RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

M. L. C. FERNANDES
F. ZANOLIN

# Repelling conditions for boundary sets using Liapunov-like functions. I. - Flow-invariance, terminal value problem and weak persistence

*Rendiconti del Seminario Matematico della Università di Padova*, tome 80 (1988), p. 95-116
<http://www.numdam.org/item?id=RSMUP_1988__80__95_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 80 (1988)

# Repelling Conditions for Boundary Sets Using Liapunov-Like Functions.

## I. – Flow-Invariance, Terminal Value Problem and Weak Persistence.

M. L. C. FERNANDES - F. ZANOLIN (*)

## 1. Introduction and notation.

Let $f: J \times \Omega \to \mathbb{R}^d$ be a continuous function, where $J$ is a nondegenerate real interval and $\Omega \subset \mathbb{R}^d$ is a nonempty open set. $\mathbb{R}^d$ is the $d$-dimensional real euclidean space endowed with the usual inner product $(\cdot|\cdot)$, norm $|\cdot| = (\cdot|\cdot)^{\frac{1}{2}}$ and distance $\operatorname{dist}(\cdot, \cdot)$. For given subsets $A, M \subset \mathbb{R}^d$, with $A \subset M$, we denote by $\operatorname{int}_M A$, $\operatorname{fr}_M A$ and $\operatorname{cl}_M A$, respectively the interior, boundary and closure of $A$, relatively to $M$. The subscript $M$ is omitted when the topological operations are considered with respect to $\mathbb{R}^d$. $B(x, r)$ is the open ball of center $x \in \mathbb{R}^d$ and radius $r > 0$ and $B[x, r] := \operatorname{cl} B(x, r)$ is its closure. For a (nonempty) set $K \subset \mathbb{R}^d$, we also define

$$B(K, r) := \{y \in \mathbb{R}^d : \exists x \in K, |y - x| < r\} = \bigcup_{x \in K} B(x, r)$$

(*) Indirizzo degli AA.: M. L. C. FERNANDES: International School for Advanced Studies, Strada Costiera 11, 34014 Trieste (Italy). On leave of absence from Faculdade de Ciências da Universidade de Lisboa with a scholarship of Fundação Calouste Gulbenkian; F. ZANOLIN: Dipartimento di Matematica e Informatica, Università, Via Zanon 6-8, 33100 Udine (Italy). Work performed under the auspices of GNAFA-CNR and partially supported by the fund ZAN 6Q4 (MPI 60%).

and

$$B[K, r] := \{y \in \mathbb{R}^d : \exists x \in K, |y - x| \leq r\} = \bigcup_{x \in K} B[x, r] .$$

We recall that $B[K, r] = \mathrm{cl}\, B(K, r)$ when $K$ is compact.

Let $a := \inf J$ and $b := \sup J$ (possibly $a = -\infty$ or $b = +\infty$). For a given noncontinuable solution $x(\cdot)$ of the equation

$$x' = f(t, x) \tag{1}$$

satisfying the initial condition

$$x(t_0) = x_0 \in \Omega \qquad (t_0 \in J) , \tag{2}$$

we denote by $I_x$ the right maximal interval of existence of $x(\cdot)$, and by $t_x := \sup I_x$. The uniqueness of solutions to (1)-(2) is not assumed. All the considered solutions are supposed noncontinuable.

In the case of the autonomous system, i.e. $f(t, x) \equiv f(x)$, $f : \Omega \to \mathbb{R}^d$, we'll assume throughout the paper $J = \mathbb{R}_+$.

Let $G$, $S \subset \mathbb{R}^d$ be nonempty sets, with $G \subset \Omega$ and $S \cap G = \emptyset$. Our purpose is to find various criteria relating the behaviour of the solutions of (1)-(2), for $x(t_0) \in G$, with the set $S$.

