}{2}\big(M(2R) + m(2R)\big)\,, \qquad k_\nu = M(2R) - \frac{M(2R) - k_0}{2^\nu}\,.$$

Then, if $w(B(k_0, R)) \leq \gamma w(B_R)$, $0 < \gamma < 1$, we have:

$$w(B(k_\nu, R)) \leq \frac{C w(B_R)}{\nu^{l(m-1)/2m}}\,. \tag{13}$$

PROOF. In inequality (12) we put $k \equiv k_{i-1}$ and $h \equiv k_i$: Hence, by (8) and by doubling property of measure $w(x)\,dx$:

$$\big(w(B(k_i, R))\big)^{2m/l(m-1)} \leq$$
$$\leq C\big(w(B_R)\big)^{(2m(1-l)+l(m+1))/l(m-1)} \Big(w(B(k_{i-1}, R)) - w(B(k_i, R))\Big)\,.$$

Summing up for $i = 1, 2, \ldots, \nu$ and using the inclusion $B(k_\nu, R) \subseteq \subseteq B(k_i, R)$, we have

$$\nu\big(w(B(k_\nu, R))\big)^{2m/l(m-1)} \leqslant C(w(B_R))^{(2m(1-l)+l(m+1))/l(m-1)} w(B(k_0, R)) .$$

So the assertion follows.

Proof of Theorem 5. We use the notations of Lemma 5.2. We may assume $w(B(k_0, R)) \leqslant \frac{1}{2} w(B_R)$. Otherwise, since

$$\{x \in B_R - u(x) > - k_0(-u)\} = \{x \in B_R \; u(x) < k_0(u)\} ,$$

we can work with $-u$. Applying Theorem 4 and replacing k by k_ν, we have

$$M(R/2) \leqslant k_\nu + C\big(w(B_R)\big)^{-\Theta/m}\Big(\int\limits_{B(k_\nu, R)} |u - k_\nu|^m w \, dx\Big)^{1/m} \big(w(B(k_\nu, R))\big)^{(\Theta-1)/m} \leqslant$$
$$\leqslant k_\nu + C\big(w(B_R)\big)^{-\Theta/m} |M(2R) - k_\nu| \big(w(B(k_\nu, R))\big)^{\Theta/m} .$$

By Lemma 5.2 we can choose a $\bar\nu$ indipendent on R, such that $C(w(B_R))^{-\Theta/m}\big(w(B(k_\nu, R))\big)^{\Theta/m} < \frac{1}{2}$. Then

$$M(R/2) \leqslant M(2R) - 2^{-(\bar\nu+1)}(M(2R) - m(2R)) + \tfrac{1}{2}(M(2R) - k_{\bar\nu}) =$$
$$= M(2R) - 2^{-(\bar\nu+2)}(M(2R) - m(2R)) .$$

We now subtract $m(R/2)$:

$$M(R/2) - m(R/2) \leqslant \big(M(2R) - m(2R)\big)(1 - 2^{-(\bar\nu+2)})$$

i.e. $\omega(R/2) = M(R/2) - m(R/2) \leqslant \beta\omega(2R)$, with $\beta \in]0, 1[$ and thus there exists a constant α such that $\omega(\varrho) \leqslant C(\varrho/R)^\alpha \omega(R)$ and u is Hölder continuous by (5).

Now we prove that the minimizers are Hölder continuous. This result follows from Theorem 5 and the following

Theorem 6. Let $u \in W^1_m(\Omega)$ be a minimizer for the functional F defined in (1), (2). Then u and $-u \in \mathfrak{DG}_m(\Omega)$.

