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Groups with finite conjugacy classes of subnormal subgroups

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1. Introduction.

In papers published in 1954-55 [10, 11], B. H. Neumann proved, together with many other results, the following fundamental Theorems:

1.1 *If in a group G every subgroup has a finite number of conjugates, then the centre $Z(G)$ has finite index in G.*

1.2. *If in a group G every subgroup has finite index in its normal closure, then the derived subgroup G' is finite.*

Sometime later, such results were in some sense specified by I. D. Macdonald [9], as follows:

1.3. *There exist functions μ, $\bar{\mu}$ of* $\mathbf{N}$ *in* $\mathbf{N}$, *such that*:

(i) *If G is a group in which every subgroup has at most m conjugates then $|G:Z(G)| \leqq \mu(m)$.*

(ii) *If G is a group in which every subgroup has index at most m in its normal closure, then $|G'| \leqq \bar{\mu}(m)$.*

Subsequently, the literature on the argument has been enriched by several authors. In the present paper, we study the case in which

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similar conditions are imposed not to all subgroups of a group, but just to subnormal subgroups. To this purpose let us introduce the following classes of groups.

Definitions. Let G be a group, m a positive integer. Then:

1) $G \in \boldsymbol{T}^*$ if every subnormal subgroup of G has finite index in its normal closure; that is if $|H^G : H| < \infty$ for every $H \operatorname{sn} G$;

2) $G \in \boldsymbol{T}_m$ if $|H^G : H| \leqq m$ for every $H \operatorname{sn} G$;

3) $V \in \boldsymbol{V}$ if every subnormal subgroup of G has a finite number of conjugates; that is if $|G : N_G(H)| < \infty$ for every $H \operatorname{sn} G$;

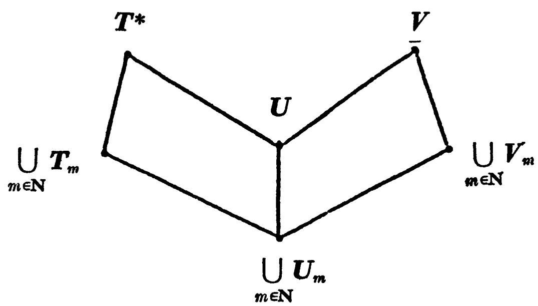
4) $G \in \boldsymbol{V}_m$ if $|G : N_G(H)| \leqq m$ for every $H \operatorname{sn} G$.

We will occasionally refer to two other classes, namely:

5) $G \in \boldsymbol{U}$ if $|H^G : H_G| < \infty$ for every $H \operatorname{sn} G$;

6) $G \in \boldsymbol{U}_m$ if $|H^G : H_G| \leqq m$ for every $H \operatorname{sn} G$.

The following diagram illustrates the inclusions among these classes.



Where all inclusions are proper, and no other inclusion holds. In fact:

(a) Let G be a group; $H \operatorname{sn} G$. If $|H^G : H_G| \leqq m < \infty$, then $|G : N_G(H)| \leqq m!$. Hence $\boldsymbol{U}_m \subseteq \boldsymbol{T}_m \cap \boldsymbol{V}_m$; moreover

$$\bigcup_{m \in \mathbf{N}} \boldsymbol{U}_m \subseteq \Big(\bigcup_{m \in \mathbf{N}} \boldsymbol{T}_m\Big) \cap \Big(\bigcup_{m \in \mathbf{N}} \boldsymbol{V}_m\Big) \quad \text{and} \quad \boldsymbol{U} \subseteq \boldsymbol{T}^* \cap \boldsymbol{V}.$$

Conversely, suppose that $|H^G:H| = n < \infty$ and $|G:N_G(H)| = r < \infty$; if $H = H_1, \ldots, H_r$ are the distinct conjugates of H in G, then:

$$|H^G:H_G| = \left| H^G:\bigcap_{i=1}^{r} H_i \right| \leqq \prod_{i=1}^{r} |H^G:H_i| = |H^G:H|^r = n^r .$$

Thus $T_n \cap V_r \subseteq U_{n^r}$ and $T^* \cap V \subseteq U$. In conclusion we have

$$U = T^* \cap V \quad \text{and} \quad \bigcup_{m\in\mathbb{N}} U_m = \Big(\bigcup_{m\in\mathbb{N}} T_m\Big) \cap \Big(\bigcup_{m\in\mathbb{N}} V_m\Big) .$$

(*b*) $U \neq \bigcup_{m\in\mathbb{N}} U_m$. In fact, the infinite dihedral group $D_\infty = \langle x, y;\ xx^y = y^2 = 2\rangle$ is an U-group, but it does not belong to any U_m, $n \in \mathbb{N}$.

(*c*) $\bigcup_{m\in\mathbb{N}} V_m \nsubseteq T^*$. For example, the standard wreath product C_{p^∞} wr C_p; where C_{p^∞} is a Prüfer group of type p^∞ and C_p is a cyclic group of order p, p a prime, belongs to V_p but not to T^*.

(*d*) $\bigcup_{m\in\mathbb{N}} T_m \nsubseteq V$. The central product of an infinite number of groups isomorphic to the quaternion group of order 8, belongs to T_2 but not to V.

Clearly, every T-group, that is every group in which each subnormal subgroup is normal, belongs to all the classes above defined. These classes can therefore be viewed as generalizations of the class of T-groups. Also, if we denote by B_n the class of groups in which every subnormal subgroup has defect at most n, then $V_n \cap T_n \subseteq B_n$, for every $n \in \mathbb{N}$. Inclusion $T_n \subseteq B_n$ is obvious, and $V_n \subseteq B_n$ follows from the fact that if H is a subnormal subgroup of a group G, and $G = H_0 \trianglerighteq H_1 \trianglerighteq \ldots \trianglerighteq H_n = H$ is the normal closure series of H in G, then $N_G(H_n) \subsetneq N_G(H_{n-1}) \subsetneq \ldots \subsetneq N_G(H_1) = G$ (by contrast, we observe that the infinite dihedral group belongs to U but not to the class of groups in which the intersection of any family of subnormal subgroups is subnormal). Furthermore, we recall that a subgroup H of G is said to be almost normal if $|H^G:H|$ is finite, and almost subnormal if $|K:H|$ is finite for some subnormal subgroup K of G. Then T^* is precisely the class of those groups in which every almost subnormal subgroup is almost normal. Indeed, T^* is the class of groups in which the relation of almost normality is transitive, see [2].

We finally observe that $\boldsymbol{IT}$-groups, that is groups in which every infinite subnormal is normal (see De Giovanni and Franciosi [4]) are $\boldsymbol{U}$-groups (see Heineken [6, Corollary 3]); and that a special subclass of $\bigcup_{m\in\mathbf{N}} \boldsymbol{U}_m$ has been recently considered by Heineken and Lennox in [7].

In common with other investigations about the subnormal structure of a group (see for instance, the treatment of $\boldsymbol{T}$-groups by D. Robinson in [12]), it is reasonable to restrict to the soluble case the classes under consideration. In this paper, we will mainly consider soluble groups belonging to $\boldsymbol{T}^*$, $\boldsymbol{V}$, and $\boldsymbol{V}_m$ ($m \in \mathbf{N}$). Therefore we will deal with non trivial abelian normal sections of a group G, over which G acts by conjugation. Hence it is of particular relevance the study of the action of a group of automorphisms Γ of an abelian group A, such that $|\Gamma : N_\Gamma(H)|$ or $|H^\Gamma : H|$ is finite (and possibly bounded) for every subgroup H of A. This is the object of Section 2, from which we quote, as an example, a single result, namely:

THEOREM 2.15. *If A is an abelian group, and $\Gamma \leqq \mathrm{Aut}\,(A)$ such that $|\Gamma : N_\Gamma(H)| \leqq m$ for every $H \leqq A$, then there exists a normal subgroup Γ_1 of Γ, such that Γ_1 normalizes every subgroup of A and the index $|\Gamma : \Gamma_1|$ is finite and bounded by a function of m.*

On the basis of the results obtained in Section 2, in Section 3 we first study hyperabelian p-groups in $\boldsymbol{T}^* \cup \boldsymbol{V}$ (Theorem 3.2); afterwards, we give some structural properties of soluble groups in $\boldsymbol{T}^*$ and $\boldsymbol{V}$. In particular, we prove (Corollaries 3,4 and 3.7) that a soluble $\boldsymbol{V}$-group (respectively $\boldsymbol{T}^*$-group) is metabelian by finite (finite by metabelian). Recalling that a soluble $\boldsymbol{T}$-group is always metabelian (Robinson [12, Theorem 2.3.1]), one might suspect that every soluble $\boldsymbol{V}$-group (or $\boldsymbol{T}^*$-group) is a finite extension of a $\boldsymbol{T}$-group (respectively is finite by $\boldsymbol{T}$-group). That this is not the case, not even for the smaller class $\boldsymbol{U}$, is shown by some examples at the end of Section 3.

In Section 4, we study soluble $\boldsymbol{V}_m$-groups, $m \in \mathbf{N}$. Our main result is the following.

THEOREM 4.8. *If G is a soluble $\boldsymbol{V}_m$-group, then $|G : \omega(G)| \leqq \gamma(m)$.*

Where γ is a function of $\mathbf{N}$ in itself, and $\omega(G)$, the Wielandt subgroup of G, is the intersection of the normalizers of the subnormal subgroups of G. It there follows that a group G has a bound on the number of conjugates of its subnormal subgroups if and only if $|G : \omega(G)|$ is finite. This result can be viewed as an analogous for soluble groups, of the quoted Theorem 1.3 (i) of I. D. Macdonald, with $\omega(G)$ instead of $Z(G)$. However, this analogy cannot be extended to Theorem 1.1 of B. Neumann; in fact, in order to have $|G:\omega(G)|$ finite, it is essential to assume that $|G:N_G(H)|$ is not only finite, but also bounded, for every $H \operatorname{sn} G$: if G is the infinite dihedral group, then $|G:N_G(H)| < \infty$ for every $H \operatorname{sn} G$, but $\omega(G) = 1$. Less obviously, it is also not possible to drop the hypothesis of solubility in Theorem 4.8. Indeed, there exist locally finite V_p-groups (p a prime) in which the Wielandt subgroup has infinite index; a family of them is constructed at the end of the paper.

NOTATIONS. Let G be a group, $n \in \mathbf{N}$. Then $Z_n(G)$ is the n-th term of the upper central series of G, $Z_\omega(G) = \bigcup_{n\in\mathbf{N}} Z_n(G)$; $\gamma_n(G)$ is the n-th term of the lower central series of G, $G^N = \bigcap_{n\in\mathbf{N}} \gamma_n(G)$; $\pi(G)$ is the set of primes p, such that G has at least one element of order p; $\omega(G) = \bigcap_{H \operatorname{sn} G} N_G(H)$ is the Wielandt subgroup of G. Further, we say that G is reduced if it does not admit non trivial normal divisible abelian subgroups. If G is an abelian group, by rank of G we mean the Prüfer rank, that is the supremum among the cardinalities of the minimal generating sets of the finitely generated subgroups of G; by total rank of G we mean the sum of the ranks of all the distinct primary components of G and of the cardinality of a maximal indipendent subset of elements of infinite order of G.

We denote by $\mathbf{P}$ the set of prime numbers; if $\pi \subseteq \mathbf{P}$, then $\pi' = \mathbf{P} \setminus \pi$.

Let A be a nilpotent group, then A_π is the π-component of A; as usual, if $\pi = \{p\}$, we write $A_\pi = A_p$ and $A_{\pi'} = A_{p'}$. If A is a p-group, $p \in \mathbf{P}$, and $n \in \mathbf{N}$, we put $\Omega_n(A) = \langle a \in A;\ |a| \leqq p^n \rangle$.

Let Γ be a group of operators on the group G, and $H \leqq G$; we write $N_\Gamma(H) = \{a \in \Gamma;\ H^\alpha = H\} \leqq \Gamma$ (and $N_\Gamma(x)$ instead of $N_\Gamma(\langle x \rangle)$ if $x \in G$). Moreover, we put $H^\Gamma = \langle H^\alpha;\ \alpha \in \Gamma \rangle$ and $H_\Gamma = \bigcap_{\alpha\in\Gamma} H^\alpha$, in particular, if $\Gamma = G$ in its action by conjugation, then H^G and H_G are, respectively, the normal closure and the normal core of H in G.

$$\operatorname{Paut}_\Gamma(G) = \{a \in \Gamma;\ H^\alpha = H, \text{ for every } H \leqq G\}.$$

Further, $\operatorname{Paut}(G) = \{\alpha \in \operatorname{Aut}(G);\ H^\alpha = H, \text{ for every } H \leqq G\}$ is the group of power automorphism of G. We will freely use the fact that $\operatorname{Paut}(G)$ is a normal abelian subgroup of $\operatorname{Aut}(G)$, and, if G is abelian, Paut $(G) \leqq Z(\mathrm{Aut}\,(G))$ (see [3] and [8] for the relevant facts on power automorphisms).

For the properties of abelian groups that we will need, we refer to the two volumes of Fuchs [5]; with the notice that we use « direct product » and « cartesian product » instead of, respectively, « direct sum » and « direct product », and that we denote by C_{p^∞}, the Prüfer group of type p^∞. The standard reference for soluble T-groups is paper [12] by D. Robinson.

We will use without any further comment the obvious fact that subnormal subgroups and homomorphic images of a group belonging to any of the classes under consideration, belong to the same class.

2. Automorphisms of abelian groups.

In this section, we collect some results on the action of those particular types of automorphisms groups of abelian groups, which are relevant in our subsequent discussion of soluble groups belonging to the class V, T^*, and V_m $(m \in \mathrm{N})$.

2.1. Lemma. *Let N be a periodic nilpotent group, $\Gamma \leqq \mathrm{Aut}\,(N)$.*

(*a*) *If $|\Gamma : N_\Gamma(H)| < \infty$ for every $H \leqq N$, then there exists a finite set π of prime numbers, such that $|\Gamma : \mathrm{Paut}_\Gamma\,(N_{\pi'})|$ is finite.*

(*b*) *If there exists $m \in \mathrm{N}$, such that $|\Gamma : N_\Gamma(x)| \leqq m$ for every $x \in N$, then there exists a set π of prime numbers, such that $|\pi| < m$, and $|\Gamma : \mathrm{Paut}_\Gamma\,(N_{\pi'})| \leqq m$.*

(*c*) *If $|H^\Gamma : H| < \infty$ for every $H \leqq N$, then there exists a finite set π of prime numbers, such that $\Gamma = \mathrm{Paut}_\Gamma\,(N_{\pi'})$.*

Proof. In view of the elementarity of these observations, we only prove (*b*). We argue by induction on m. If $m = 1$, then $\Gamma = \mathrm{Paut}_\Gamma\,(N)$ and $\pi = \emptyset$. Let $m > 1$; then there exists a prime p, such that Γ does not fix all cyclic subgroups of N_p. Let $x \in N_p$ be such that $r = |\Gamma : N_\Gamma(x)| > 1$ and write $\Gamma_1 = N_\Gamma(x)$. If $y \in N_{p'}$, since x and y commute and have coprime order, we have:

$$N_\Gamma(xy) = N_\Gamma(x) \cap N_\Gamma(y) = N_{\Gamma_1}(y)\,.$$

Hence, for any $y \in N_{p'}$:

$$m \geqq |\Gamma : N_\Gamma(xy)| = |\Gamma : \Gamma_1|\, |\Gamma_1 : N_{\Gamma_1}(y)| = r|\Gamma_1 : N_{\Gamma_1}(y)|\,,$$

that is:

$$|\Gamma_1 : N_{\Gamma_1}(y)| \leqq [m/r] \leqq m-1\,.$$

By inductive hypothesis, there exists a set of primes π_1, such that $|\pi_1| < m-1$ and $|\Gamma_1 : \mathrm{Paut}_{\Gamma_1}(N_{p',\pi_1'})| \leqq [m/r]$. Now, setting $\pi = \{p\} \cup \pi_1$, we have $|\pi| < m$ and, clearly, $\mathrm{Paut}_\Gamma(N_{\pi'}) \geqq \mathrm{Paut}_{\Gamma_1}(N_{\pi'})$, so:

$$|\Gamma : \mathrm{Paut}_\Gamma(N_{\pi'})| \leqq |\Gamma : \mathrm{Paut}_{\Gamma_1}(N_{\pi'})| =$$
$$= |\Gamma : \Gamma_1|\, |\Gamma_1 : \mathrm{Paut}_{\Gamma_1}(N_{\pi'})| \leqq r[m/r] \leqq m\,. \qquad ■$$

The following is essentially Proposition 34.1 in Fuchs [5].