More precisely, we'll consider the following situations:

1) solutions of (1) never reach $S$ from the set $G$;

2) there are not solutions $x(\cdot)$ of (1), with $x(t) \in G$ for all $t \in I_x$ and such that

$$\lim_{t \to t_x^-} x(t) = u \in S ;$$

3) solutions of (1), with $x(t) \in G$ for all $t \in I_x$, are asymptotically far from $S$;

4) $S$ is a repeller for the solutions of (1) which remain in $G$.

The cases listed above are significant for studying, respectively, flow-invariance for $G$ ([8]), terminal value problems ([20]), weak persistence ([17]) and uniform persistence ([5]). In each of these examples the most interesting situation is when the set $S$ is a piece, or even the whole boundary, of $G$. The set $G$ will be assumed open with respect to a suitable flow-invariant set $N \subset \Omega$ (possibly $N = \Omega$). A motivation for such a choice is the following:

It is well known that flow-invariant sets play a fundamental role in the study of the qualitative behaviour of the solutions of (1) and they represent an useful tool for investigating global existence, stability and periodicity problems ([26], [43]). For this reason, so far the theory has been extensively developed for closed (or relatively closed with respect to $\Omega$) sets ([42], [8], [33], [40]), beginning with the pioneering work of M. Nagumo ([31]). In more recent years, various concepts stronger than flow-invariance, like e.g. persistence, have been proposed in order to deal with differential systems modeling ecological or biochemical phenomena ([14], [18], [37]). A common feature of all these definitions, related to persistence-type problems is that they require, for any solution $x(\cdot)$ of (1)-(2) with $x(t_0) = x_0 \in \operatorname{int} M$ ($M$ being a suitable set), that $x(t)$ remains far away from the boundary of $M$ as time evolves. Indeed, the ecological explaination of this requirement is that, in the concrete models, points of fr $M$ represent extintion states for some of the considered species. Then, from this point of view, conditions ensuring the flow-invariance of open sets, like $G = \operatorname{int} M$, or the repulsivity of the boundary, are significant as well. On the other hand, dealing with the terminal value problem the assumption that $G$ is open (for instance, $G = \Omega$) is very natural and commonly considered in the literature ([20], [39], [3]). In this framework conditions of repulsivity of the set $S$ with respect to the solutions of (1) will be used to get nonexistence results for the terminal value problems with end point in $S$.

The proof of the results makes use of families of Liapunov-like functions and differential inequalities. This method, which is very classical for studying stability ([36], [35]), flow-invariance of closed sets ([27], [2]) or existence of periodic solutions ([34], [24], [2]), has been recently proposed also in the study of various definitions of persistence ([37], [22], [23], [13]), but, in all these latter quoted papers only one Liapunov function is considered.

Our approach is more linked to the concepts of bound set and of bounding functions introduced by R. E. Gaines (see [28]) and employed by R. E. Gaines and J. Mawhin [15] and J. Mawhin [29] in the search of periodic solutions of equation (1) via the coincidence degree theory. However, we stress the fact that compared with [15] and [29], our conditions on the bounding functions are not contained in the preceding ones. In particular, the assumptions considered here are more general than the analogous ones concerning the «attractive bound sets» ([28]) and, when applied to the periodic problem, allow us to get new results. Moreover, these tools, already exploited in our recent paper ([10]), dealing with persistence for nonautonomous systems, can be also applied for getting weak flow-invariance of closed sets. We show, by examples, that classical theorems can be redemonstrated or improved along this way.

Dealing with the terminal value problem, we get nonexistence results, while, in what concerns weak persistence (see sect. 6 for the pertinent definition), we obtain new conditions for nonautonomous systems as well as we improve the previous ones.

We recall that, in [10], we have shown that many natural theorems which hold for the autonomous systems cannot be extended to equation (1). From this point of view, our result seems to be significant.

Finally, we point out that although we restrict ourselves to the assumption that $f$ is continuous, most of the results of the paper could be extended to the case of a function $f(\cdot,\cdot)$ verifying the Carathéodory assumptions.

In a subsequent paper ([12]), using the same technique, we study persistence and existence of periodic solutions.