Proof. Let B_R be the d-ball with centre $\bar x \in \Omega$, such that $B_R \subset \Omega$. Moreover let u a minimizer (see (7)). Let now $\eta \in C_0^\infty(B_R)$, $\eta = 1$ in

B_ϱ, $0 < \varrho < R$ and such that $|D_\lambda u| \leqslant C/(R-\varrho)$ (see [FL2], proof of Lemma 4.2). For a fixed $k \in \mathbf{R}$, we put $\Phi = -\eta \max(u-k, 0)$. Since $B(k, \varrho) = \{x \in B_\varrho / u(x) > k\} \subseteq \operatorname{supp} \Phi$, we have

$$M^{-1}\int\limits_{B(k,\varrho)} |D_\lambda u|^m w\, dx \leqslant M^{-1}\int\limits_{\operatorname{supp}\Phi} |D_\lambda u|^m w\, dx \leqslant \int\limits_{\operatorname{supp}\Phi} F(x, Du)\, dx \leqslant$$

$$\leqslant \int\limits_{\operatorname{supp}\Phi} F(x, D(u+\Phi))\, dx \leqslant M\int\limits_{\operatorname{supp}\Phi} |D_\lambda(u+\Phi)|^m w\, dx \leqslant$$

$$\leqslant C\int\limits_{\operatorname{supp}\Phi} (1-\eta)^m |D_\lambda u|^m w\, dx + C\int\limits_{\operatorname{supp}\Phi} |u-k|^m |D_\lambda \eta|^m w\, dx \leqslant$$

$$\leqslant C\int\limits_{B(k,R)-B(k,\varrho)} |D_\lambda u|^m w\, dx + \frac{C}{(R-\varrho)^m}\int\limits_{B(k,R)} |u-k|^m w\, dx\,.$$

In the last step we used $\operatorname{supp}\Phi \subseteq \{x \in B_R / u(x) > k\}$ and $\eta = 1$ in B_ϱ. From this inequality adding $C \int\limits_{B(k,\varrho)} |D_\lambda u|^m w\, dx$, we obtain

$$(M^{-1} + C)\int\limits_{B(k,\varrho)} |D_\lambda u|^m w\, dx \leqslant C\int\limits_{B(k,R)} |D_\lambda u|^m w\, dx + \frac{C}{(R-\varrho)^m}\int\limits_{B(k,R)} |u-k)^m w\, dx\,.$$

This, by Lemma 1.1 in [GG1], implies

$$\int\limits_{B(k,\varrho)} |D_\lambda u|^m w\, dx \leqslant \frac{C}{(R-\varrho)^m}\int\limits_{B(k,R)} |u-k|^m w\, dx\,;$$

that is $u \in \mathfrak{DG}_m(\Omega)$.

At this point, to prove that $-u \in \mathfrak{DG}_m(\Omega)$, it is enough to observe that $-u$ is a minimizer of the functional $\tilde{\mathbf{F}}(v, \Omega) = \int\limits_\Omega \tilde{F}(x, Dv)\, dx$ where $\tilde{F}(x, Dv) = F(x, D(-v))$. It is evident that $\tilde{F}$ satisfies the sane hypotheses as F.

§ 2. Here we prove the Harnack inequality for the nonnegative minimizers of F. To this aim we prove a Krylow-Safanov covering type Lemma:

LEMMA 7. Let B_R be a d-ball, $E \subseteq B_R$, E measaurable, $\delta \in]0, 1[$,

$$B = \{B_{4\varrho}(x) \cap B_R / x \in B_R,\ \varrho > 0,\ w(E \cap B_\varrho(x)) \geqslant \delta w(B_\varrho(x) \cap B_R)\}$$

($B_\varrho(x)$ is the d-ball of centre x and radius ϱ). We consider

$$E_\delta = \bigcup_{B \in \mathfrak{B}} B .$$

Then either i) $E_\delta = B_R$, or ii) $w(E) \leqslant C\delta w(E_\delta)$.

Here $C > 1$ is dependent on the doubling constant for the measure $w(x)\, dx$.

PROOF. If $w(E) \geqslant \delta w(B_R)$ then, since $E \subseteq B_R$, we have $w(E) = w(E \cap B_R) \geqslant \delta w(B_R)$ and hence $B_R \in B$ and hence i) is valid.