2.2. LEMMA. *Let A be a reduced abelian p-group; let B be a subgroup of A such that A/B is divisible. Assume that α is an automorphism of A leaving B invariant and acting as a power automorphism on it. Then α is a powerr automorphism of A.*

PROOF. We observe that $\exp(B) = \exp(A)$. In fact, if $\exp(B) = p^n$, then $B \leqq \Omega_n(A)$, and so $A^{p^n} \simeq A/\Omega_n(A)$ is divisible; since A is reduced, we get $\Omega_n(A) = A$, that is $\exp(A) = p^n$. If $\exp(B) = \infty$, clearly $\exp(A) = \infty$. We can therefore define a power automorphism ν of A by putting, for $a \in A$, $\nu(a) = a^{\nu_k}$, where ν_k is a positive integer such that $\alpha(b) = b^{\nu_k}$ if b is an element of B of the same order p^k of a. Now, the kernel K of the endomorphism $\nu - \alpha$ of A contains B. Since A/B is divisible and A is reduced, this implies $K = A$ and so $\alpha = \nu$ is a power automorphism of A. ■

We discuss now the case in which Γ is a group of automorphisms of an abelian group A, and $|\Gamma : N_\Gamma(H)|$ is finite for every $H \leqq A$. The description of the general case is preceded by some particular cases.

2.3. LEMMA. *Let A be a divisible abelian p-group, $\Gamma \leqq \mathrm{Aut}(A)$ such that $|\Gamma : N_\Gamma(H)| < \infty$ for every $H \leqq A$. Then $|\Gamma : \mathrm{Paut}_\Gamma(A)|$ is finite. If $|\Gamma : N_\Gamma(x)| \leqq m$ $(m \in \mathbb{N})$ for every $x \in A$, then $|\Gamma : \mathrm{Paut}_\Gamma(A)| \leqq m$.*

PROOF. Suppose that there exists a sequence $x_1, x_2, \ldots$ of elements of A, such that, for all i: $x_{i+1}^{p^{n_i}} = x_i$ for some positive integer n_i, and $N_\Gamma(x_{i+1}) \lneqq N_\Gamma(x_i)$. Then $H = \langle x_i;\, i \in \mathbf{N}\rangle$ is a subgroup of A isomorphic to a Prüfer group of type p^∞; whence $N_\Gamma(H) = \bigcap_{i \in \mathbf{N}} N_\Gamma(x_i)$, and the index $|\Gamma : N_\Gamma(H)|$ is infinite, contradicting our hypotheses. Thus, there exists an element $x_1 \in A$, such that

$$\text{(1)} \qquad \text{if } y \in A \text{ and } x_1 \in \langle y\rangle\,, \text{ then } N_\Gamma(x_1) = N_\Gamma(y)\,.$$

(If $|\Gamma : N_\Gamma(x)| \leqq m$ for every $x \in A$, just take $x_1 \in A$ such that $|\Gamma : N_\Gamma(x_1)|$ is maximal).

Put $\Gamma_1 = N_\Gamma(x_1)$, then, $|\Gamma : \Gamma_1|$ is finite (and $|\Gamma : \Gamma_1| \leqq m$ in the second case). We prove that $\Gamma_1 \leqq \mathrm{Paut}\,(A)$.

Let $y \in A$; if $x_1 \in \langle y\rangle$ then Γ_1 normalizes $\langle y\rangle$ by (1). Otherwise, let $\langle x_1\rangle \cap \langle y\rangle = \langle y^{p^k}\rangle \lneqq \langle x_1\rangle$, $k \in \mathbf{N}$. Take $x_2, x_3 \in A$ such that $x_3^{p^k} = x_2$ and $x_2^{p^k} = x_1$. By (1), Γ_1 normalizes $\langle x_3\rangle$ (and $\langle x_2\rangle$). Also $\langle (x_2 y)^{p^k}\rangle = \langle x_1 y^{p^k}\rangle = \langle x_1\rangle$ and so Γ_1 normalizes $\langle x_2 y\rangle$. In particular, Γ_1 acts on the group $V = \langle x_3, y\rangle / \langle x_2\rangle$. Moreover, Γ_1 induces on V a group of power automorphisms; in fact, if $a, b \in \mathbf{N}$ and $a \not\equiv 0 \pmod{p^k}$: $\langle (x_3^a y^b)^{p^k}\rangle = \langle x_2^a y^{bp^k}\rangle \geqq \langle x_1\rangle$, whence $\langle x_2^a y^b\rangle$ is normalized by Γ_1.

Let now $g \in \Gamma_1$, then:

$$\text{(2)} \qquad y^g = y^s x_2^u \text{ and } x_3^g = x^l\,, \text{ where } s, u, l \in \mathbf{N} \text{ and } l \equiv s \pmod{p^k}$$

and, since $x_2 y$ is normalized by Γ_1:

$$\text{(3)} \qquad x_2^l y^g = x_2^g y^g = (x_2 y)^g = x_2^t y\;, \qquad t \in \mathbf{N}\,.$$

By comparing (2) and (3), we get: $x_2^{t-l} y^t = x_2^u y^s$, hence

$$y^{t-s} = x_2^{u+l-t} \in \langle x_2\rangle \cap \langle y\rangle = \langle y^{p^k}\rangle\,.$$

This yields $t \equiv s \pmod{p^k}$ and $u + l \equiv t \pmod{p^k}$; together with $l \equiv s \pmod{p^k}$, we have $u \equiv 0 \pmod{p^k}$. Thus:

$$y^g = y^s x_2^u = y^s x_2^{cp^k} = y^s x_1^c\,, \qquad \text{for } c \in \mathbf{N}\,.$$

Let now $y_1 \in A$ such that $y_1^{p^i} = y$, where $p^i = |x_1|$. Then $(x_1 y_1)^{p^i} = y$.

Now, for some $a, b \in \mathbf{N}$:

$$y^q = (y_1^{p^t})^q = (y_1^q)^{p^t} = (y_1^a x_1^b)^{p^t} = (y_1^{p^t})^a = y^a .$$

This proves that $\Gamma_1 \leqq \mathrm{Paut}(A)$. ■

2.4. THEOREM. *Let A be a periodic abelian group, $\Gamma \leqq \mathrm{Aut}(A)$ such that $|\Gamma : N_\Gamma(H)| < \infty$ for every $H \leqq A$. Then $|\Gamma : \mathrm{Paut}_\Gamma(A)|$ is finite.*

PROOF. A is the direct product of its primary components A_p. By 2.1 (*a*), there exists a subgroup of finite index in Γ, which acts as a group of power automorphisms on each component A_p, except at most a finite number of them. Since, clearly,

$$\mathrm{Paut}_\Gamma(A) = \bigcap_{p \in \pi(A)} \mathrm{Paut}_\Gamma(A_p),$$

it is sufficient to prove the Theorem in the case in which A is a p-group, for some prime p.

1) *If A is divisible*, then we apply Lemma 2.3.

2) *Let A be reduced.* In this case, by Lemma 2.2, $\mathrm{Paut}_\Gamma(A) = \mathrm{Paut}_\Gamma(B)$, where B is a basic subgroup of A. Put $\Gamma_1 = N_\Gamma(B)$, then $|\Gamma : \Gamma_1| < \infty$. Assume that $\Gamma_1 = \mathrm{Paut}_{\Gamma_1}(M)$, for some Γ_1-invariant subgroup M of B, of finite index in B. If Y is a transversal (i.e. a set of representatives) of M in B, then $\langle Y \rangle^{\Gamma_1}$ is finite, say of exponent p^r; then $\langle Y \rangle^{\Gamma_1} \leqq \Omega_r(B)$. Now, $\Gamma_1 = \mathrm{Paut}_{\Gamma_1}(M)$ implies that $\Gamma_1 / C_{\Gamma_1}(\Omega_r(M))$ is finite, and so $\Gamma_2 = C_{\Gamma_1}(\Omega_r(M)) \cap C_{\Gamma_1}(\langle Y \rangle^{\Gamma_1})$ has finite index in Γ_1 (and therefore in Γ); moreover, Γ_2 acts as a group of power automorphisms on B (and, therefore, on A).

Assume now, by contradiction, that $|\Gamma : \mathrm{Paut}_\Gamma(B)| = \infty$; then, by what we observed above, no subgroup of finite index of Γ_1 acts as a group of power automorphisms on a subgroup of finite index of B. Let $x_1 \in B$ such that $H_1 = N_{\Gamma_1}(x_1) \neq \Gamma_1$, put $K_1 = \langle x_1 \rangle^{\Gamma_1}$ and let M_1 be a subgroup of finite index in B, such that $K_1 \cap M_1 = 1$ and M_1 is Γ_1-invariant (this is possible because B is in particular residually finite and, if L is a subgroup of finite index in B, then L_Γ has again finite index in B). Let $x_2 \in M_1$ such that $N_{H_1}(x_2) \neq H_1$; $K_2 = \langle x_2 \rangle^{\Gamma_1}$ (hence

$K_2 \leqq M_1$) and M_2 a Γ_1-invariant subgroup of finite index in B such that $\langle K_1, K_2\rangle \cap M_2 = 1$, and $M_2 \leqq M_1$. Continuing in this way, with the obvious notation, we obtain a sequence x_i ($i \in \mathrm{N}$) of elements of B, such that if $X = \langle x_1, x_2, \ldots\rangle$, then $X \simeq \operatorname{Dir}_{i\in\mathrm{N}} \langle x_i\rangle$ and $X \cap K_i = \langle x_i\rangle$ for every $i \in \mathrm{N}$ (in fact, $X \cap K_1 = \langle x_1\rangle\langle x_2, \ldots\rangle \cap K_1 \leqq \langle x_1\rangle M_1 \cap K_1 = \langle x_1\rangle(M_1 \cap K_1) = \langle x_1\rangle$, and, if $i > 1$: $X \cap K_i \leqq \langle x_1, \ldots, x_i\rangle M_i \cap K_i = \langle x_1, \ldots, x_i\rangle$, but $K_i \leqq M_{i-1}$, so: $X \cap K_i \leqq \langle x_1, \ldots, x_i\rangle \cap M_{i-1} = \langle x_i\rangle$).

Finally, we get $N_{\Gamma_1}(X) \leqq \bigcap_{i\in\mathrm{N}} N_{\Gamma_1}(x_i)$, which implies, by our choice of the x_i's: $|\Gamma_1 : N_{\Gamma_1}(X)| = \infty$. A contradiction.

3) *The general case.* Let D be the divisible radical of A, C a complement of D in A. Then C is reduced and $|\Gamma : N_\Gamma(C)|$ is finite. Hence, by the two cases discussed before, we may assume that $D \neq 1$ and $\Gamma = \operatorname{Paut}_\Gamma(D) = \operatorname{Paut}_\Gamma(C)$.

If $\exp(C)$ is finite, say p^n, then $C_\Gamma(C) \cap C_\Gamma(\Omega_n(D))$ has finite index in Γ and acts as a group of power automorphisms on A.

If $\exp(C) = \infty$, then (see Fuchs [5, 35.4]) C admits a basic subgroup $B \neq C$, and, therefore, a subgroup H such that $H/B \simeq C_{p^\infty}$. Set $H/B = \langle \bar{b}_i;\ i \in \mathrm{N}\rangle$ with $\bar{b}_0 = 1$ and, if $i \geqq 1$, $\bar{b}_i^p = \bar{b}_{i-1}$. Since B is a pure subgroup of C, we may select a set of representatives $\{b_i;\ i \in \mathrm{N}\}$ of the cosets $\bar{b}_i$, in such a way that $|b_i| = |\bar{b}_i| = p^i$ (thus $\langle b_i\rangle \cap B = 1$) for every $i = 0, 1, \ldots$. Let K be a subgroup of D, isomorphic to C_{p^∞}; $K = \langle a_i;\ i \in \mathrm{N}\rangle$ with $a_0 = 1$ and, for $i \geqq 1$, $a_i^p = a_{i-1}$. Consider the subgroup $L = \langle B, a_i b_i;\ i \in \mathrm{N}\rangle$ of A. Then $\Gamma_0 = N_\Gamma(L)$ has finite index in Γ. We show that $\Gamma_0 \leqslant \operatorname{Paut}_\Gamma(A)$. Observe first that L is isomorphic to H, which is reduced; moreover, $L/B \simeq C_{p^\infty}$. Now, $\Gamma_0 = \operatorname{Paut}_{\Gamma_0}(B)$, hence, by Lemma 2.2, $\Gamma_0 = \operatorname{Paut}_{\Gamma_0}(L)$. Thus, if $g \in \Gamma_0$, then G induces a power on D, C and L; hence for any $i \in \mathrm{N}$, there exist positive integers δ_i, γ_i, t_i such that:

$$a_i^{t_i} b_i^{t_i} = (a_i b_i)^{t_i} = (a_i b_i)^g = a_i^g\, b^g = a_i^{\delta_i} b_i^{\gamma_i}.$$

Since $a_i \in D$, $b_i \in C$ and $D \cap C = 1$, we get $a_i^{\delta_i} = a_i^{t_i}$ and $b_i^{\gamma_i} = b_i^{t_i}$. This shows that $g \in \Gamma_0$ induces the same power t_i on elements of D and C of the same order p^i. This is true for every $g \in \Gamma_0$, whence $\Gamma_0 \leqq \operatorname{Paut}_\Gamma(A)$, concluding the proof. ∎

2.5. LEMMA. *Let A be a torsion free abelian group, $\Gamma \leqq \operatorname{Aut}(A)$ such that $|\Gamma : N_\Gamma(H)| < \infty$ for every $H \leqq A$. Then Γ is finite.*

PROOF. Let $x \in A$, $0 \neq m \in \mathbf{N}$ and $g \in C_\Gamma(x^m)$; then $(x^g)^m = x^m$, whence $(x^{-1}x^g)^m = 1$. Since A is torsion free, we have $x = x^g$, that is $g \in C_\Gamma(x)$. Hence $C_\Gamma(x) = C_\Gamma(x^m)$ for every $x \in A$ and $0 \neq m \in \mathbf{N}$.

Assume, by contradiction, that Γ is infinite. We construct a sequence of elements $y_i \in A$ and normal subgroups Γ_i, of finite index in Γ, such that, for every $i = 1, 2, \ldots$, $\Gamma_{i+1} \lneqq \Gamma_i$, $\langle y_1, \ldots, y_i\rangle \cong \cong \langle y_1\rangle \times \ldots \times \langle y_i\rangle$ is centralized by Γ_i, but $\langle y_1, \ldots, y_i, y_{i+1}\rangle$ is not normalized by Γ_i. We proceed by induction on i. Let $y_1 \in A$ such that $C_\Gamma(y_1) \neq \Gamma$. We put $\Gamma_1 = (C_\Gamma(y_1))_\Gamma$ and observe that, since $|\Gamma : N_\Gamma(y_1)|$ is finite, $|\Gamma : C_\Gamma(y_1)|$ is also finite and therefore, $|\Gamma : \Gamma_1|$ is finite.

Let now $i \geqq 2$, and assume that we have already constructed $y_1, \ldots, y_i \in A$ and Γ_{i-1} satisfying the desired properties. Let $B_{i-1} = = C_A(\Gamma_{i-1})$, then B_{i-1} is Γ-invariant, and, since Γ_{i-1} has finite index in Γ, $B_{i-1} \neq A$; moreover, by what observed above, A/B_{i-1} is torsion free. Let $g \in C_{\Gamma_{i-1}}(A/B_{i-1})$ and $y \in A$, then $[y, g, g] \in [B_{i-1}, g] = 1$; on the other hand, there exists, by hypothesis, a $k \in \mathbf{N}$ such that $g^k \in C_\Gamma(y)$. Since $[y, g]$ and g commute, we get $[y, g]^k = [y, g^k] = 1$, yielding $[y, g] = 1$, as A is torsion free. This holds for every $y \in A$, hence $g \in C_\Gamma(A) = 1$. This implies, in particular, that

$$\Gamma_{i-1} \neq \mathrm{Paut}_{\Gamma_{i-1}}(A/B_{i-1}), \quad \text{for otherwise } \Gamma_{i-1} = \Gamma_{i-1}/C_{\Gamma_{i-1}}(A/B_{i-1})$$

would have order at most two, and Γ would be finite. Thus, there exists $y_i \in A$, such that $\langle B_{i-1}, y_i\rangle$ is not normalized by Γ_{i-1} (observe that this implies that $\langle y_1, \ldots, y_i\rangle$ is not normalized by Γ_{i-1}). Now, $\langle y_i\rangle \cap B_{i-1} = 1$, since A/B_{i-1} is torsion free; in particular:

$$\langle y_1, \ldots, y_{i-1}\rangle \cap \langle y_i\rangle = 1$$

and so

$$\langle y_1, \ldots, y_i\rangle \cong \langle y_1, \ldots, y_{i-1}\rangle \times \langle y_i\rangle \cong \langle y_1\rangle \times \ldots \times \langle y_{i-1}\rangle \times \langle y_i\rangle\,.$$

We take now $\Gamma_i = (C_{\Gamma_{i-1}}(y_i))$; then $\Gamma_i \lneqq \Gamma_{i-1}$ and Γ_i is a normal subgroup of finite index in Γ, moreover $\langle y_1, \ldots, y_i\rangle \leqq C_A(\Gamma_i) = B_i$. Let now $H = \langle y_i;\ i \in \mathbf{N}\rangle$; then $|\Gamma : N_\Gamma(H)|$ is finite by hypothesis; thus there exists an index $n \in \mathbf{N}$ such that $\Gamma_n N_\Gamma(H) = \Gamma_{n+1} N_\Gamma(H)$. Now, if $B_{n+1} = C_A(\Gamma_{n+1})$, then B_{n+1} is Γ-invariant, and so $\Gamma_{n+1}N_\Gamma(H)$ normalizes $H \cap B_{n+1}$. But, by construction, $H \cap B_{n+1} = \langle y_1, \ldots, y_{n+1}\rangle$.