## 2. Basic definitions.

Let $N \subset \Omega$ be a nonempty set. We say that $N$ is a *flow-invariant* set (or a positively invariant set) for equation (1) if for each $(t_0, x_0) \in \in J \times N$ and each $x(\cdot)$ solution of (1), with $x(t_0) = x_0 \in N$, we have $x(t) \in N$, for every $t \in I_x$. In a similar manner, one defines $N$ negatively invariant.

Troughout the paper we assume:

$N \subset \Omega$ *is a flow-invariant set for* (1),

$G \subset N$ *is nonempty and open relatively to* $N$,

$S \subset \mathbb{R}^d$ *is nonempty with* $S \cap G = \emptyset$.

In what follows we give various conditions ensuring that solutions of (1) do not reach or approach $S$ from the set $G$. Of course, the most significant situation (which will be considered in all the applications) is that $S \cap \operatorname{fr} G \neq \emptyset$. Repulsivity conditions for $S$, with respect to $G$, will be obtained by means of families of Liapunov-like functions. To this purpose, we recall some basic facts from Liapunov theory.

For a function $V = V(t, x) \in C(J' \times \Omega', \mathbb{R})$ (the space of continuous functions from $J' \times \Omega'$ into $\mathbb{R}$), where $J' \subset J$, $\Omega' \subset \Omega$ and $J' \times \Omega'$ is an open subset of $\mathbb{R}^{d+1}$, we define

$$\dot{V}(t, x) := \liminf_{h \to 0^+} [V(t + h, x + hf(t, x)) - V(t, x)]/h .$$

It is known that for $x(t)$ a solution of (1) we have

$$\dot{V}(t, x(t)) = D_+ V(t, x(t))$$

provided that $V$ is locally lipschitzian with respect to $x$ (see [43]). Here, $D_+$ denotes the lower right Dini derivative (however, in all the results of the paper, any other Dini derivative could be considered instead of $D_+$). We also recall that, if $V$ is of class $C^1$, then

$$\dot{V}(t, x) = (\partial V/\partial t)(t, x) + ((\partial V/\partial x)(t, x)|f(t, x)) .$$

In the particular case in which $V = V(x)$ is independent on the $t$-variable, we denote by $\nabla V$ the gradient of $V$, wherever it is defined and, for $c \in \mathbb{R}$, by $[V < c] := \{x \in \operatorname{dom} V \colon V(x) < c\}$. The sets $[V \leqslant c]$, $[V = c]$ and $[V > c]$ are defined analogously.

## 3. Nonreachable sets and flow-invariance.

In this section we propose some sufficient criteria which guarantee that points of $S$ are not reached from $G$ along the solutions of equation (1). Such conditions are then used in order to get the flow-invariance of $G$, by choosing $S = \operatorname{fr}_N G$.

DEFINITION 1. Let $u \notin G$. We say that $u$ is *reachable through* $G$ if there is a solution $x(\cdot)$ of equation (1) such that $x(t_0) = x_0 \in G$, $x(t_1) = u$ for some $t_0 < t_1 \in I_x$ and $x(t) \in G$ for all $t \in [t_0, t_1[$. The set $S$ is said to be *nonreachable through* $G$ if there is no point $u \in S$ which is reachable through $G$.

Since $G \subset N$, with $N$ a flow-invariant set, we have that $x(t) \in N$ for each $t \in I_x$, and $x(\cdot)$ solution to (1)-(2) with $x_0 \in G$. Accordingly, no point belonging to $S \setminus \mathrm{fr}_N G$ is reachable through $G$. Then it will be enough to find sufficent conditions for the nonreachability of the set

$$S^* := S \cap \mathrm{fr}_N G . \tag{3}$$

Some obvious consequences of the definition are the following:

(*a*) $G$ is flow-invariant if and only if there is $S \supset \mathrm{fr}_N G$ with $S$ nonreachable through $G$.

(*b*) If $S \cap N$ is negatively invariant, then $S$ is nonreachable through $G$.