Let now $w(E) < \delta w(B_R)$. If $w(E) = 0$ then ii) is valid. Let so $w(E) > 0$. We say that a d-ball of centre x and radius ϱ intersects substantially E(B i.s. E) if $w(E \cap B_\varrho(x)) \geqslant \delta w(B_\varrho(x) \cap B_R)$. We suppose that every point of E let be a Lebesgue point ([C]) that is

$$\lim_{r \to 0^+} \frac{w(E \cap B_r(x))}{w(B_r(x))} = 1 . \tag{14}$$

Afterwards, we cover B_R by means of d-balls $B_{R/2}(x_i^{(1)})$ of radius $R/2$ which are centred at the points of a maximal set $\{x_1^{(1)}, \ldots, x_{I_1}^{(1)}\} \subset B_R$ with $d(x_i^{(1)}, x_j^{(1)}) > R/2$, $\forall i \neq j$ ([CW]). Moreover we cover B_R by means of d-ball of radius $R/4$ centred at the points of a maximal set $\{x_1^{(2)}, \ldots, x_{I_2}^{(2)}\} \subset B_R$ with $d(x_i^{(2)}, x_j^{(2)}) > R/4$ and so on; in this way we obtain a class of familes

$$R^{(p)} = \{B_{2^{-(p-1)}R}(x_i^{(p)}) / i = 1, \ldots, I_p\} \quad \text{of covering of } B_R.$$

Then we consider

$$S_p = \Big\{B_{R/2^{p-1}}(x_i^{(p)})\ i = 1, \ldots, I_p,\ B \text{ i.s. } E,$$
$$x_i \notin \bigcup_{h=1}^{p-1} \bigcup_{B \in R^p} B,\ B_{R/2^{h-1}}(x_j^{(h)}) \text{ i.s. } E\Big\}$$

It results that for every $x \in E$, $\exists p \in \mathrm{N}$, $i \in \mathrm{N}$ such that $x \in B_{R/2^p}(x_i^{(p)})$ and $w(E \cap B_{R/2^{p-1}}(x_i^{(p)})) \geqslant \delta w(B_{R/2^{p-1}}(x_i^{(p)}))$. In fact, by (14), taken

$\sigma > \delta/A^2$ (A is the doubling constant for the measure $w(x)\,dx$), $\sigma < 1$, it exists $r_x \in \mathbf{R}_+$ such that $\forall r < r_x$ it holds $w(E \cap B_r(x)) \geqslant \sigma w(B_r(x))$. Let be $x \in E$, there exist $p, i \in \mathbf{N}$ such that $x \in B_{R/2^p}(x_i^{(p)})$ with $R/2^p < r_x$, then

$$w(E \cap B_{R/2^{p-1}}(x_i^{(p)})) \geqslant w(E \cap B_{R/2^p}(x)) \geqslant \sigma w(B_{R/2^p}(x)) \geqslant A\sigma w(B_{R/2^{p-1}}(x)) \geqslant$$
$$\geqslant A\sigma w(B_{R/2^p}(x_i^{(p)})) \geqslant A^2 \sigma w(B_{R/2^{p-1}}(x_i^{(p-1)})) \geqslant \delta w(B_{R/2^{p-1}}(x_i^{(p)}))\ .$$

To any $B \in \bigcup_{p\in\mathbf{N}} S_p$, we associate $\tilde{B}_B$ as following manner: let $B \in \bigcup_{p\in\mathbf{N}} S_p$ then $\exists p \in \mathbf{N}, \exists i \in \{1, \dots, I_p\}$ such that $B \equiv B_{R/2^{p-1}}(x_i^{(p)}) \in S_p$ with $x_i^{(p)} \in \in B_{R/2^{p-1}}(x_j^{(p-1)})$, $j \in \{1, \dots, I_{p-1}\}$. We put $\tilde{B}_B = B_{R/2^{p-2}}(x_j^{(p-1)})$. It results $B \subset \tilde{B}_B$ and $\tilde{B}_B$ not intersects substantially E. Then we put

$$\tilde{E}_\delta = \bigcup_{p\in\mathbf{N}} \Big(\bigcup_{B\in S_p} (\tilde{B}_B \cap B_R)\Big)\ .$$

Itresults $\tilde{E}_\delta \subseteq E_\delta$. In fact: let $y \in \tilde{E}_\delta$, then $\exists p \in \mathbf{N}, \exists B_{R/2^{p-1}}(x_i^{(p)}) \in S_p$ that intersects substantially E, such that $y \in \tilde{B}_B = B_{R/2^{p-2}}(x_j^{(p-1)})$ and $B_{R/2^{p-2}}(x_j^{(p-1)}) \subset B_{4R/2^{p-1}}(x_i^{(p)})$ (2), and $B_{4R/2^{p-1}}(x_i^{(p)}) \cap B_R \subset E_\delta$.