In particular, $\Gamma_n \leqq \Gamma_{n+1} N_\Gamma(H)$ normalizes $\langle y_1, \dots, y_{n+1}\rangle$, contradicting our choice of y_{n+1}. We therefore conclude that Γ is finite. ■

2.6. THEOREM. *Let A be an abelian group, $\Gamma \leqq \mathrm{Aut}(A)$. Then $|\Gamma : N_\Gamma(H)| < \infty$ for every $H \leqq A$ if and only if one of the following holds:*

1) *$|\Gamma : \mathrm{Paut}_\Gamma(A)|$ is finite; or*

2) *$\mathrm{Tor}(A)$ (the torsion subgroup of A) has infinite exponent, and there exists a Γ-invariant, free subgroup of A, of finite rank, such that A/F is periodic, and $|\Gamma : C_\Gamma(F)|$ and $|\Gamma : \mathrm{Paut}_\Gamma(A/F)|$ are both finite.*

PROOF. Assume that $|\Gamma : N_\Gamma(H)| < \infty$ for every $H \leqq A$. Let $T = \mathrm{Tor}(A)$. By Theorem 2.4, $|\Gamma : \mathrm{Paut}_\Gamma(T)|$ is finite. If $\exp(T)$ is finite, say $\exp(T) = r$, then $|\Gamma : C_\Gamma(T)|$ is finite. Moreover, T has, in this case, a complement U in A, which is torsion free. Since, by assumption, $|\Gamma : N_\Gamma(U)| < \infty$, it follows from Lemma 2.5 that $|\Gamma : C_\Gamma(U)|$ is finite. Hence:

$$|\Gamma : C_\Gamma(A)| = |\Gamma : (C_\Gamma(U) \cap C_\Gamma(T))| \leqq |\Gamma : C_\Gamma(U)|\,|\Gamma : C_\Gamma(T)| < \infty$$

and we are in case 1).

Let $\exp(A) = \infty$ and assume that $|\Gamma : \mathrm{Paut}_\Gamma(A)| = \infty$. Let $\mathcal{F}_A$ be the set of non trivial free subgroups of finite rank of A, then $\mathcal{F}_A \neq \emptyset$, otherwise A is periodic, and so, by Theorem 2.4, $|\Gamma : \mathrm{Paut}_\Gamma(A)| < \infty$. Assume that $F_1 \leqq F_2 \leqq \dots$ is an infinite chain of elements of $\mathcal{F}_A$, such that $C_\Gamma(F_{i+1}) \subsetneqq C_\Gamma(F_i)$ for every $i = 1, 2, \dots$; then, if $K = \bigcup_{i \in \mathbf{N}} F_i$, we have $|\Gamma : C_\Gamma(K)| = \infty$, which is not possible by Lemma 2.5, since $|\Gamma : N_\Gamma(K)| < \infty$ and K is torsion free. Hence there exists an $F_0 \in \mathcal{F}_A$, such that, for any $V \in \mathcal{F}_A$, and $F_0 \leqq V$, it is $C = C_\Gamma(F_0) = C_\Gamma(V)$. In particular, if yF_0 is an element of infinite order in A/F_0, then $\langle F_0, y\rangle \in \mathcal{F}_A$ and so $C = C_\Gamma(\langle F_0, y\rangle) \leqq C_\Gamma(y)$. Hence C centralizes every element of A whose order modulo F_0 is infinite. If A/F_0 is not periodic, then $A = \langle x \in A;\, |xF_0| = \infty\rangle$, whence C centralizes A, contradicting our assumption (recall that $|\Gamma : C|$ is finite). Thus A/F_0 is periodic; now, since F_0 has a finite number of conjugates under Γ, $F = (F_0)_\Gamma$ is a Γ-invariant free subgroup of finite rank of A, and A/F is periodic. Finally, $|\Gamma : C_\Gamma(F)|$ is finite by Lemma 2.5 and $|\Gamma : \mathrm{Paut}_\Gamma(A/F)|$ is finite by Theorem 2.4.

Conversely, let A be an abelian group, $\Gamma \leqq \mathrm{Aut}(A)$. If $H \leqq A$, clearly $N_\Gamma(H) \geqq \mathrm{Paut}_\Gamma(A)$; hence if 1) holds then $|\Gamma{:}N_\Gamma(H)| < \infty$ for every $H \leqq A$.

Assume now that condition 2) holds, with F a free subgroup of finite rank of A. Firstly, since $|\Gamma{:}C_\Gamma(F)|$ and $|\Gamma{:}\mathrm{Paut}_\Gamma(A/F)|$ are both finite, we may assume, with no loss in generality, that $C_\Gamma(F) = \mathrm{Paut}_\Gamma(A/F) = \Gamma$. Therefore, if $H \leqq A$, then HF and $H \cap F$ are Γ-invariant. In particular $H^\Gamma \leqq HF$.

Let $R/H \cap F$ be the torsion subgroup of $HF/H \cap F$, then $R/H \cap F$ is finite, since F has finite rank. Now, A/F is periodic, so $H/H \cap F \simeq HF/F$ is periodic, and $RH/H \cap F$ is the torsion subgroup of $HF/H \cap F$. Hence RH is Γ-invariant; in particular, $H^\Gamma \leqq RH$. Now, $|RH{:}H| = |R{:}R \cap H| = |R{:}H \cap F|$ is finite, whence $|H^\Gamma{:}H|$ is finite, say $|H^\Gamma{:}H| = r$. It follows that $(H^\Gamma)^r \leqq H$. Let now T be the torsion subgroup of A; then our hypotheses imply that $\Gamma = \mathrm{Paut}_\Gamma(T)$, whence $T \cap H$ is Γ-invariant and so $H_\Gamma \geqq T \cap H$. Moreover, since F has finite rank and A/F is periodic, A/T has finite rank. Now, H/H_Γ is a homomorphic image of $H/H \cap T \simeq HT/T$ of finite exponent; we therefore conclude that H/H_Γ is finite. Hence H^Γ/H_Γ is finite and so $N_\Gamma(H) \geqq C_\Gamma(H^\Gamma/H_\Gamma)$ has finite index in Γ. ■

EXAMPLE 1. We show that, in the hypotheses of Theorem 2.6, case 2) can actually occur, yet $|\Gamma{:}\mathrm{Paut}_\Gamma(A)| = \infty$. Let $A = C \times K$, where $C \simeq C_{p^\infty}$ for a prime p, and $K = \langle x \rangle$ is a cyclic group of infinite order. Let $\Gamma = \langle \alpha \rangle \leqq \mathrm{Aut}(A)$, where α centralizes x and induces on C a power automorphism of infinite order (for instance $z^\alpha = z^{p+1}$ for every $z \in C$). It is easy to check that $|\Gamma{:}N_\Gamma(H)| < \infty$ for every $H \leqq A$ (indeed $|H^\Gamma{:}H_\Gamma| < \infty$ for every $H \leqq A$), but no power of α fixes every subgroup of A.

We now turn to the case in which Γ is a group of automorphisms of an abelian group A, such that $|H^\Gamma{:}H|$ is finite, for every $H \leqq A$. Again, we split the discussion of the general case into a number of steps, each of those dealing with a particular case.

2.7. LEMMA. *Let N be a torsion free group, $\Gamma \leqq \mathrm{Aut}(N)$ such that $|H^\Gamma{:}H| < \infty$ for every $H \leqq N$, then $\Gamma = \mathrm{Paut}_\Gamma(N)$. In particular $\Gamma = 1$ if N is not abelian; and $\Gamma = 1$ or $\Gamma = \langle \alpha \rangle$, where α is the inversion map, if N is abelian.*

PROOF. Let $1 \neq y \in N$, and set $K = \langle y \rangle^\Gamma$. Then, by hypothesis, $|K{:}\langle y \rangle|$ is finite. Hence $|K{:}\langle y \rangle_K|$ is finite. Now $1 \neq \langle y \rangle_K$ is infinite cyclic, put $\langle y \rangle_K = \langle z \rangle$. Assume that there exists $t \in K$, such that $z^t \neq z$; then $z^t = z^{-1}$. But, for some $n \in \mathbf{N}$, $t^n \in \langle z \rangle$ and so $t^n = (t^n)^t = t^{-n}$, yielding $t^{2n} = 1$; since N is torsion free, it follows $t = 1$, contradiction. Hence $\langle z \rangle \leqq Z(K)$ and so $|K:Z(K)|$ is finite. By a Theorem of I. Schur (see [13, 10.1.4]), K' is finite; that is $K' = 1$. Thus, K is a finitely generated torsion free abelian group; since $|K:\langle y \rangle| < \infty$, we have that K is cyclic, whence $K = \langle y \rangle$, proving that $\Gamma \leqq \mathrm{Paut}\,(N)$.

The final claims follow, for instance, from Cooper [3, Corollary 4.2.3]. ■

2.8. Lemma. *Let A be a divisible abelian group, $\Gamma \leqq \mathrm{Aut}\,(A)$ such that $|H^\Gamma : H| < \infty$ for every $H \leqq A$. Then $\Gamma \leqq \mathrm{Paut}\,(A)$.*

Proof. We observe first that if H is a divisible subgroup of A, then $H^\Gamma = H$. In fact, if $g \in \Gamma$, then

$$|H : H \cap H^g| = |H^g H : H| \leqq |H^\Gamma : H| < \infty ;$$

but H is divisble and so it does not admit any subgroup of finite index; hence $H \cap H^g = H$, yielding $H = H^g$. This is true for any $g \in \Gamma$, whence $H = H^\Gamma$. Let now $x \in A$. If $|x| = \infty$ choose a subgroup K of A, $K \simeq \mathbf{Q}$ and $x \in K$. By what observed above, K is Γ-invariant; moreover, it is torsion free, hence, by Lemma 2.7, $\Gamma = \mathrm{Paut}_\Gamma\,(K)$; in particular, $\langle x \rangle^\Gamma = \langle x \rangle$. Otherwise x has finite order. Since the torsion subgroup of A is the direct product of its primary components, we may assume that $|x| = p^n$, for some $p \in \mathbf{P}$ and $n \in \mathbf{N}$. Then choose a subgroup $L \simeq C_{p^\infty}$ of A, with $x \in L$. Again $L^\Gamma = L$ and so $\langle x \rangle^\Gamma = \langle x \rangle$. ■

2.9. Lemma. *Let A be a reduced abelian p-group, $\Gamma \leqq \mathrm{Aut}\,(A)$ such that $|H^\Gamma : H| < \infty$ for every $H \leqq A$. Then there exists a finite Γ-invariant subgroup N of A, such that $\Gamma = \mathrm{Paut}_\Gamma\,(A)$.*

Proof. Assume, for the moment, that A is residually finite. Suppose further, by contradiction, that for any finite Γ-invariant subgroup N of A, there exists $x \in A$, such that $\langle N, x \rangle$ is not Γ-invariant. By induction on $i \in \mathbf{N}$, we construct a sequence of elements x_i of A such that, if $K_i = \langle x_1, \ldots, x_i \rangle$, then $\langle x_i \rangle \cap K_{i-1}^\Gamma = 1$ and $\langle K_{i-1}, x_i \rangle$ is not Γ-invariant, for every $i \in \mathbf{N}$ (and $K_0 = 1$). Let $x_1 \in A$ be such that $\langle x_1 \rangle$ is not Γ-invariant. Assume now that we have already found $x_1, \ldots, x_{n-1}$ satisfying the desired properties. Then $M = K_{n-1}^\Gamma =$

$= \langle x_1, \dots, x_{n-1}\rangle^{\Gamma}$ is finite and so, since A is residually finite, there exists $B \leqq A$ of finite index in A, such that $M \cap B = 1$. Suppose that, for each $y \in B$, $\langle y, M\rangle$ is Γ-invariant; then BM is Γ-invariant and $\Gamma = \mathrm{Paut}_{\Gamma}(BM/M)$; now, BM has finite index in A, hence, if Y is a transversal of BM in A, then Y^{Γ} is finite and $R = MY^{\Gamma}$ is finite, $BR = A$ and $\Gamma = \mathrm{Paut}_{\Gamma}(A/R)$, contradicting our initial assumption. Therefore, there exists $x_n \in B$, such that $\langle M, x_n\rangle$ is not Γ-invariant. We have also:

$$\langle x_n\rangle \cap K_{n-1}^{\Gamma} = \langle x_n\rangle \cap M \leqq B \cap M = 1 .$$

In this way we construct the infinite sequence $x_1, x_2, \dots$.

Let now $K = \langle x_i; i \in \mathrm{N}\rangle \cong \mathrm{Dir}\,\langle x_i\rangle$, then $|K^{\Gamma} : K|$ is finite. On the other hand $K^{\Gamma} = \langle K_i^{\Gamma}; i \in \mathrm{N}\rangle = \bigcup K_i^{\Gamma}$. Hence, there exists $n \in \mathrm{N}$, such that $K^{\Gamma} = KK_n^{\Gamma}$. We consider x_{n+1}; by construction, there exists $g \in \Gamma$ such that $x_{n+1}^g \notin \langle K_n^{\Gamma}, x_{n+1}\rangle$. Now:

$$x_{n+1}^g \in K^{\Gamma} = KK_n^{\Gamma} \cong K_n^{\Gamma} \times \Big(\mathop{\mathrm{Dir}}_{0 \neq i \in \mathrm{N}} \langle x_{n+i}\rangle \Big) ;$$

let $x_{n+1}^g = kb_1 \dots b_r$, with $k \in K_n^{\Gamma}$, $b_i \in \langle x_{n+i}\rangle$ $(i = 1, \dots, r)$ and $b_r \neq 1$. If $r > 1$, then:

$$b_r = x_{n+1}^g k^{-1} b_1^{-1} \dots b_{r-1}^{-1} \in K_{n+r-1}^{\Gamma} \cap \langle x_{n+r}\rangle = 1 ,$$

hence $r = 1$, that is:

$$x_{n+1}^g = kb_1 \in \langle K_n, x_{n+1}\rangle ,$$

against the choice of $g \in \Gamma$. This contradiction shows that there exists a finite Γ-invariant subgroup N of A such that $\Gamma = \mathrm{Paut}_{\Gamma}(A/N)$.

We now go back to the general case; thus let A be reduced (and not necessarily residually finite), and let B be a basic subgroup of A. Now, B^{Γ} is a finite extension of B, and B is a direct product of cyclic groups; hence B^{Γ} is residually finite. By the case discussed before, there exists a finite Γ-invariant subgroup N of B^{Γ}, such that $\Gamma = \mathrm{Paut}_{\Gamma}(B^{\Gamma}/N)$. Now, since N is finite and A is reduced, A/N is also reduced. Moreover A/B^{Γ} is divisible. Then, by applying Lemma 2.2, we conclude that $\Gamma = \mathrm{Paut}_{\Gamma}(A/N)$. ■

We illustrate the previous Lemma by an example, in which we show that it may happen that no subgroup of finite index of Γ acts as a group of power automorphisms on A.

Example 2. Let A be an infinite elementary abelian 3-group, with generators $x_1, x_2, \ldots$. For $j = 2, 3, \ldots$ let α_j be the automorphism of A defined by:

$$\alpha_j(x_i) = \begin{cases} x_1 & \text{if } i = 1 , \\ x_1 x_i & \text{if } 1 < i \leqq j , \\ x_1^{-1} x_i & \text{if } i > j . \end{cases}$$

Then the subgroup $\Gamma = \langle \alpha_j ; j = 2, 3, \ldots \rangle$ of $\mathrm{Aut}(A)$ is also an infinite elementary abelian 3-group, and it is clear that no subgroup of finite index of Γ acts as a group of power automorphisms (in this case it would centralize every $a \in A$) on A. Moreover $[A, \Gamma] = \langle x_1 \rangle = C_A(\Gamma)$, whence $\langle H, x_1 \rangle$ is Γ-invariant, for every $H \leqq A$, so $|H^\Gamma : H| \leqq 3$ for every $H \leqq A$ (by contrast, if $x_1 \notin H \leqq A$, then $H_\Gamma = 1$). We observe that the natural semidirect product $W = A \rtimes \Gamma$ is a nilpotent 3-group in $\boldsymbol{T}_3$, such that $W' = Z(W)$ has order 3.

2.10. Lemma. *Let A be an abelian p-group, D the divisible radical of A, and $\Gamma \leqq \mathrm{Aut}(A)$ such that $|H^\Gamma : H| < \infty$ for every $H \leqq A$. Then:*

1) *there exists a finite Γ-invariant $N \leqq A$, such that*

$$\Gamma = \mathrm{Paut}_\Gamma (A/N) \; ; \qquad \textit{or}$$

2) *D has finite rank, $\exp(A/D)$ is finite, $\Gamma = \mathrm{Paut}_\Gamma (D)$ and there exists a finite Γ-invariant $N \leqq A$, such that*

$$\Gamma = \mathrm{Paut}_\Gamma (\Omega_n(A)/N) \, ,$$

where $p^n = \exp(A/D)$.

Proof. Let $A = D \times R$, where D is the divisible radical of A, and R is a reduced complement of D in A. Then, by Lemma 2.8, $\Gamma = \mathrm{Paut}_\Gamma (D)$. Put $L = R^\Gamma$, then L is a finite extension of a reduced group, and so it is reduced.

If L is finite, put $L = N$ and conclude with 1); in fact A/L is divisible, so by Lemma 2.8, Γ acts as a group of power automorphisms on it.

Assume now L to be infinite; then, by Lemma 2.9, there exists a Γ-invariant finite subgroup N of L, such that $\Gamma = \mathrm{Paut}_\Gamma(L/N)$. Observe that, since $|L \cap D| = |L:R|$ is finite, we may choose N in such a way that $N \geqq L \cap D$. If $\exp(L) = \infty$, there exists a basic subgroup H/N of L/N with $H \neq L$ (see Fuchs [5, 35.4]). Then H is Γ-invariant and L/H is a non trivial divisible group; moreover A/H is divisible and so $\Gamma = \mathrm{Paut}_\Gamma(A/H)$. Arguing as in the proof of Theorem 2.4 (case 3)), we have $\Gamma = \mathrm{Paut}_\Gamma(A/N)$.

Hence suppose that $\exp(L) = p^n$ is finite. Put $C = \Omega_n(A)$; then $L \leqq C$ and C is reduced. By 2.9, there exists a finite Γ-invariant subgroup M of C, such that $\Gamma = \mathrm{Paut}_\Gamma(C/M)$. Assume that $M \ngeqq \Omega_1(D)$, and let $a \in \Omega_1(D) \setminus M$ and $b \in D$ be such that $b^{p^{n-1}} = a$; then $\langle b \rangle \cap \cap M = 1$, hence Mb is an element of order p^n in MD/M. Now, Γ acts as a group of power automorphisms both on C/M and $MD/M \simeq \simeq D/D \cap M$, and Mb is an element of order p^n in C/M as well as in MD/M. Thus each $g \in \Gamma$ induces the same power in C/M and in $\Omega_n(MD/M)$. It follows that Γ acts as a group of power automorphisms on $A/M = (MD/M)(C/M)$, and this is case 1). Otherwise, $M \geqq \Omega_1(D)$; thus, M being finite, it follows that D has finite rank, and this is case 2). ∎

EXAMPLE 3. Let $A = D \times R$ be an abelian group, where $D \simeq C_{p^\infty}$, p odd, and R is an infinite elementary abelian p-group. Let $\Gamma = \langle \alpha \rangle \leqq \leqq \mathrm{Aut}(A)$ where α is the automorphism of A, which maps every element of D into its inverse, and fixes every element of R. Then $|H^\Gamma : H| \leqq p$ for every $H \leqq A$, but there exists no subgroup N of A, such that α acts as a power automorphism on A/N.

In the example above, it is easy to check directly that $|H^\Gamma : H|$ is finite for every $H \leqq A$, but this more generally follows from the fact that Lemma 2.10 can be inverted. This is itself a particular aspect of our next result.

2.11. THEOREM. *Let A be a periodic abelian group, $\Gamma \leqq \mathrm{Aut}(A)$. Then $|H^\Gamma : H| < \infty$ for every $H \leqq A$ if and only if there exist Γ-invariant subgroups N, D of A, such that $N \leqq D$, N is finite, D/N is divisible of finite total rank, $\Gamma = \mathrm{Paut}_\Gamma(A/D) = \mathrm{Paut}_\Gamma(D/N)$ and, if $p \in \pi(D/N)$, then the p-component of A/D has finite exponent.*

PROOF. Assume first that $|H^{\Gamma}:H|$ is finite for every $H \leqq A$. By 2.1 (c), there exists a finite set of primes π, such that $\Gamma = \mathrm{Paut}_{\Gamma}(A_{\pi'})$. Let π_1 be the set of those primes $p \in \pi$ such that Γ does not act as a group of power automorphisms on any quotient of A_p over a finite Γ-invariant subgroup. For each $p \in \pi$, let N_p be a finite Γ-invariant subgroup of A_p, such that 1) or 2) of Lemma 2.10 holds (where $N = N_p$) according to $p \notin \pi_1$ or, respectively, $p \in \pi_1$. Let

$$N = \langle N_p;\, p \in \pi \rangle \qquad \text{and} \qquad D = \langle D_p N;\, p \in \pi_1 \rangle$$

(where D_p is the obvious subgroup of A_p, according to Lemma 2.10). Then $N \leqq D$ and, since π is finite, N is finite, and Lemma 2.10 implies that D/N has finite total rank. Moreover, $\Gamma = \mathrm{Paut}_{\Gamma}(A/D) = \mathrm{Paut}_{\Gamma}(D/N)$ and, if $p \in \pi(D/N) = \pi_1$, then the p-component of A/D is $A_pD/D \simeq A_p/A_p \cap D = A_p/D_p$ and has finite exponent.

Conversely, let N, D be Γ-invariant subgroups of A, satisfying the conditions of the statement, and let H be a subgroup of A. We want to show that $|H^{\Gamma}:H|$ is finite. Now, since N is finite, $|HN:H| = |N:H \cap N|$ is finite and $H^{\Gamma} \leqq (HN)^{\Gamma}$ Thus if we prove that every subgroup of A containing N has finite index in its Γ-closure, then the same is true for every subgroup of A. Hence we may assume $N = 1$.

Observe now that $H \cap D$ and HD are Γ-invariant, since Γ acts as a group of power automorphisms on both D/N and A/D; in particular, $H^{\Gamma} \leqq HD$. Let $\pi_1 = \pi(D)$ and $R/H \cap D = (H/H \cap D)_{\pi_1'} = (HD/H \cap D)_{\pi_1'}$. Then R is Γ-invariant and $H/R \simeq HD/RD$ is a π_1-group; but the exponent of the π_1-component of A/D is finite (this follows from our hypotheses, since D/N has finite total rank so, in particular, π_1 is finite). Thus, the exponent of H/R is finite. This implies that $\exp(H^{\Gamma}/R)$ is finite and, in particular, $\exp(H^{\Gamma}/H)$ is finite. But $H^{\Gamma}/H \leqq HD/H \simeq D/D \cap H$ is a divisible group of finite total rank; hence $|H^{\Gamma}/H|$ is finite. This completes the proof. ■

The next result, together with Theorem 2.11, completely describes the action of Γ on the abelian group A.

2.12. THEOREM. *Let A be an abelian group; $\Gamma \leqq \mathrm{Aut}(A)$. Then $|H^{\Gamma}:H| < \infty$ for every $H \leqq A$ if and only if one of the following conditions holds.*

1) *There exists a finite Γ-invariant subgroup N of A such that $\Gamma = \mathrm{Paut}_{\Gamma}(A/N)$; or*

2) *there exist Γ-invariant subgroups $N \leqq R$ of A, such that N is finite, R/N is free of finite rank,* $\Gamma = \mathrm{Paut}_\Gamma(R/N)$, *$A/R$ is periodic and $|H^\Gamma : H|$ is finite for every subgroup H with $R \leqq H \leqq A$.*

PROOF. Suppose first that $|H^\Gamma : H| < \infty$ for every $H \leqq A$, and assume that A is not periodic. Let T be the torsion subgroup of A and let B be a torsion free subgroup of A; then $B \cap T = 1$. Now, A/T is torsion free, hence, by 2.7, $\Gamma = \mathrm{Paut}_\Gamma(A/T)$; in particular, BT is Γ-invariant and so $B^\Gamma \leqq BT$, which gives:

$$B^\Gamma = B^\Gamma \cap BT = B(B^\Gamma \cap T) .$$

Let now $\mathcal{F}_A$ be the set of finitely generated free subgroups of A. Suppose that there exists an infinite ascending chain $H_1 \leqq H_2 \leqq \ldots$ of elements of $\mathcal{F}_A$, such that $H_{i+1}^\Gamma \cap T \supsetneqq H_i^\Gamma \cap T$ for every $i \in \mathbb{N}$. Put $H = \bigcup_{i\in\mathbb{N}} H_i$; then $H^\Gamma = \bigcup_{i\in\mathbb{N}} H_i^\Gamma$ and $H^\Gamma \cap T \geqq \bigcup_{i\in\mathbb{N}} (H_i^\Gamma \cap T)$ is infinite. On the other hand, H is torsion free, hence, by what observed above $H^\Gamma = (H^\Gamma \cap T)H$ and so $|H^\Gamma \cap T| = |H^\Gamma/H|$ is finite by our hypothesis, a contradiction. Therefore no such a chain exists in $\mathcal{F}_A$. This implies that there exists an $R_0 \in \mathcal{F}_A$ such that, for any $S \in \mathcal{F}_A$, if $R_0 \leqq S$, then $R_0^\Gamma \cap T = S^\Gamma \cap T$, that is

$$S^\Gamma = (S^\Gamma \cap T)S = (R_0^\Gamma \cap T)S .$$

Put $R = R_0^\Gamma$ and $N = R \cap T$. Then $N \simeq R/R_0$ is finite and Γ-invariant; $R/N \simeq R_0$ is torsion free of finite rank, and so, by 2.7, $\Gamma = \mathrm{Paut}_\Gamma(R/N)$. If A/R is periodic we are in case 2).

Assume that A/R is not periodic, and let $y \in A$ be such that the order of y modulo R is infinite. Then $\langle y \rangle \cap R_0 = 1$ and $\langle R_0, y\rangle \in \mathcal{F}_A$. By choice of R_0, we have

$$\langle R_0, y\rangle^\Gamma = (R_0^\Gamma \cap T)\langle R_0, y\rangle = (R \cap T)\langle R_0, y\rangle = N(R_0, y) = \langle R, y\rangle ;$$

hence $\langle R, y\rangle$ is Γ-invariant. Moreover, $\langle R, y\rangle/N$ is torsion free and so Γ acts as a group of power automorphisms on it; in particular, $\langle N, y\rangle$ is fixed by Γ. Since $\Gamma = \mathrm{Paut}_\Gamma(R/N)$, it easily follows that either Γ centralizes or it inverts each cyclic subgroup of A/N, whose order modulo R/N is infinite. Now, A/R is not periodic, so A/N is generated by the set of those elements whose order modulo R/N is infinite; thus, in any case, $\Gamma = \mathrm{Paut}_\Gamma(A/N)$.

We now prove the converse, namely that if the couple $A, \Gamma \leqq \leqq \mathrm{Aut}(A)$ satisfies 1) or 2), then $|H^\Gamma{:}H| < \infty$ for every $H \leqq A$. Case 1) is easy, so assume that 2) is verified. As in the proof of Theorem 2.11, we may assume $N = 1$.

Let $H \leqq A$; then $H \cap R$ is Γ-invariant, since $\Gamma = \mathrm{Paut}_\Gamma(R)$. Moreover $|H^\Gamma R{:}HR| = |(HR)^\Gamma{:}HR|$ is finite by the hypotheses in 2). Now, $H/H \cap R \simeq HR/R$ is periodic, hence $H^\Gamma/H \cap R \leqq \mathrm{Tor}(A/H \cap R)$. In particular, $H^\Gamma \cap R/H \cap R$ is a periodic factor of R. Since R is free of finite rank, it follows that $|H^\Gamma \cap R/H \cap R| = |(H^\Gamma \cap RH)/H|$ is finite. On the other hand, $|H^\Gamma{:}H^\Gamma \cap RH| = |H^\Gamma R{:}HR|$ is finite. We have therefore that $|H^\Gamma{:}H| = |H^\Gamma{:}H^\Gamma \cap RH|\,|H^\Gamma \cap RH{:}H|$ is finite. This completes the proof of the Theorem. ■

We now turn our attention to the case in which $|\Gamma{:}N_\Gamma(H)|$ is boundedly finite for $H \leqq A$. We begin by recalling a related known result.

2.13. Proposition (Baer [1]; Robinson and Wiegold [14]). *Let G be an FC-group and let $n = \sup_{g \in G} |G{:}C_G(g)|$ (n may be infinite). If Γ is a group of automorphisms of G, such that $|\Gamma{:}C_\Gamma(x)| \leqq m$ ($m \in \mathbb{N}$) for every $x \in G$, then there exists a finite Γ-invariant normal subgroup N of G such that $|\Gamma{:}C_\Gamma(G/N)|$ is finite. If $n < \infty$ then $|N| \leqq a_1(m, n)$ and $|\Gamma{:}C_\Gamma(G/N)| \leqq a_2(m, n)$, where a_1, a_2 are functions of $\mathbb{N}\times\mathbb{N}$ in $\mathbb{N}$.*

2.14. Lemma. *Let A be an abelian group of finite exponent, $\Gamma \leqq \mathrm{Aut}(A)$ such that $|\Gamma{:}N_\Gamma(x)| \leqq m$ for every $x \in A$. Then there exists a Γ-invariant subgroup M of A, which can be generated by at most $a_3(m)$ elements, such that $|\Gamma{:}\mathrm{Paut}_\Gamma(A/M)| \leqq a_4(m)$, where a_3, a_4 are functions of $\mathbb{N}$ in $\mathbb{N}$.*

Proof. Let A, Γ be as in the hypotheses. Then A is in particular periodic, and, by 2.1 (b), a suitable subgroup of index at most m in Γ acts as a group of power automorphisms on all but at most $m - 1$ primary components of A. Thus we may assume that A is a p-group, for some prime p.

We proceed by induction on $m = \max_{x \in A} |\Gamma{:}N_\Gamma(x)|$. Let $x \in A$ be of maximal order such that $|\Gamma{:}N_\Gamma(x)| = m$.