It results $w(\tilde{E}_\delta \cap E) = w(E)$. In fact $E \subseteq \tilde{E}_\delta$ because every point of E is Lebesgue point.

Let now $0 < \alpha < \frac{1}{4}$. We can prove that

$$B_{\alpha R/2^{p-1}}(x_i^{(p)}) \cap B_{\alpha R/2^{q-1}}(x_j^{(q)}) = \emptyset\ ,$$
$$\forall p, q \in \mathbf{N},\ \forall i = 1, \dots, I_p,\ \forall j = 1, \dots, I_q\ .$$

In fact, if (let $q \geqslant p$) $B_{\alpha R/2^{p-1}}(x_i^{(p)}) \cap B_{\alpha R/2^{q-1}}(x_j^{(q)}) \neq \emptyset$, because $B_{R/2^p}(x_i^{(p)}) \in \in R^{(p)}$, $B_{R/2^q}(x_j^{(q)}) \in R^{(q)}$, if $p = q$, then $d(x_i^{(p)} x_j^{(q)}) > R/2^p$. If $q > p$ then $B_{R/2^{p-1}}(x_i^{(p)}) \in S_p$, $B_{R/2^{q-1}}(x_j^{(q)}) \in S_q$ and $x_j^{(q)} \notin B_{R/2^p}(x_i^{(p)})$ then $d(x_i^{(p)}, x_j^{(q)}) > > R/2^p$.

In every case $d(x_i^{(p)}, x_j^{(q)}) > R/2^p$. Let $z \in B_{\alpha R/2^{p-1}}(x_i^{(q)}) \cap B_{\alpha R/2^{q-1}}(x_j^{(q)})$ then

$$R/2^p < d(x_i^{(p)}, x_j^{(q)}) \leqslant d(x_i^{(p)}, z) + d(z, x_j^{(q)}) \leqslant \alpha R/2^{p-1} + \alpha R/2^{q-1} \Leftrightarrow \alpha \geqslant \tfrac{1}{4}\ .$$

Then, by taking $0 < \alpha < \frac{1}{4}$ and by recalling that the doubling prop-

(2) In fact, let $z \in B_{R/2^{p-2}}(x_j^{(p-1)})$, then

$$d(z, x_i^{(p)}) \leqslant d(z, x_j^{(p-1)}) + d(x_j^{(p-1)}, x_i^{(p)}) < R/2^{p-2} + R/2^{p-1} = 3R/2^{p-1}\ .$$

erty holds also for $B \cap B_R$ (see Prop. 2.10 of [FL2] and Lemma 4 of [C]).

$$
\begin{aligned}
w(E) &= w(\tilde{E}_\delta \cap E) \leqslant \sum_{B \in \cup S_p} w(\tilde{B}_B \cap E) < \\
&< \delta \sum_{B \in \cup S_p} w(\tilde{B}_B \cap B_R) \leqslant^{(3)} \delta D \sum_{B \in \cup S_p} w(B \cap B_R) \leqslant^{(3)} \\
&\leqslant C\delta \sum_{B \in \cup S_p} w((\alpha B) \cap B_R) = C\delta w\Big(\bigcup_{B \in \cup S_p} (\alpha B) \cap B_R\Big) \leqslant C\delta w\Big(\bigcup_{B \in \cup S_p} B \cap B_R\Big) \leqslant \\
&= C\delta w\Big(\bigcup_{B \in \cup S_p} (\tilde{B}_B \cap B_R)\Big) = C\delta w(\tilde{E}_\delta) \leqslant C\delta w(E_\delta).
\end{aligned}
$$

Now we prove the Harnack inequality. We have, in fact:

THEOREM 8. Let u and $-u \in \mathfrak{DG}_m(\Omega)$, $u \geqslant 0$. Then it exists a constant C dependent on the functional and on $\bar{x}$ such that for any d-ball $B_R(\bar{x}) \equiv B_R$, $B_{8R} \subset \Omega$, we have:

$$\sup_{B_{R/2}} u \leqslant C \inf_{B_{R/2}} u .$$

The proof follows by following propositions:

PROPOSITION 8.1. Let $u \in \mathfrak{DG}_m(\Omega)$, $u \geqslant 0$. Then for any $q > 0$, it exists $C = C(\bar{x}, q)$ such that

$$\sup_{B_{R/2}} u \leqslant C \left(\frac{1}{w(B_R)} \int_{B_R} u^q w \, dx\right)^{1/q} . \tag{15}$$

PROOF. Let $q \geqslant m$. The (15) follows by (10) and Hölder inequality. Let now $0 < q < m$. By (10) for $\sigma \in]0, 1[$ fixed and $k = 0$

$$
\begin{aligned}
\sup_{B_{\sigma R}} u &\leqslant \frac{C}{(1-\sigma)^{l\Theta/(l-1)}} \left(\frac{1}{w(B_R)} \int_{B_R} u^m w \, dx\right)^{1/m} \leqslant \\
&\leqslant \frac{C}{(1-\sigma)^{l\Theta/(l-1)}} \left(\frac{1}{w(B_R)} \int_{B_R} u^q \,(\sup_{B_R} u)^{m-q}\, w \, dx\right)^{1/m} \leqslant \\
&\leqslant \varepsilon \sup_{B_R} u + \frac{C(\varepsilon)}{(1-\sigma)^{l\Theta m/q(l-1)}} \left(\frac{1}{w(B_R)} \int_{B_R} u^q w \, dx\right)^{1/q}
\end{aligned}
$$

(3) By doubling property.

with $0 < \varepsilon < 1$ fixed. (Here we have utilized the inequality $ab \leqslant \leqslant \varepsilon a^{m/(m-q)} + C(\varepsilon) b^{m/q}$, $a, b \geqslant 0$). Now by Lemma 1.1 in [GG1] we have

$$\sup_{B_{\sigma R}} u \leqslant \frac{C}{(1-\sigma)^{l\Theta m/q(l-1)}} \left(\frac{1}{w(B_R)} \int_{B_R} u^q w \, dx \right)^{1/q}$$

by which, for $\sigma = \frac{1}{2}$, the thesis follows.

PROPOSITION 8.2. Let $u \geqslant 0$, $-u \in \mathfrak{D}\mathfrak{G}_m(\Omega)$, $\tau > 0$, $\gamma \in]0,1[$. If $w(\{x \in B_R / u < \tau\}) \leqslant \gamma w(B_R)$, then:

$$\text{(16)} \qquad \text{i)} \; w(\{x \in B_R / u < \tau/2^{\nu+1}\}) \leqslant C(\gamma)(\gamma/\nu)^{l(m-1)/2m} w(B_R), \quad \nu \in \mathbf{N},$$

$$\text{(17)} \qquad \text{ii)} \; \inf_{B_{\frac{1}{2}R}} u \geqslant \lambda(\gamma)\tau, \text{ with } 0 < \lambda(\gamma) < 1.$$

PROOF. We observe that

$$w(\{x \in B_R / - u > -\tau\}) = w(\{x \in B_R / u < \tau\}) \leqslant \gamma w(B_R).$$

So we can apply the (12) of Lemma 5.1 to the function $-u$, for $h > k > -\tau$:

$$|h-k|^{m(m+1)/(m-1)} w(B(h,R))^{2m/l(m-1)} \leqslant \leqslant C(w(B_R))^{2m(1-l)/l(m-1)} \left(\int_{B(k,2R)} |-u-k|^m w \, dx \right)^{(m+1)/(m-1)} \cdot \cdot \left(w(B(k,R)) - w(B(h,R)) \right).$$