Let $p \neq 2$. Set $K = \langle x \rangle^\Gamma$ and $\Gamma_1 = N_\Gamma(x)$; then $|\Gamma{:}\Gamma_1| = m$ and K is generated by at most m elements. We show that $|\Gamma_1{:}N_{\Gamma_1}(Ky)| \leqq \leqq m - 1$ for every $y \in A$. Suppose, by contradiction, that this is not true, and take $y \in A$ such that $|\Gamma_1:N_{\Gamma_1}(Ky)| \geqq m$. Now, since K is Γ-invariant:

$$N_{\Gamma_1}(Ky) = \Gamma_1 \cap N_\Gamma(Ky) \geqq \Gamma_1 \cap N_\Gamma(y) = N_{\Gamma_1}(y)$$

and so

$$m \geqq |\Gamma:N_\Gamma(y)| \geqq |\Gamma_1:N_{\Gamma_1}(Ky)| \geqq |\Gamma_1:N_{\Gamma_1}(Ky)| \geqq m\,,$$

yielding:

$$N_\Gamma(y) = N_\Gamma(Ky) \quad \text{and} \quad |\Gamma:N_\Gamma(y)| = m\,.$$

In particular, by our choice of x, $|y| \leqq |x|$: Therefore, we can find $z \in \langle x, y\rangle$ such that $\langle x, y\rangle = \langle x, z\rangle$ and $\langle x\rangle \cap \langle z\rangle = 1$; it also follows $Ky = Kz$ and so $N_\Gamma(y) = N_\Gamma(Kz) = N_\Gamma(z)$, which in turn implies $N_\Gamma(az) = N_\Gamma(z)$ for every $a \in K$. Now, let $g \in N_\Gamma(z)$; then there exist positive integers r, s, t such that:

$$(1) \qquad x^r z^r = (xz)^g = x^g z^t \quad \text{and} \quad x^{-s} z^s = (x^{-1}z)^g = (x^{-1})^g z^t\,.$$

Multiplying these equations, we get: $x^{r-s}z^{r+s} = z^{2t}$; thus $x^{r-s} = z^{2t-r-s} \in \langle x\rangle \cap \langle z\rangle = 1$ and so $r \equiv s \pmod{|x|}$ and $2t \equiv r+s \pmod{|z|}$. Since $|x| \geqq |z|$, we have $2t \equiv 2r \pmod{|z|}$ and so, p being an odd prime, $t \equiv r \pmod{|z|}$. Then (1) becomes $x^r z^r = x^g z^r$, yielding $x^g = x^r$, whence $g \in N_\Gamma(x) = \Gamma_1$. We have therefore $N_\Gamma(z) \leqq \Gamma_1$ which leads to the contradiction $N_\Gamma(z) = \Gamma_1$. Thus $|\Gamma_1:N_{\Gamma_1}(Ky)| \leqq m-1$ for every $y \in A$. By inductive hypothesis, there exists an $a_3(m-1)$-generated Γ_1-invariant subgroup M_0/K of A/K such that $\mathrm{Paut}_{\Gamma_1}(A/M_0)$ has index at most $a_4(m-1)$ in Γ_1. We finish by taking $M = (M_0)^\Gamma$.

Let now $p = 2$ and assume that, for every $x \in A$ such that $|\Gamma:N_\Gamma(x)| = m$, there exists $y \in A$ such that $|\Gamma_1:N_{\Gamma_1}(Ky)| = m$, where $\Gamma_1 = N_\Gamma(x)$ and $K = \langle x\rangle^\Gamma$. Then choose $x \in A$ of maximal order such that $|\Gamma:N_\Gamma(x)| = m$, and $y \in A$ with $|\Gamma_1:N_{\Gamma_1}(Ky)| = m$ and $\langle K, y\rangle/K$ of maximal order. Then we may replace y by y_0 such that $\langle K, y_0\rangle = \langle K, y\rangle$ and $\langle y_0\rangle \cap \langle x\rangle = 1$. Now, if $\Gamma_2 = N_\Gamma(x) \cap N_\Gamma(y_0)$ and $T = \langle x, y_0\rangle^\Gamma = \langle x, y\rangle^\Gamma$, it is not hard to see, by an argument similar to that used in the case $p \neq 2$, that $|\Gamma_2:N_{\Gamma_1}(Tz)| \leqq m-1$ for every $z \in A$. Then apply induction.

2.15. THEOREM. *There exists a function* $\alpha\colon \mathbf{N} \to \mathbf{N}$ *such that, if* G *is a group all of whose subgroups have at most* m *conjugates, and* Γ *is a group of automorphisms of* G *satisfying* $|\Gamma : N_\Gamma(H)| \leqq m$ *for every* $H \leqq G$, *then* $|\Gamma : \mathrm{Paut}_\Gamma(A)| \leqq \alpha(m)$.

PROOF. Assume first that G is abelian and periodic. By 2.1 (*b*), it is enough to prove our claim in the case in which G is a p-group, for some prime p. For any $r \in \mathbf{N}$, let $G_r = \langle a \in G;\ a^{p^r} = 1\rangle$. By Lemma 2.14, for every $r \in \mathbf{N}$, there exists a Γ-invariant subgroup B_r of G_r, such that $|\Gamma : \mathrm{Paut}_\Gamma(G_r/B_r)| \leqq a_4(m)$ and $rk(B_r) \leqq a_3(m)$ (by $rk(A)$ we mean here the minimal number of generators of a finite group A). Fixing $r \in \mathbf{N}$, there exists $C \leqq G_r$, such that $C \cap B_r = 1$ and $rk(G_r/C) \leqq a_3(m)$. Since $|\Gamma : N_\Gamma(C)| \leqq m$, we have $\mathrm{Paut}_\Gamma(G_r/B_r) \cap \cap N_\Gamma(C) \leqq \mathrm{Paut}_\Gamma(C)$ and $|\Gamma : \mathrm{Paut}_\Gamma(C)| \leqq m a_4(m)$. Let $\{Cg_1, \ldots, Cg_k\}$ be a generating set of G_r/C with $k \leqq a_3(m)$; let also g_0 be an element of maximal order in C, and set $Y = \langle g_0, g_1, \ldots, g_k\rangle$. Now, if

$$T = \bigcap_{i=0}^{k} N_\Gamma(g_i)\,,$$

then $|\Gamma : T| \leqq m^{k+1}$ and $T/C_T(Y)$ has rank at most $2(k+1)$; since $\Gamma^{m!} \leqq \mathrm{Paut}(G)$, it follows:

$$|\Gamma : \mathrm{Paut}_\Gamma(Y)| \leqq m^{k+1}(m!)^{2(k+1)} = a_5(m)\,.$$

Now, $\mathrm{Paut}_\Gamma(Y) \cap \mathrm{Paut}_\Gamma(C)$ has index at most $m a_4(m) a_5(m) = a_6(m)$ in Γ, and contains $\mathrm{Paut}_\Gamma(G_r)$, as it is easily checked. Put $\Gamma_r = \mathrm{Paut}_\Gamma(G_r)$, then for every $r \in \mathbf{N}$, $|\Gamma : \Gamma_r| \leqq a_6(m)$. On the other hand, it is clear that $\Gamma_i \leqq \Gamma_j$ if $i \geqq j$. Thus, if $n \in \mathbf{N}$ is such that $|\Gamma : \Gamma_n|$ is maximal, then $\Gamma_{n+1} = \Gamma_n$ for every $i \in \mathbf{N}$. Since $G = \bigcap_{r \in \mathbf{N}} G_r$, we have $\Gamma_n = \mathrm{Paut}_\Gamma(G)$, and this completes the proof when G is a periodic abelian group.

We now turn to the general case. By 1.3 (i), $|G : Z(G)| \leqq \mu(m)$, put $Z = Z(G)$.

Assume, firstly, that G is periodic, and let π be the set of prime divisors of $|G : Z|$ (observe that π depends on m). Then we can choose a set of generators $\{Zg_1, \ldots, Zg_r\}$ of G/Z such that, if c is the least common multiple of $|g_1|, \ldots, |g_r|$, then c is a π-number. For any prime p dividing c, let h_p be a p-element of maximal order in Z, if the p-component Z_p of Z has finite exponent, or an element of Z whose order is greater than the p-part of c if $\exp(Z_p) = \infty$. Let Y be the subgroup of G generated by the g_i's and the h_p's; then Y can be generated by a number of elements of prime power order that does not exceed a bound depending ultimately on m. Arguing as in the abelian case, we find that $\mathrm{Paut}_\Gamma(Y)$ has index bounded by a function of m in Γ. Now, $|\Gamma{:}\mathrm{Paut}_\Gamma(Z)| \leqq a_7(m)$ by the abelian case, whence $\Gamma_1 = \mathrm{Paut}_\Gamma(Z) \cap \mathrm{Paut}_\Gamma(Y)$ has bounded index in Γ. Since $ZY = G$, and by our choice of the g_i's and the h_p's, it is now easy to check that Γ_1 acts as a group of power automorphisms on G.

Suppose now that G is not a periodic group. If y is an element of infinite order in G, then the automorphism group of $\langle y\rangle$ has order two, thus $|\Gamma{:}C_\Gamma(y)| \leqq 2m$. If a is an element of finite order in G, consider an element z in Z infinite order (such a z exists because G/Z is finite); then az has infinite order and so:

$$|\Gamma{:}C_\Gamma(a)| \leqq |\Gamma{:}C_\Gamma(x)|\,|\Gamma{:}C_\Gamma(ax)| \leqq 4m^2 .$$

The same argument, applied to the action by conjugation of G on itself, yields $|G{:}C_G(x)| \leqq 4m^2$ for every $x \in G$. We are therefore in a position to apply Proposition 2.13, obtaining a Γ-invariant subgroup M of G, such that $|M| \leqq a_1(4m^2, 4m^2)$ and $|\Gamma{:}C_\Gamma(G/M)| \leqq a_2(4m^2, 4m^2)$. Let now H be a maximal torsion free subgroup of Z and put

$$K = N_\Gamma(H) \cap C_\Gamma(G/M) ;$$

then $|G{:}K| \leqq a_8(m)$ and, since $M \cap K = 1$, K centralizes H. Furthermore, G/H is periodic, so by the preceding case, $|K{:}\mathrm{Paut}_K(G/H)| \leqq$ $\leqq a_9(m)$. But $|\Gamma{:}C_\Gamma(x)| \leqq 4m^2$ for every $x \in G$, so we actually have that $|K{:}C_K(G/H)| \leqq a_{10}(m)$. We conclude the proof by observing that $C_K(G/H) \leqq C_\Gamma(G/H) \cap C_\Gamma(G/M) \leqq C_\Gamma(G)$. ■

3. Soluble groups in the classes T^* and V.

We begin by considering locally nilpotent groups. We recall that a Baer group is a group in which all finitely generated subgroups are subnormal.

3.1. LEMMA. *Every Baer group in $T^* \cup V$ is nilpotent.*

PROOF. Using a technique of K. W. Scott (see [15, Theorem 15.1.15]), we first show that a Baer group in V is a T^*-group. Let G be a a Baer V-group, and let $x \in G$; then $\langle x \rangle$ is subnormal in G; hence $|G:N_G(x)|$ is finite, and so $|G:C_G(x)|$ is finite. Therefore, G is an FC-group. Let H be a subnormal subgroup of G, and let $H = H_1, H_2, \dots, H_n$ be the conjugates of H in G, with $H_i = H^{x_i}$, $x_i \in G$ for $i = 1, 2, \dots, n$. Now, if $h \in C_H(x_i)$, then $h = h^{x_i} \in H \cap H_i$; hence $C_H(x_i) \leqq H \cap H_i$ for any $i = 1, 2, \dots, n$ and, consequently, $|H:H \cap H_i| \leqq |H:C_H(x_i)| < \infty$. Thus, if $R = H_G$, then H/R is finite. As G/R is an FC-group, this implies that H^G/R is finite; in particular, $|H^G:H|$ is finite. This holds for each subnormal subgroup of G and so G is a T^*-group.

We now prove that a Baer T^*-group is nilpotent. Let G be such a group. Then G is an FC-group and so (see Robinson [13, 14.5.6]) $G/Z(G)$ is a residually finite torsion group; without loss of generality, we may therefore assume that G is a residually finite torsion group. Assume, by contradiction, that G is not nilpotent; then, since G is a Baer group, it does not admit nilpotent subgroups of finite index. By induction on i, we construct a sequence of non-Dedekind, finite normal subgroups R_i of G, such that $\langle R_i;\ i \in \mathbf{N} \rangle \simeq \operatorname{Dir}_{i \in \mathbf{N}} R_i$. Since G is not nilpotent, there exists a finitely generated (hence finite, because G is periodic) subgroup A_1 of G, such that A_1 is non-Dedekind. We put $R_1 = A_1^G$; then R_1 is finite, since both A_1 and $|R_1:A_1|$ are finite. Assume now that we have already constructed non-Dedekind finite normal subgroups $R_1, \dots, R_{i-1}$, such that $B_{i-1} = \langle R_1, \dots, R_{i-1} \rangle \simeq \simeq R_1 \times \dots \times R_{i-1}$. Now, B_{i-1} is a finite subgroup of the residually finite group G; thus there exists a normal subgroup N of finite index in G, such that $N \cap B_{i-1} = 1$. By what we observed above, N is not nilpotent and so it has a finite subgroup A_i which is not Dedekind. Put $R_i = A_i^G$, then R_i is finite and $R_i \cap B_{i-1} \leqq N \cap B_{i-1} = 1$, whence $B_i = \langle B_{i-1}, R_i \rangle \simeq R_1 \times \dots \times R_{i-1} \times R_i$.

Now, by construction, for any $i \in \mathbf{N}$, there exists a subgroup H_i of R_i which is subnormal of defect exactly 2 in R_i (this is because the R_i's are nilpotent and non-Dedekind). Write $R = \langle R_i;\ i \in \mathbf{N} \rangle \simeq \simeq \operatorname{Dir}_{i \in \mathbf{N}} R_i$ and $H = \langle H_i;\ i \in \mathbf{N} \rangle$. Then H is subnormal of defect 2 in R, in particular it is subnormal in G. Thus $|H^G:H|$ is finite. But $H^G \leqq R$, and so there exists an $n \in \mathbf{N}$ such that $H^G \leqq HB_n$, where $B_n = \langle R_1, \dots, R_n \rangle$. Then, if $j > n$ and π_j is the canonical projection of

R on R_j, we get $\pi_j(H^G) \leqq \pi_j(HB_n) = \pi_j(H) = H_j$, whence $\pi_j(H^G) = H_j$, yielding $H_j \trianglelefteq R_j$ and contradicting our choice of H_j. This implies that G is nilpotent and concludes the proof. ■

3.2. THEOREM. *Let G be a hyperabelian p-group (p a prime) in $T^* \cup V$; let D be the abelian divisible radical of G and $C = C_G(D)$. Then G/D is nilpotent, $|G:C|$ is finite and $C \leqq Z_\omega(G)$. Moreover:*

a) if $G \in T^*$ and $p \neq 2$, then G is nilpotent;

b) if $G \in T^*$ and $p = 2$, then $|G:C| \leqq 2$;

c) if $G \in V_m$ ($m \in \mathbb{N}$), then $|G:C| \leqq 2m$.

PROOF. 1) *Assume that G is soluble and reduced.* We proceed by induction on the derived length n of G. Let A be a maximal normal abelian subgroup of G containing $G^{(n-1)}$. We claim that G/A is reduced. In fact, let L/A be a normal divisible abelian subgroup of G/A, then L belongs to $T^* \cup V$ and so, for every $x \in A$, we have $|L:C_L(x)| < \infty$; but $A \leqq C_L(x)$ and L/A has no non trivial finite quotient; hence $A \leqq Z(L)$ and L is nilpotent. Now, $L/Z(L)$ is divisible and it is also residually finite, because L is an FC-group. It there follows that $L = Z(L)$; and maximality of A implies $A = L$, whence G/A is reduced. By inductive hypothesis, G/A is nilpotent.

We now prove that G is a Baer group. It will follow from Lemma 3.1 that G is nilpotent. We have therefore to show that, for every $x \in G$, $\langle x \rangle$ is subnormal in G. If $x \in A$, then $\langle x \rangle \trianglelefteq A \trianglelefteq G$. Assume $x \notin A$, then $\langle A, x \rangle \operatorname{sn} G$, since G/A is nilpotent. Write $B = [A, x]$ and let $|x| = p^m$; then:

$$[B, {}_{p^m} x] = B^{(x-1)^{p^m}} \leqq B^p .$$

Hence, $\langle B^p, x \rangle \operatorname{sn} \langle B, x \rangle \trianglelefteq \langle A, x \rangle \operatorname{sn} G$, and so $\langle B^p, x \rangle$ is subnormal in G.

If $G \in V$, then $K = N_G(B^p)$ has finite index in G and $|K:N_K(\langle B^p, x \rangle)|$ is also finite. Since $\langle B^p, x \rangle / B^p$ is certainly finite, we get that $\langle B^p, x \rangle^K / B^p$ is finite. But: $\langle B^p, x \rangle^K \geqq \langle B^p, x \rangle^A \geqq \langle [A, x], x \rangle = \langle B, x \rangle$. Hence, $|\langle B, x \rangle : \langle B^p, x \rangle|$ is finite, and consequently, $|B:B^p|$ is finite.

If $G \in T^*$, then $|\langle B^p, x \rangle^G : \langle B^p, x \rangle|$ is finite and, by the same argument as before, $|B:B^p|$ is finite.

Since B is a reduced abelian p-group, we deduce that B is finite. Thus $\langle x \rangle \operatorname{sn} \langle B, x \rangle \operatorname{sn} G$ and so $\langle x \rangle \operatorname{sn} G$. Hence G is a Baer group.

2) *G reduced, general case.* Let $(A_\alpha)_{\alpha \leqq \lambda}$ be an ascending normal series of G, with abelian factors. We argue by transfinite induction on the ordinal number λ. If $\lambda = \beta + 1$, then A_β, which is certainly reduced, is nilpotent by inductive hypothesis, whence G is soluble; by case 1), G is therefore nilpotent. Otherwise, λ is a limit ordinal and $G = \bigcup_{\alpha<\lambda} A_\alpha$; again by inductive hypothesis, A_α is nilpotent, for every ordinal $\alpha < \lambda$. But then G is a Baer group and so it is nilpotent by Lemma 3.1.

3) *Conclusion.* Let G be a hyperabelian p-group in $\boldsymbol{T}^* \cup \boldsymbol{V}$, and let D be the maximal normal abelian divisible subgroup of G. If $x \in D$, then $\langle x \rangle \operatorname{sn} G$ and so $|G:C_G(x)|$ is finite. It there follows that, if L/D is a normal abelian divisible subgroup of G/D, then $D \leqq Z(L)$, which in turn implies that L is abelian. Thus $L = D$ and G/D is therefore reduced. By the preceding case, G/D is nilpotent.

Put $C = C_G(D)$ and observe that C is nilpotent. Now, G acts by conjugation on D, and every subgroup of D is subnormal in G. If $G \in \boldsymbol{T}^*$, then, by Lemma 2.8, $G = \operatorname{Paut}_G(D)$. Since G is periodic we have that $|G:C|$ divides $p-1$ if $p \neq 2$, and $|G:C| \leqq 2$ if $p = 2$. But G is a p-group; hence $G = C$, and so G is nilpotent, if $p \neq 2$; and $|G:C| \leqq 2$ if $p = 2$. If $G \in \boldsymbol{V}$, then, by Lemma 2.3,

$$|G:\operatorname{Paut}_G(D)| \leqq m\,.$$

Arguing as before, we have $|G:C| \leqq m$ if $p \neq 2$, and $|G:C| \leqq 2m$ if $p = 2$.

It remains to prove that $C \leqq Z_\omega(G)$. Let $x \in C$; since C is nilpotent, $\langle x \rangle$ is subnormal in G. Thus $\langle x \rangle^G$ is finite in both cases. Because G is a p-group, this implies $\langle x \rangle^G \leqq Z_n(G)$ for some $n \in \mathbb{N}$, and so $C \leqq Z_\omega(G)$. ■

Observe that non nilpotent p-groups in $\boldsymbol{V}_p$ do exist. For example, the standard wreath product $C_{p^\infty} \operatorname{wr} C_p$ is a $\boldsymbol{V}_p$ group and it is not nilpotent.

We now give necessary conditions for a soluble group to be a $\boldsymbol{V}$-group. Just to this purpose, let us introduce another class of soluble groups.

DEFINITION. A group G is said to be of type I if G' is periodic and every subgroup of G' is normal in G (observe that this implies that G is metabelian and hypercyclic).

3.3. THEOREM. *Let G be a soluble V-group. Then there exists a normal subgroup N, of finite index in G, which is a group of type I.*

PROOF. Let G be a soluble V-group, and let F be the Fitting subgroup of G (that is the subgroup of G generated by the normal nilpotent subgroups). Then, by 3.1, F is nilpotent. Let $A = Z(F)$ be the centre of F. By result 1.1 of B. Neumann, A has finite index in F; thus $C_G(F/A)$ has finite index in G.

Let T be the torsion subgroup of A. We now apply the results of Sect. 2, with Γ the group of automorphisms induced by conjugation by G on A/T and on T. By Lemma 2.5 and Theorem 2.4, respectively, he have that $C_G(A/T)$ and $\mathrm{Paut}_G(T)$ have finite index in G.

Write $L = C_G(F/A) \cap C_G(A/T) \cap \mathrm{Paut}_G(T)$; then L is normal in G and $|G:L|$ is finite. Now, recalling that the group of power automorphisms of any group is abelian, we get: $L' \leq C_G(F/A) \cap C_G(A/T) \cap$ $\cap\, C_G(T)$. Hence $[F, L', L', L'] = 1$, and solubility of G implies $L' \leq F$, so $\gamma_4(L) = [L', L, L] \leq [F, L, L] \leq [A, L] \leq T$. Thus L/T is nilpotent. But it is also a V-group, hence, again by 1.1, if $N/T = Z(L/T)$, then $|L:N|$ is finite, which in turn implies that $|G:N|$ is finite; moreover N is normal in G.

It is now easy to see that N is a group of type I. In fact, $N' \leq T$ and so N' is a periodic abelian group; furthermore, $N \leq L \leq \mathrm{Paut}_G(T)$, hence every subgroup of N' is normal in N. ■

3.4. COROLLARY. *Every soluble V-*group *is metabelian by finite.*

3.5. COROLLARY. *A finitely generated soluble V-*group *is abelian by finite* (see Robinson [12, 13.4.9]).

PROOF. Let G be a finitely generated soluble V-group, and let N be a normal subgroup of finite index in G, which is a group of type I. Then N also is finitely generated, say $N = \langle x_1, \ldots, x_n \rangle$. Moreover, if $d_{ij} = [x_i, x_j]$ $(i, j = 1, \ldots, n)$, then $\langle d_{ij} \rangle \trianglelefteq N$. Hence N' is an abelian group generated by the set of d_{ij}'s. Since N' is periodic, it there follows that N' is finite. Because N is finitely generated, this implies that $Z(N)$ has finite index in N, and so the corollary is true. ■

In order to give a first description of soluble T^*-groups, we introduce one more class of soluble groups.

DEFINITION. A soluble group G is of type II if:

1) $G^{(2)} = 1$ and either G' or G/G' is periodic;

2) every subgroup of G' is normal in G, or there exist normal subgroups $R \leqq D \leqq G'$ of G, such that every subgroup of G contained in any of the sections G'/D, D/R, $R/1$ is normal in G and R is a finitely generated torsion free abelian group, G'/R is periodic, D/R is divisible of finite total rank.

3.6. THEOREM. *Every soluble T^*-group G has a finite normal subgroup M, such that G/M is a group of type II.*

PROOF. Let G be a soluble T^*-group, and let F be the Fitting subgroup of G. Then, by 3.1, F is nilpotent and, by result 1.2 of B. Neumann, F' is finite. Now, by a well known nilpotency criterion of P. Hall, Fit $(G/F') = F/F'$; we may therefore assume that $F' = 1$.

Every subgroup of F is subnormal in G, whence $|H^G:H| < \infty$ for every $H \leqq F$. By Theorems 2.11 and 2.12, there exist subgroups $N_1 \leqq R_1 \leqq D_1$ of F, all normal in G, such that N_1 is finite, and G acts, by conjugation, as a group of power automorphisms on the sections R_1/N_1, D_1/R_1, F/D_1, where one or more of these sections might well be trivial (observe that, in the statement of Theorem 2.12, A/R is periodic; by Theorem 2.11, we can therefore find a finite subgroup N/R of A/R, and a divisible subgroup D/N of A/N such that $\Gamma = \mathrm{Paut}_\Gamma(A/D) = \mathrm{Paut}_\Gamma(D/N)$. In our present case, we take as N_1 the torsion subgroup of N, $R_1 = N$ and $D_1 = D$).

Since the group of power automorphisms of any group is abelian, we have that G' stabilizes a finite series of F/N_1. Let $C = C_G(N_1)$; then $G' \cap C$ stabilizes a finite series of F. Since G is soluble, this implies that $G' \cap C$ is contained in F. Now, G/C is finite, hence $|G'/G' \cap F| \leqq |G'/G' \cap C| = |G'C/C|$ is finite. But also we have

$$G' \cap F \leqq Z_3(G') \quad \text{mod. } N_1 .$$

By a Theorem of R. Baer (see Robinson [13, 14.5.2]) $\gamma_4(G')N_1/N_1$ is finite and so, since N_1 is finite, $\gamma_4(G')$ is finite. Applying once more 1.2, we have that $G^{(2)}/\gamma_4(G')$ is finite, whence $G^{(2)}$ is finite.

Now, arguing on the section $G'/G^{(2)}$ as we did before on F, we find a finite normal subgroup M of G, containing $G^{(2)}$, and subgroups R/M, D/M of G'/M such that part 2) of the definition of a soluble group of type II is satisfied by G/M.

To conclude, we prove that if G is any soluble T^*-group, then either G' or G/G' is periodic. Since we have proved before that $G^{(2)}$ is finite, we may assume that G is metabelian. Let T be the torsion subgroup of G' and assume that $G'/T \neq 1$. Then, by Lemma 2.7, G acts by conjugation as a group of power automorphisms on G'/T. If $K = C_G(G'/T)$, then $|G:K| \leqq 2$. Suppose, by contradiction, that G/G' is not periodic; then K/G' is not periodic. Let xG' be an element of infinite order in K/G'. Then $\langle G', x\rangle/T$ is an abelian torsion free group. Again by 2.7, G acts as a group of power automorphisms on $\langle G', x\rangle/T$. Since G obviously centralizes $\langle G', x\rangle/G'$, it there follows that G centralizes $\langle G', x\rangle/T$, which in turn implies that G' is contained in T, a contradiction. ∎

3.7. COROLLARY. *Every soluble* $\boldsymbol{T}^*$-group *is finite by metabelian.*

Note that a dual of Corollary 3.5 does not hold for finitely generated $\boldsymbol{T}^*$-groups. In fact, the infinite dihedral group is a 2-generated soluble $\boldsymbol{T}^*$-group, but its derived subgroup is infinite. On the other hand Corollary 3.5 itself remains true.

3.8. COROLLARY. *Every finitely generated soluble* $\boldsymbol{T}^*$-group *is abelian by finite.*

PROOF. Let G be a finitely generated soluble $\boldsymbol{T}^*$-group and let M be a finite normal subgroup of G, such that G/M is a group of type II. In particular $\bar{G} = G/M$ is a finitely generated metabelian group. If $\{g_1, \dots, g_r\}$ is a set of generators of $\bar{G}$, then $\bar{G}'$ is generated by the subgroups $\langle[g_i, g_j]\rangle^{\bar{G}}$, $i, j = 1, \dots, r$. Now $\bar{G}'$ is abelian, thus $|\langle[g_i, g_j]\rangle^{\bar{G}}:\langle[g_i, g_j]\rangle|$ is finite, for every $i, j = 1, \dots, r$. It follows that $\bar{G}'$ is finitely generated, and also that, if $C/M = C_{\bar{G}}(\bar{G}')$, then $|G:C|$ is finite. Moreover C/M is a nilpotent $\boldsymbol{T}^*$-group; by 1.2, $C'M/M$ is finite and, consequently, C' is finite. Now, since $|G:C|$ is finite, C is finitely generated, so C' finite implies $|C:Z(C)|$ finite. Then $Z(C)$ is a noımal abelian subgroup of finite index in G. ∎

The following rwo examples show that soluble groups in the class $\boldsymbol{T}^* \cap \boldsymbol{V} = \boldsymbol{U}$ need not be finite extensions of $\boldsymbol{T}$-groups, nor they need admit a normal finite subgroup, such that the quotient is a $\boldsymbol{T}$-group. The constructions are similar; the first one provides a group with elements of infinite order, while the second one gives a periodic group.

EXAMPLE 4. Let $p_1, p_2, \dots$ be an infinite sequence of distinct prime numbers. Let $C_i = \langle x_i\rangle$ be a cyclic group of order p_i for every $i = 1, 2, \dots$; then, for any i, C_i admits an automorphism θ_i of order $p_i - 1$.

Let $A = \operatorname{Dir}_{i \in \mathbb{N}} C_i$, and let α be the automorphism of A defined by:

$$x_i^\alpha = x_i^{\theta_i} \qquad \text{for every } i = 1, 2, \ldots .$$

Then $\alpha \in \operatorname{Paut}(A)$ and it has infinite order. Let G be the natural semidirect product $A \rtimes \langle\alpha\rangle$. Note that, for each $i = 1, 2, \ldots$; $[C_i, \alpha] = [C_i, \theta_i] = C_i$ and so $[A, \alpha] = A$. Moreover:

(1) $\quad [C_i, \alpha^n] = 1 \quad \text{if } p_i - 1 | n ; \qquad [C_i, \alpha^n] = C_i \quad \text{otherwise}.$

Let S be a subnormal subgroup of G. If $S \leqq A$, then $S \trianglelefteq G$. Otherwise, $AS \gneqq A$, say $AS = \langle A, \alpha^n\rangle$ for some $0 \neq n \in \mathbb{N}$. By (1), we have:

$$[A, S] = [A, AS] = [A, \alpha^n] = \langle [C_i, \alpha^n]; 0 \neq i \in \mathbb{N}\rangle = \langle C_i; p_i - 1 \nmid n\rangle .$$

It follows that $A/[A, S]$ is finite; also

$$[A, S, S] = [A, \alpha^n, \alpha^n] = [A, \alpha^n] = [A, S] .$$

Since $S \operatorname{sn} AS$, this last identity means that S is normal in AS and $[A, S] \leqq S \cap A$. Hence

$$|G : N_G(S)| \leqq |G : AS| = n \text{ and } |S^G : S| \leqq |AS : S| \leqq |A : [A, S]| < \infty .$$

This holds for any subnormal subgroup of G, whence $G \in \boldsymbol{T}^* \cap \boldsymbol{V}$.

By a direct verification, or by refering to Robinson [12, Theorem 4.3.1], one easily checks that G does not have normal $\boldsymbol{T}$-subgroups of finite index, nor a finite normal subgroup M, such that G/M is a $\boldsymbol{T}$-group.

EXAMPLE 5. Let $p_0 < p_1 < p_2 < \ldots$ be an infinite sequence of prime numbers, such that, for each $i = 0, 1, 2, \ldots$, $p_0 p_1 \ldots p_i$ divides $p_{i+1} - 1$ (such a sequence exists by Dirichlet's Theorem). For every $j = 1, 2, \ldots$, let $C_j = \langle x_j \rangle$ be a cyclic group of order p_j. Now, if $i, j \in \mathbb{N}$, and $i < j$ then $p_i | p_j - 1$; hence C_j admits an automorphism σ_{ji} of order exactly p_i.

Let $A = \operatorname{Dir}_{0 \neq i \in \mathbb{N}} C_i$ and, for each $i = 0, 1, \ldots$, let y_i be the (power) automorphism of A defined by:

$$\begin{cases} x_j^{y_i} = x_j & \text{if } j \leqq i\,, \\ x_j^{y_i} = x_j^{\alpha_{ji}} & \text{if } j > i\,. \end{cases}$$

Then y_i has order p_i and, if $j > i$, then $[C_j, y_i] = C_j$.

Hence $[A, y_i] = \langle C_j; i > j \rangle$ and also $[A, y_i] \geqq [A, y_k]$ if $i \leqq k$.

Let $H = \langle y_i; i \in \mathrm{N} \rangle \leqq \mathrm{Paut}(A)$, and let $G = A \rtimes H$ be the natural semidirect product. Then G is periodic and metabelian. Let S be a normal subgroup of G. If $S \leqq A$, then S is normal in G. Otherwise let K be a non trivial subgroup of H, such that $AS = AK$. Then there exists a minimal $i_0 \in \mathrm{N}$, such that $y_{i_0} \in K$. Since S sn G, there is also an $r \in \mathrm{N}$ such that $[A, {}_rS] \leqq S$. Now:

$$[A, {}_rS] = [A, {}_rAS] = [A, {}_rK] = [A, {}_ry_{i_0}] = [A, y_{i_0}] = [A, S]\,;$$

whence $A \cap S \geqq [A, S] = [A, y_{i_0}] \simeq \mathrm{Dir}_{j > i_0}\, C_j$ and S is normal in AS.

Now, $A/[A, S]$ is a finite cyclic group; since $A \cap S \geqq [A, S]$ we get that $AS/S \simeq A/A \cap S$ is a finite cyclic group; in particular, S^G/S, is finite. Since G/A is locally finite cyclic, it there follows that S/S_G, which is a quotient of finite exponent of $S/A \cap S \simeq AS/A$, is finite. Hence $|S^G : S_G|$ is finite. This holds for each subnormal subgroup of G and so $G \in \boldsymbol{U} = \boldsymbol{T}^* \cap \boldsymbol{V}$.

Again refering to Robinson [12, Theorem 4.2.2], we observe that G does not have any normal $\boldsymbol{T}$-subgroup of finite index, nor any finite normal subgroup M, such that G/M is a $\boldsymbol{T}$-group.

In sect. 4, we will show, in particular, that every soluble $\boldsymbol{V}_m$-group ($m \in \mathrm{N}$) admits a normal $\boldsymbol{T}$-subgroup of finite (bounded) index. Our next example shows that a dual statement does not hold for $\boldsymbol{T}_m$-groups.