We put $h = -\tau/2^{s+1}$, $k = -\tau/2^s$, $s \in \mathbf{N}$ in the last inequality. We observe that on $B(-\tau/2^s, 2R)$ it is $0 \leqslant -\tau + \tau/2^s \leqslant \tau/2^s$. By doubling property for the measure $w(x)\,dx$, we obtain:

$$\left(w\left(\left\{ x \in B_R / u < \frac{\tau}{2^{s+1}} \right\} \right) \right)^{2m/l(m-1)} \leqslant \leqslant C(\gamma)(w(B_R))^{(2m(1-l)+l(m+1))/l(m-1)} \left(w\left(B\left(\frac{-\tau}{2^s}, R \right) \right) - w\left(B\left(\frac{-\tau}{2^{s+1}}, R \right) \right) \right).$$

Summing up for $s = 0, \ldots, \nu$, we have

$$\nu\left(w\left(\left\{x \in B_R/u < \frac{\tau}{2^{\nu+1}}\right\}\right)\right)^{2m/l(m-1)} \leqslant$$

$$\leqslant C(\gamma)\big(w(B_R)\big)^{(2m(1-l)+l(m+1))/l(m-1)} w(\{x \in B_R\ u < \tau\}) \leqslant$$

$$\leqslant C(\gamma)\big(w(B_R)\big)^{(2m(1-l)+l(m+1))/l(m-1)} \gamma w(B_R)\,.$$

From which it follows that

$$w\left(\left\{x \in B_R/u < \frac{\tau}{2^{\nu+1}}\right\}\right) \leqslant C(\gamma)\left(\frac{\gamma}{\nu}\right)^{l(m-1)/2m} w(B_R)$$

that is (16).

At last we prove the ii). By (10) applied to $-u$, with $k = -\tau$, $\sigma = \frac{1}{2}$, we have

$$\inf_{B_{R/2}} u \geqslant \tau - C\big(w(B_R)\big)^{-\Theta/m}\Bigg(\int\limits_{B(-\tau,R)} |-u+\tau|^m w\, dx\Bigg)^{1/m}\Big(w\big(B(-\tau, R)\big)\Big)^{(\Theta-1)/m} \geqslant$$

$$\geqslant \tau\left(1 - C\left(\frac{w(B(-\tau, R))}{w(B_R)}\right)^{\Theta/m}\right) \geqslant \tau(1 - C\gamma^{\Theta/m})$$

(we observe that on $B(-\tau, R)$ it is $0 \leqslant -u + \tau \leqslant \tau$).

Put $\gamma_0 = (1/2C)^{m/\Theta}$. Then

$$\inf_{B_{R/2}} u \geqslant \tfrac{1}{2}\tau \qquad \gamma \in\,]0, \gamma_0[\,, \tag{18}$$

Let nox $\nu \in \mathbf{N}$ fixed such that $C(\gamma)(\gamma/\nu)^{l(m-1)/2m} < \gamma_0$. $C(\gamma)$ is the constant that appears in (16). Then by (16) and (18) it is

$$\inf_{B_{R/2}} u \geqslant \frac{1}{2}\,\frac{\tau}{2^{\nu+1}} = \lambda(\gamma)\,\tau\,. \tag{19}$$

We note that $\lambda(\gamma) < 1$. The conclusion of proof of ii) follows making use of the doubling property of the measure $w(x)\,dx$. It exists $\alpha > 0$ (see Lemma 2.10 in [FS]) such that

$$w(\{x \in B_{8R}/u \geqslant \tau\}) \geqslant (1-\gamma)(1/8)^{\alpha}\, w(B_{8R})\,,$$

that is equivalent to $w(\{x \in B_{8R}/u < \tau\}) \leqslant \gamma' \cdot w(B_{8R})$. So by (19) follows (17).