Example 6. Let A be an infinite elementary abelian group of exponent 3 and $H = \langle x \rangle$ an infinite cyclic group, acting on A in such a way that $a^x = a^{-1}$ for every $a \in A$. Let $G = A \rtimes H$ be the semidirect product defined by this action. It is easy to see that $G \in \boldsymbol{T}_3$ (indeed, it is not hard to check that $G \in \boldsymbol{U}_9$) and that G is not a $\boldsymbol{T}$-group. If N is a finite normal subgroup of G, then $N \leqq A$ and $G/N \simeq G$ is not a $\boldsymbol{T}$-group.

4. Soluble groups in the classes V_m ($m \in \mathbf{N}$).

We now turn to the study of soluble V_m-groups, $m \in \mathbf{N}$; that is soluble groups in which every subnormal subgroup has at most m conjugates.

Let G be a group, and π a set of prime numbers; then we denote by $O_\pi(G)$ the maximal normal π-subgroup of G; as usual, if $\pi = \{p\}$, we simply write $O_p(G)$ and $O_{p'}(G)$ instead of, respectively, $O_\pi(G)$ and $O_{\pi'}(G)$. Furthermore, we denote by $\omega(G)$ the intersection of the normalizers of all subnormal sugbroups of G; $\omega(G)$ is called the Wielandt subgroup of G. We also use the following notation: $\omega_p(G)/O_{p'}(G) = \omega(G/O_{p'}(G))$, for any prime p.

The following observation will be useful.

4.1. Lemma. *Let G be a group. Then* $\omega(G) = \bigcap_{p \in \mathbf{P}} \omega_p(G)$.

Proof. It is clear that, for any subnormal subgroup N of G, $\omega(G)N/N \leqq \omega(G/N)$, whence $\omega(G) \leqq \bigcap_{p \in \mathbf{P}} \omega_p(G)$.

Conversely, let S be a subnormal subgroup of G, and put

$$B = \bigcap_{p \in \mathbf{P}} O_{p'}(G)S\,.$$

Then, for every $p \in \mathbf{P}$:

$$N_B(S) \leqq O_{p'}(G)S \cap N_G(S) = S(O_{p'}(G) \cap N_G(S))\,.$$

Hence $N_B(S)/S$ is a p'-group, for every prime p. It follows that $N_B(S) = S$. But S is subnormal in B, so $N_B(S) = S$ implies $B = S$. Now, if $L = \bigcap_{p \in \mathbf{P}} \omega_p(G)$, we get:

$$[L, S] \leqq \bigcap_{p \in \mathbf{P}} [\omega_p(G), S] \leqq \bigcap_{p \in \mathbf{P}} O_{p'}(G)S = S\,.$$

Hence L normalizes S. This holds for any S sn G, and consequently $L \leqq \omega(G)$, thus $L = \omega(G)$. ∎

In the sequel, we will often refer to properties of soluble T-groups. These are to be found in D. Robinson's paper [12]. We recall here that Robinson splits the class of soluble $\boldsymbol{T}$-groups into four mutually disjoint subclasses, namely:

1) the class of abelian groups;

2) the class of non abelian periodic soluble $\boldsymbol{T}$-groups;

3) the class of soluble $\boldsymbol{T}$-groups of type 1. A soluble $\boldsymbol{T}$-group is of type 1 if G is non abelian and $C_G(G') = \mathrm{Fit}\,(G)$ is non periodic;

4) the class of soluble $\boldsymbol{T}$-groups of type 2. A soluble $\boldsymbol{T}$-group is of type 2 if it is non abelian, non periodic, and $C_G(G')$ is periodic.

We shall use the following property of soluble $\boldsymbol{T}$-groups of type 2, that does not appear explicitly in [12].

4.2. Lemma. *Let G be a soluble $\boldsymbol{T}$-group of type 2, S a (sub)normal subgroup of G. Then either S is periodic, or $S \geqq G'$.*

Proof. We refer to Robinson [12, Theorem 4.3.1], from which it follows, in particular, that the set of periodic elements of G is a subgroup T of G. If $S \nleqq T$, then there exists $x \in S$ of infinite order. By point (iii) of the quoted Theorem by D. Robinson, x does not centralize any primary component of G'. But, by point (ii) of the same Theorem, G' is a periodic divisible abelian group, and x acts, by conjugation, as a power automorphism on it. Thus we have:

$$G' = [G', x] \leqq [G', S] \leqq S\,, \qquad \text{as we wanted.} \qquad \blacksquare$$

Another elementary, but useful, observation is the following:

4.3. Lemma. *Let G be a soluble p-group in V_m; where $p \neq 2$ and $p > m$. Then G is abelian.*

Proof. If $H \leqq G$, then $H = G$ or $|G:H| \geqq p > m$. Hence the normalizer of any subnormal subgroup of G, coincides with G; that is G is a $\boldsymbol{T}$-group. Since G is soluble, and $p \neq 2$, it follows that G is abelian (see Robinson [12, Theorem 4.2.1]). $\blacksquare$

Our first aim is to prove that a soluble V_m-group admits a normal $\boldsymbol{T}$-subgroup, whose index in G is boundedly finite (and depending only on m). We begin with a very particular case.

4.4. LEMMA. *Let G be a periodic V_m-group; A a normal-abelian p-subgroup (p a prime) of G, such that G/A is abelian. Then there exists a normal $\boldsymbol{T}$-subgroup N of G, with $|G:N| \leqq b_1(m)$; where b_1 is a function of* $\mathbf{N}$ *in* $\mathbf{N}$.

PROOF. Let P/A be the p-component of G/A; then P is a p-group in V_m. By Theorem 3.2, $F = \text{Fit}(G)$ is nilpotent and $|P:F| \leqq 2m$. Now, F is normal in G and, by 1.3 (i), $|F:Z(F)| \leqq \mu(m)$. Let $B = AZ(F)$, then B is a normal abelian subgroup of G and $|P:B| \leqq \leqq 2m\mu(m)$. Let $L = \text{Paut}_G(B)$; by Theorem 2.15, $|G:L| \leqq \alpha(m)$. If N/B is the p'-component of L/B, then

$$|G:N| = |G:L|\,|L:N| \leqq \alpha(m)|P \cap L:B| \leqq 2m\mu(m)\alpha(m) = b_1(m)\,.$$

Now, N normalizes every subgroup of the p-group B, and N/B is an abelian p'-group. By a result of Robinson [12, Lemma 5.2.2], N is a $\boldsymbol{T}$-group. ∎

4.5. LEMMA. *Let G be a soluble V_m-group; then G has a normal subgroup H, such that H is a $\boldsymbol{T}$-group, and $|G:H| \leqq b_2(m)$; where b_2 is a function of* $\mathbf{N}$ *in itself.*

PROOF. Let G be a soluble V_m-group, F the Fitting radical of G. Then F is nilpotent, by 3.1, and G acts on F in such a way that $|G:N_G(H)| \leqq m$ for every $H \leqq F$. By Theorem 2.15, it follows that $L = \text{Paut}_G(F)$ has index at most $\alpha(m)$ in G. Since G is soluble, we have therefore: $L' \leqq C_G(F) = Z(F)$. In particular, $L^{(2)} = 1$ and all subgroups of $L' \leqq F$ are normal in L. Without loss of generality, we may therefore assume that G is metabelian, and that every subgroup $A = G'$ is normal in G. Let us also assume $m > 1$ (if $m = 1$, G is a $\boldsymbol{T}$-group).

We distinguish three cases:

CASE A) *G is periodic.* Let $A = G'$ and consider the following subsets of $\pi(A)$.

$$\pi_1 = \{p \in \pi(A);\ p > m \text{ and } |G:C_G(A_p)| > m\}\,;$$

$$\pi_2 = \{p \in \pi(A);\ p > m \text{ and } |G:C_G(A_p)| \leqq m\}\,;$$

$$\pi_3 = \{p \in \pi(A);\ p \leqq m\}\,.$$

It is clear thet $\pi(A)$ is the disjoint union $\pi_1 \cup \pi_2 \cup \pi_3$.

1) Suppose that $\pi_1 \cap \pi(G/A) \neq \emptyset$. Let $p \in \pi_1 \cap \pi(G/A)$ and let $P_0/A_{p'}$ be the Sylow p-subgroup of $G/A_{p'}$. Since $p > m \geqq 2$, $P_0/A_{p'}$ is abelian by 4.3. Hence, if $C = C_G(A_p) = C_G(A/A_{p'})$, we have $P_0 \leqq C$ and so $p \notin \pi(G/C)$. But G acts as a group of power automorphisms on A and it is periodic; thus G/C is cyclic of order a divisor of $p-1$. It therefore follows $C = C_G(\bar{x})$ for any $1 \neq x \in \bar{A} = A/A_{p'}$, and $[\bar{A}, G] = \bar{A}$. Take $\bar{x} \in \bar{A}$ with $|\bar{x}| = p$ and $y \in P_0 \setminus A$. Consider the finite abelian p-group $\bar{T} = \langle \bar{x}, \bar{y} \rangle^G$, where $\bar{y} = yA_{p'}$. Since $C = C_G(\bar{T})$ and G/C is a p'-group, we have:

$$\bar{T} = [\bar{T}, G] \times C_{\bar{T}}(G) = (\bar{T} \cap \bar{A}) \times \langle \bar{y}_0 \rangle = \bar{A}_0 \times \langle \bar{y}_0 \rangle$$

(observe that $\langle A, y \rangle \trianglelefteq G$, whence $\bar{A}\bar{T} \leqq \langle \bar{A}, \bar{y} \rangle$).

If $g \in N_G(\bar{x}\bar{y}_0)$, then, for some $n \in \mathbb{N}$, $1 \leqq t \leqq p-1$:

$$\bar{x}^n \bar{y}_0^n = (\bar{x}\bar{y}_0)^g = \bar{x}^g \bar{y}_0^g = \bar{x}^t \bar{y}_0$$

which yields:

$$\bar{x}^{n-t} = \bar{y}_0^{1-n} \bar{A}_0 \cap \langle \bar{y}_0 \rangle = 1 ;$$

hence $\bar{y}_0^n = \bar{y}_0$ and so $\bar{x}^n = \bar{x}$. We conclude that $N_G(\bar{x}\bar{y}_0) \leqq C$. On the other hand $\langle \bar{x}\bar{y}_0 \rangle \operatorname{sn} \bar{G} = G/A_{p'}$ and so, by hypothesis, $|G : N_G(\bar{x}\bar{y}_0)| \leqq m$. This implies $|G : C| \leqq m$, against the choice of $p \in \pi_1$. Therefore $\bar{T} = [\bar{T}, G]$ and $P_0 = A$, a contradiction.

Thus $\pi_1 \cap \pi(G/A) = \emptyset$. Since both $A/A_{\pi_1'}$ and G/A are abelian, and all subgroups of A are normal in G, it follows from Robinson [12, Lemma 5.2.2] that $G/A_{\pi_1'}$ is a $\boldsymbol{T}$-group.

2) We consider now the set π_2. Let $d = m!$. Then $R = G^d$ normalizes every subnormal subgroup of G; in particular, R is a $\boldsymbol{T}$-group. Moreover R centralizes $A_{\pi_2} \simeq A/A_{\pi_2'}$, whence $\bar{R} = RA_{\pi_2'}/A_{\pi_2'}$ is nilpotent (we use a bar to denote subgroups of G modulo $A_{\pi_2'}$). Now, $\bar{R}$ is abelian, because $2 \notin \pi_2$ and $\bar{R}/\bar{A}$ is abelian (see Robinson [12, 4.2.1]). By Theorem 2.15,

$$|\bar{G} : \operatorname{Paut}_{\bar{G}}(\bar{R})| \leqq \alpha(m) ; \qquad \text{let } \bar{V} = \operatorname{Paut}_{\bar{G}}(\bar{R}) .$$

If $p \in \pi_2 \cap \pi(V/A) = \pi(\bar{A}) \cap \pi(\bar{V}/\bar{A})$, let P/A be the p-component of V/A. We note that, since $m < p$, $P/A \leqq RA/A$ and so V acts as a group of power automorphisms on $P/A_{p'}$. Also, $p \in \pi(V/A)$, thus P/A

is non trivial and it is centralized by V/A. It there follows that $V/C_V(P/A_{p'})$ is a cyclic group of order a power of p. But

$$|V:C_V(A/A_{p'})| \leqq m < p .$$

Hence V centralizes $A/A_{p'} \simeq A_p$. Furthermore, $P/A_{p'}$ is abelian by 4.3. We conclude that $\bar{V}/\bar{A}_{p'}$ is abelian, that is $V' \leqq A_{p'}$ and $p \notin \pi(V')$. Thus we have shown that $\pi(\bar{V}') \cap \pi(\bar{V}/\bar{V}') = \emptyset$. Again by Lemma 5.2.2 in Robinson [12], we have that $\bar{V} = V/A_{\pi_2'}$ is a $\boldsymbol{T}$-group. Also $V \trianglelefteq G$ and $|G:V| = |\bar{G}:\bar{V}| \leqq \alpha(m)$.

3) Let now $p \in \pi_3$: By Lemma 4.4, there exists a normal subgroup N_p of G, such that $N_p/A_{p'}$ is a $\boldsymbol{T}$-group, and $|G:N_p| \leqq b_1(m)$. Let $N = \bigcap_{p\in\pi_3} N_p$; then $N \trianglelefteq G$, and $|G:N| \leqq \prod_{p\in\pi_3} |G:N_p| \leqq b_1(m)^{|\pi_3|} \leqq b_1(m)^m$. Furthermore, $N \geqq A$ and $N/A_{p'} \leqq N_p/A_{p'}$ is a $\boldsymbol{T}$-group for every $p \in \pi_3$. From Lemma 4.1 it now easily follows that $N/A_{\pi_3'}$ is a $\boldsymbol{T}$-group.

4) Conclusion. Let $H = N \cap V$; then $H \trianglelefteq G$ and $H/A_{\pi_1'}$, $H/A_{\pi_2'}$, $H/A_{\pi_3'}$ are $\boldsymbol{T}$-groups. Recalling that $\pi(A)$ is the disjoint union $\pi_1 \cup \cup \pi_2 \cup \pi_3$, we conclude by Lemma 4.1 that H is a $\boldsymbol{T}$-group.
Moreover: $|G:H| \leqq |G:V|\,|G:N| \leqq \alpha(m)\, b_1(m)^m = b_3(m)$.

Case B) *A is not periodic.* Since G acts as a group of power automorphisms on A, we have $|G:C_G(A)| \leqq 2$. Let $K = C_G(A)$. Then K is nilpotent and, by Macdonald's Theorem 1.3 (i), $|K:Z(K)| \leqq \mu(m)$. We put $H = Z(K)$, which is an abelian normal subgroup of G; moreover $|G:H| = |G:K|\,|K:H| \leqq 2\mu(m)$.

Case C) *G is not periodic, A is periodic.* Let $d = m!$ and $R = G^d$. Then R is a $\boldsymbol{T}$-group and it is not periodic, otherwise G would be periodic.

If R is abelian, let F be the Fitting subgroup of G. Then F is nilpotent by 3.1, and $|G:N_G(H)| \leqq m$ for every $H \leqq F$. By Theorem 2.15, we have $|G:\mathrm{Paut}_G(F)| \leqq \alpha(m)$. Moreover, $F \geqq R$; hence F is not periodic, thus the group of power automorphisms of F has order at most 2. Hence: $|G:Z(F)| = |G:C_G(F)| \leqq 2\alpha(m)$; and we may take $H = Z(F)$.

If R is not abelian, then it is a $\boldsymbol{T}$-group of type 2. In fact R is not periodic but $D = R' \leqq A$ is periodic. Thus D is divisible (see Robinson [12, Theorem 4.3.1]). Moreover, R/D is not periodic and abelian. We can apply on G/D the same argument used above, obtaining an abelian normal subgroup M/D of G/D, such that $|G:M| \leqq$

$\leqq 2\alpha(m)$. Let P be the set of torsion elements of M, then $D \leqq P \trianglelefteq M$. By case A), there exists a normal T-subgroup L of P, with $|P:L| \leqq \leqq b_3(m)$ (observe that, since D is divisible, $L \geqq D$). Let D_2 be the 2-component of D, $C = C_L(D_2)$; then $|L:C| \leqq 2$ and C is a periodic T-group, such that $\pi(C') \cap \pi(C/C') = \emptyset$ (see Robinson [12, Theorem 4.2.2], in fact $C' \leqq D$ implies $C' = [C', C] = [D, C]$ and $2 \notin \pi(C')$). Now, P/C is the torsion subgroup of the abelian group M/C. Since $|P/C| = |P:L|\,|L:C| \leqq 2b_3(m)$ is finite, P/C has a torsion free complement K/C in M/C. But K sn G, whence $K \in V_m$; by Theorem 2.15, if $K_0 = \mathrm{Paut}_K(C/C')$, then $|K:K_0| \leqq \alpha(m)$. We take $H = (K_0)_G$. Now,

$$|G:K_0| \leqq |G:M|\,|M:K|\,|K:K_0| = |G:M|\,|P:C|\,|K:K_0|$$

is boundedly finite, whence

$$|G:H| \leqq |G:K_0| \leqq b_4(m)\,.$$

We put $b_2(m) = \max\{b_3(m), 2\mu(m), b_4(m)\}$.