PROPOSITION 8.3. Let $u \geqslant 0, -u \in \mathfrak{D}\mathfrak{S}_m(\Omega)$. Then $\forall q > 0$, $q < 1/C_0$ (see proof), $\exists C > 0$ dependent on q such that

$$(20) \qquad \left(\frac{1}{w(B_R)} \int_{B_R} u^q w \, dx\right)^{1/q} \leqslant C \inf_{B_{4R}} u .$$

PROOF. If $u = 0$ the (20) is evident. Now let $u \neq 0$. Then it exists $t > 0$ such that $w(\{x \in B_R/u(x) \geqslant t\}) > 0$. Let $\delta \in]0, 1[$ fixed, we consider $E = \{x \in B_R/u(x) \geqslant \lambda^{i-1} t\} \equiv A_t^{i-1}$ ($\lambda = \lambda(\gamma)$ is the constant that appears in (17)). We can apply the Lemma 7 to the set E. There exist $z \in B_R$ and $\varrho > 0$ such that $w(E \cap B(z, \varrho)) \geqslant \delta w(B(z, \varrho))$. Then by (17) replacing $\tau = \lambda^{i-1}$, it follows $u(x) = \lambda^i t$, $\forall x \in B(z, 4\varrho)$. Hence or

$$w(\{x \in B_R/u(x) \geqslant \lambda^i t\}) \geqslant \frac{1}{C\delta} w(\{x \in B_R/u(x) \geqslant \lambda^{i-1} t\})$$

or

$$w(\{x \in B_R/u(x) \geqslant \lambda^i t\}) = w(B_R) .$$

In both case we conclude: if $w(A_t^0) \geqslant C^{s-1} \delta^s w(B_R)$ for some s (we can suppose $C\delta < 1$) then

$$w(A_t^{s-1}) \geqslant \frac{1}{C\delta} w(A_t^{s-2}) \geqslant \ldots \geqslant \frac{1}{(C\delta)^{s-1}} w(A_t^0) \geqslant \delta w(B_R) ,$$

and hence by (17) we have $\inf_{B_{4R}} u \geqslant \lambda^s t$.

Choose s so that $w(A_t^0) \geqslant w(\{x \in B_R/u(x) \geqslant t\}) \geqslant C^{s-1} \delta^s$ i.e.

$$s \geqslant \frac{\log (C w(A_t^0)/w(B_R))}{\log (C\delta)} .$$

And so $\inf_{B_{4R}} u \geqslant C_1 t (w(A_t^0)/w(B_R))^{C_0}$ i.e.

$$w(A_t^0)/w(B_R) \leqslant C_1^{-1} t^{-1/C_0} \left(\inf_{B_{4R}} u\right)^{1/C_0} .$$

Now, since

$$\frac{1}{w(B_R)}\int_{B_R} u^q w\,dx = \frac{1}{w(B_R)}\left(q\int_{\xi}^{+\infty} t^{q-1} w(A_t^0)\,dt + q\int_0^{\xi} t^{q-1} w(A_t^0)\,dt\right),$$

setting $\xi = \inf\limits_{B_{4R}} u$ and $q < 1/C_0$, it holds

$$\frac{1}{w(B_R)}\int_{B_R} u^q w\,dx = \frac{q}{w(B_R)}\int_{\xi}^{+\infty} t^{q-1} w(A_t^0)\,dt + \xi^q \leqslant$$

$$\leqslant C_1^{-1} q \int_{\xi}^{+\infty} t^{q-1} t^{-1/C_0}\left(\inf_{B_{4R}} u\right)^{1/C_0} dt + \left(\inf_{B_{4R}} u\right)^q = \frac{C_1^{-1} q}{(1/C_0) - q}\left(\inf_{B_{4R}} u\right)^q + \left(\inf_{B_{4R}} u\right)^q .$$

From this it follows

$$\left(\frac{1}{w(B_R)}\int_{B_R} u^q w\,dx\right)^{1/q} \leqslant C \inf_{B_{4R}} u \leqslant C \inf_{B_{R/2}} u .$$

The conclusion of Theorem 8 now follows by (15).

REMARK. The previous results are true for *quasi-minima* (see [G], [GG2], [BT]). Recall that $u \in W_m^1(\Omega)$ is a quasi-minimum for the functional (1) with constant Q if, for any $\Phi \in W_m^1(\Omega)$, with $\operatorname{supp}\Phi \subset \Omega$

$$F(u, \operatorname{supp}\Phi) \leqslant QF(u + \Phi, \operatorname{supp}\Phi) . \tag{7'}$$

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