We have to show that H is a T-group. Since $C' \leqq D \leqq A$, H normalizes every subgroup of C' (remember that every subgroup of A is normal in G); furthermore, every periodic subgroup of H/C' is contained in C/C' and so it is normal in H. Since $\pi(C/C') \cap \pi(C') = \emptyset$, we may apply Lemma 5.2.2 of Robinson [12], concluding that each periodic subnormal subgroup of H is normal in H. Let now S be a non periodic subnormal subgroup of H; then $S \cap R$ is a non periodic subnormal subgroup of R. Because R is a T-group of type 2, we have, by Lemma 4.2, that $S \cap R$ contains $R' = D$. In particular $S \geqq D$ and so $S \trianglelefteq H$, since H/D is abelian. This proves that H is a T-group, and concludes the proof of the Theorem. ■

4.6. Lemma. *Let $G = MS$ be a group, with M a normal nilpotent subgroup of G, and S a subnormal subgroup of G. Then $S \geqq G^N$.*

Proof. We argue by induction on the defect $d(S; G)$ of S in G. If $d(S; G) = 1$, then $S \trianglelefteq G$ and $G/S = SM/S \simeq M/M \cap S$ is nilpotent, whence $S \geqq G_N$.

Let $d = d(S; G) > 1$, $G_1 = S^G$, then $G_1 \cap M$ is a nilpotent normal subgroup of $S(G_1 \cap M) = G_1 \cap G = G_1$; moreover, $d(S; G_1) = d - 1$. By inductive hypothesis, $S \geqq G_1^N$. Now, G_1^N car $G_1 \trianglelefteq G$, hence $G_1^N \trianglelefteq G$

and $G_1^N M \trianglelefteq G$. We have that $G_1^N M/G_1^N \simeq M/M \cap G_1^N$ is nilpotent. On the other hand, G_1/G_1^N is residually nilpotent and therefore

$$G_1 G_1^N M/G_1^N = G_1 M/G_1^N = G/G_1^N$$

is residually nilpotent and so $G^N \leq G_1^N \leq S$, as we wanted. ∎

4.7. LEMMA. *Let G be a V_m-group, N a normal nilpotent subgroup of G, with $|G:N| \leq n$. Then $|G:\omega(G)| \leq \delta(m, n)$ where δ is a function of* N *in* N.

PROOF. Let $\mathfrak{L} = \{L \leq G;\ N \leq L \text{ sn } G\}$; then $|\mathfrak{L}| \leq 2^n$ is finite. For each $L \in \mathfrak{L}$ define a subgroup Z_L of G by $Z_L/L^N = Z(L/L^N)$. Since L is subnormal in G, L/L^N is a V_m-group; hence it is nilpotent (in fact nilpotent groups in V_m have bounded class). By 1.3 (i), $|L:Z_L| \leq \mu(m)$ and so $|G:Z_L| \leq n\mu(m)$. Let $A = \bigcap_{L \in \mathfrak{L}} Z_L$, and observe that, N being normal in G, $A \trianglelefteq G$ (in fact, if $L \in \mathfrak{L}$, $g \in G$, then $Z_L^g = Z_{L^g}$ and $L^g \in \mathfrak{L}$). Moreover $|G:A| \leq (n\mu(m))^{|\mathfrak{L}|} \leq (n\mu(m))^{2^n} = \delta(m, n)$.

Let S be a subnormal subgroup of G; then $K = NS \in \mathfrak{L}$ and so, by Lemma 4.6, $S \geq L^N$, whence $[A, S] \leq [Z_L, S] \leq [Z_L, L] \leq L^N \leq S$. Thus $A \leq N_G(S)$. We have therefore $A \leq \omega(G)$ and, consequently, $|G:\omega(G)| \leq \delta(m, n)$. ∎

We are now in a position to prove our main result.

4.8. THEOREM. *There exists a function γ:* N → N, *such that, if G is a soluble V_m-group, then $|G:\omega(G)| \leq \gamma(m)$.*

PROOF. Let G be a soluble V_m-group. By Lemma 4.5, there exists a normal T-subgroup H of G, such that $|G:H| \leq b_2(m)$. We distinguish two cases:

A) H is abelian, or of type 1;

B) H is non abelian periodic, or of type 2.

Case A) If H is a soluble T-group of type 1, then, by Robinson [12, Theorem 3.1.1], H admits a characteristic abelian subgroup of index two. Hence, in both cases that we are considering, there exists a normal abelian subgroup of G whose index in G is at most $2b_2(m)$. By Lemma 4.7, we have $|G:\omega(G)| \leq \delta(m, 2b_2(m)) = b_5(m)$.

Case B) In this case, the set of elements of finite order of H is a subgroup, that we denote by P. Moreover (see Robinson [12]), $C_H(H') \leqq P$. Put $D = H'$, and $C = C_G(D)$. Now, D is normal in G, and $H \leqq \mathrm{Paut}_G(D)$; hence $HC/C \leqq Z(G/C)$, and so $[G, H] \leqq C \cap \cap H \leqq P$. Since $|G:H|$ is finite, we have that the elements of finite order of G form a (normal) subgroup T of G.

For each prime $p \in \pi(T)$, we put

$$K_p = O_{p'}(T) = O_{p'}(G)\ ; \qquad N_p = O_{p',p}(T) \quad \text{and} \quad \omega_p(G)/K_p = \omega(G/K_p)\,.$$

By Lemma 4.1, we have $\omega(G) = \bigcap_{p \in \pi(T)} \omega_p(G)$.

Now, G/D is one of the groups treated in case A); let $W/D = = \omega(G/D)$; then $|G:W| \leqq b_5(m)$. Put $\pi_0 = \pi(G/H)$; then $|\pi_0| \leqq |G/H| \leqq \leqq b_2(m)$ (actually $|\pi_0| \leqq m - 1$).

1) Let p be a prime, $p \notin \pi(D)$. Then $K_p \geqq D$; whence $\omega_p(G)/K_p \geqq \geqq WK_p/K_p$. In particular $\omega_p(G) \geqq W$.

2) Let $p \in \pi(D)$, $p \notin \pi_0$.

If S/K_p is subnormal in G/K_p, then $(S \cap H)K_p \trianglelefteq HK_p$, because $HK_p/K_p \simeq H/H \cap K_p$ is a T-group. Also, $(S \cap H)K_p \trianglelefteq HS$. Now, $HK_p \geqq N_p$, since $p \notin \pi_0$; in particular $HS \geqq N_p$ and so $(S \cap H)K_p \trianglelefteq N_pS$. Let $R = (S \cap H)N_p$ and $U = (S \cap H)K_p$. Thus we have $N_pS/U = = (R/U)(S/U)$, where $R \trianglelefteq N_pS$ and $S \operatorname{sn} N_pS$. But

$$R/U \simeq N_p/(U \cap N_p) = N_p/K_p(S \cap H \cap N_p)$$

is a p-group, while

$$S/U = S/(S \cap K_pH) \simeq SH/K_pH$$

is a p'-group. It follows that S/U is characteristic in N_pS/U. Now, since $N_p \geqq D$, we have that W normalizes N_pS; moreover $W \cap H$ normalizes U. Hence $W \cap H$ normalizes S and thus $\omega_p(G) \geqq W \cap H$.

3) Let $p \in \pi(D) \cap \pi_0$. In order to simplify notations, let us assume $K_p = 1$; hence $N_p = O_p(G) = O_p(T) = \mathrm{Fit}\,(T)$.

Now, $N_p \cap H$ is a normal p-subgroup of G, and it is also a T-group. If $p \neq 2$, it is abelian, and we put $L_p = N_p \cap H$; if $p = 2$, there exists a maximal nilpotent subgroup L_2 of index at most 2 in $N_2 \cap H$ (see Robinson [12, Theorem 4.2.1]). Observe that, in any case, $L_p \trianglelefteq G$.

Suppose that H is periodic. Then $L_p = \mathrm{Fit}\,(H)$; hence, H being soluble, $C_H(L_p) \leqq L_p$: But H acts as a group of power automorphisms on L_p; thus $|H:L_p|$ divides $p-1$ if $p>2$, and $|H:L_2| \leqq 2$. Consequently, $|G:L_p| \leqq (p-1)b_2(m)$ if $p>2$, and $|G:L_2| \leqq 2b_2(m)$. In any case, since $p \in \pi_0$ (and so, certainly, $p \leqq b_2(m)$), we get $|G:L_p| \leqq$ $\leqq b_2(m)^2$. We now apply Lemma 4.7, obtaining:

$$|G:\omega_p(G)| \leqq \delta(m, b_2(m)^2) = b_6(m)\,.$$

Suppose, now, that H is a soluble $\boldsymbol{T}$-group of type 2. Again, L_p is the Fitting subgroup of $H \cap T$. Arguing as in the previous case, we have therefore $|T:L_p| = |T:(H \cap T)|\,|(H \cap T):L_p| \leqq b_2(m)^2$. Let $\mathfrak{L}_p$ be the set of subnormal subgroups of T that contain L_p. Then $|\mathfrak{L}_p| \leqq b_7(m)$.

Let $S \in \mathfrak{L}_p$ and write $R = S^N$. Then S/R is a nilpotent $\boldsymbol{V}_m$-groip. Now, since G is a $\boldsymbol{V}_m$-group, $Q = N_G(S)$ has index at most m in G. Moreover, since R is characteristic in S, $Q \leqq N_G(R)$. Hence Q acts on the nilpotent $\boldsymbol{V}_m$-group S/R in such a way that the normalizer in Q of any subgroup of S/R has index at most m in Q. By Theorem 2.15: $|Q:\mathrm{Paut}_Q\,(S/R)| \leqq \alpha(m)$. Let $V_S = \mathrm{Paut}_Q\,(S/R) \cap H$; then $|H:V_S| \leqq m\alpha(m)$. Do the same for each $S \in \mathfrak{L}_p$ and put $V_p = \bigcap_{S \in \mathfrak{L}_p} V_S$. Then:

$$|G:V_p| \leqq |G:H|\,|H:V_p| \leqq b_2(m) \prod_{S \in \mathfrak{L}_p} |H:V_S| \leqq b_2(m)(m\alpha(m))^{b_7(m)} = b_8(m)\,.$$

Consider now $W \cap V_p$. Let E be a subnormal subgroup of G. If $E \leqq T$, then $E_1 = L_p E \in \mathfrak{L}_p$ and, by Lemma 4.6, $E \geqq E_1^N$. Consequently E is normalized by V_p. If $E \nleqq T$, then $E \cap T \nleqq T \cap H = P$ (P is the torsion subgroup of H), whence, by Lemma 4.2, $E \geqq H' = D$ and so E is normalized by W.

Hence

$$\omega_p(G) \geqq W \cap V_p \quad \text{and} \quad |G:\omega_p(G)| \leqq |G:W|\,|G:V_p| \leqq b_5(m)\,b_8(m)\,.$$

Summarizing, if $p \in \pi_0 \cap \pi(D)$, we have

$$|G:\omega_p(G)| \leqq \max\,\{b_6(m),\, b_5(m)\,b_8(m)\} = b_9(m)\,.$$

4) In conclusion, by Lemma 4.1 and the cases above discussed, we have:

$$\omega(G) = \bigcap_{p\in\pi(T)} \omega_p(G) \geq W \cap H \cap \Big(\bigcap_{p\in\pi_0\cap\pi(D)} \omega_p(G)\Big)$$

and, therefore:

$$|G:\omega(G)| \leq |G:W|\,|G:H| \prod_{p\in\pi_0\cap\pi(D)} |G:\omega_p(G)| \leq b_5(m)\,b_2(m)\,b_9(m)^{|\pi_0\cap\pi(D)|}\,.$$

This concludes case B), and the proof of the Theorem. ■

Finally, we show that Theorem 4.8 is no longer true when we drop the assumption of solubility. In fact, for any prime p, there exist (non soluble) locally finite V_p-groups such that $|G:\omega(G| = \infty$. The examples that we give, have been constructed by H. Heineken, and appear in his paper [6] on IT-groups (that is groups in which every infinite subnormal subgroup is normal).

EXAMPLE 7. Let p be a prime. For every positive integer i let $X_i \cong Y_i$ be groups such have:

i) $X' \geq Z(X_i)$ and $X_i'/Z(X_i)$ is a non abelian finite simple group;

ii) X_i/X_i' is a cyclic group of order p^{i+1};

iii) $Z(X_i)$ is a cyclic group of order p^i.

If p^{i+1} divides $q^t - 1$ (q a prime); then a suitable normal section of $GL(p^{i+1}, q^t)$ will do for X_i.

Let

$$X_i = \langle X_i', x_i\rangle\,, \quad H_i = \langle Y_i', y_i\rangle\,, \quad Z(X_i) = \langle u_i\rangle\,, \quad Z(H_i) = \langle v_i\rangle$$

and let

$$W_i = \langle s_i, t_i;\ s_i^{p+1} = t_i^{p+1} = [s_i, t_i, t_i] = [s_i, t_i, s_i] = 1\rangle\,.$$

In the direct product $X_i \times Y_i \times W_i$ take:

$$D_i = \langle X_i', Y_i', x_i s_i, y_i t_i\rangle / \langle u_i v_i, u_i[s_i, t_i]^p\rangle\,.$$

Then, $Z(D_i)$ is cyclic of order p^{i+1}, let $Z(D_i) = \langle d_i \rangle$ (where d_i is the omage of $[s_i, t_i]$ in D_i). If R_i is the nilpotent residual of D_i, then $|D_i'/R_i| = p$ and D_i/R_i, which is generated by the images of $x_i s_i$ and $y_i t_i$, is a nilpotent V_p-group. Moreover, $D_i' \geqq Z(D_i)$ and $D_i'/Z(D_i) \simeq \simeq X_i'/Z(X_i) \times Y_i'/Z(Y_i)$.

Now, examination of the subnormal subgroups of D_i shows that:

(1) if $A \trianglelefteq B \operatorname{sn} D_i$ and B/A is nilpotent, then either B/A is central in D_i or $A \geqq R_i$.

Let $K = \operatorname{Dir}_{i \in \mathbb{N}} D_i$ and $V = \langle d_i^{-1} d_{i+1}^p;\ i \in \mathbb{N} \rangle \leqq Z(K)$. The group we want to consider is the quotient $G = K/V$. We show that G is a V_p-group.

Let $H/V \operatorname{sn} K/V$; then $H \operatorname{sn} K$. Consider the natural projections $\pi_i(H)$ of H on D_i, for each i. Since H is subnormal in K, $\pi_i(H) \operatorname{sn} D_i$, $D_i \cap H \trianglelefteq \pi_i(H)$ and $\pi_i(H)/D_i \cap H$ is nilpotent, for every $i = 1, 2, \ldots$. If $\pi_i(H)/D_i \cap H$ is a central factor of D_i for every $i = 1, 2, \ldots$, write $n = n(H) = 0$. Otherwise let $n = n(H)$ be the minimal integer such that $\pi_n(H)/D_n \cap H$ is not central in D_n.

Let $n \neq 0$ and $j \in \mathbb{N}$, $j > n$. If $\pi_j(H)/D_j \cap H$ were not central in D_j, then by (1), $R_j \leqq D_j \cap H$; in particular $D_j \cap H \geqq \langle d_i^p \rangle$. Since $H \geqq V$ we have therefore $H \geqq \langle d_l;\ l < j \rangle$, whence $D_n \cap H \geqq \langle d_n \rangle = Z(D_n)$ and, consequently, $\pi_n(H)/D_n \cap H$ would be central in D_n, against our choice of n. Thus, for every $j \neq n$, $\pi_j(H)/D_j \cap H$ is central in D_j (this holds also when $n = 0$), and so, if $j \neq n = n(H)$:

$$[H, D_j] = [\pi_j(H), D_j] \leqq D_j \cap H \leqq H$$

whence $D_j \leqq N_K(H)$. It follows that $|K : N_K(H)| = |D_n : N_{D_n}(H)| \leqq p$.

This holds for every subnormal subgroup H of K, containing V; thus $G = K/V$ is a V_p-group. On the other hand it is easy to check that G has no normal T-subgroups of finite index, and so

$$|G : \omega(G)| = \infty .$$

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