The main result of this section is:

THEOREM 1. *Suppose that for each* $t \in J \setminus \{a\}$ *and* $u \in S^*$ *there are an* $\varepsilon = \varepsilon_{t,u} > 0$ *and two continuous functions* $V = V_{t,u}$ *and* $\psi = \psi_{t,u}$, *with*

$$V = V(s, x): [t - \varepsilon, t] \times B(u, \varepsilon) \to \mathbb{R}, \qquad \textit{locally lipschitzian in } x$$

*and*

$$\psi = \psi(s, x): [t - \varepsilon, t[ \times ]-\infty, 0[ \to \mathbb{R},$$

*such that*

($i_1$) $V(t, u) = 0$ ,

($i_2$) $V(s, x) < 0$, *for all* $s \in ]t - \varepsilon, t[$ *and* $x \in G \cap B(u, \varepsilon)$

*and*

($i_3$) $\dot{V}(s, x) \leqslant \psi(s, V(s, x))$, *for all* $s \in ]t - \varepsilon, t[$ *and* $x \in G \cap B(u, \varepsilon)$ ,

*hold.*

*Then* $S$ *is nonreachable through* $G$ *provided that there is* $\eta = \eta_{t,u}$, *with* $0 < \eta$ $(< \varepsilon)$, *such that*

($c_1$) *for every* $0 < h$, $k < \eta$, *the problem*

$$w' = \psi(s, w) , \qquad w(t - h) = -k \tag{4}$$

*has a maximal solution* $r = r(s)$, *with* $\liminf_{s \to t_r^-} r(s) < 0$.

PROOF. Suppose, by contradiction, thath there exists a point $u_1 \in S$ which is reachable through $G$. This means that equation (1) has a solution $x(\cdot)$ with $x(t_0) = x_0 \in G$ and $x(t_1) = u_1$, for some $t_0 < t_1 \in I_x$, such that $x(s) \in G$ for all $s \in [t_0, t_1[$. Obviously, $t_1 > a = \inf J$, and $x(s) \in N$ for all $s \in [t_0, t_1]$ (as $N$ is flow-invariant), so that $u_1 \in S^*$ (defined by (3)).

Let $\varepsilon$, $\eta$, $V$ and $\psi$ be chosen according to the assumptions for the pair $(t_1, u_1)$. By $(i_1)$ and $(i_2)$ there is $\varepsilon_1$, with $0 < \varepsilon_1 < \varepsilon$, such that $-\eta < V(s, x) < 0$ for all $s \in ]t_1 - \varepsilon_1, t_1[$ and $x \in G \cap B(u, \varepsilon_1)$.

Let $h_1$, with $0 < h_1 < \min\{\eta, \varepsilon_1, t_1 - t_0\}$, be such that $x(s) \in B(u_1, \varepsilon_1)$ for each $s \in [t_1 - h_1, t_1[$.

Then the scalar function $v\colon [t_1 - h_1, t_1[ \to \mathbb{R}$, defined by

$$v(s) := V(s, x(s)) , \tag{5}$$

satisfies the following conditions:

$$-\eta < v(s) < 0 \qquad \text{for each } s \in [t_1 - h_1, t_1[ , \tag{6}$$

$$\lim_{s \to t_1^-} v(s) = 0 . \tag{7}$$

Moreover, by $(i_3)$, we can compute the right Dini derivative of $v(\cdot)$ and have

$$D_+ v(s) = D_+ V(s, x(s)) = \dot{V}(s, x(s)) \leqslant \psi\big(s, V(s, x(s))\big) = \psi(s, v(s)) , \tag{8}$$

for all $s \in [t_1 - h_1, t_1[$.

Set $-k_1 := v(t_1 - h_1)$. By (6), $-k_1 > -\eta$.

Let $r = r(s)$ be a maximal solution of the problem

$$w' = \psi(s, w) , \qquad w(t_1 - h_1) = -k_1 ,$$

according to $(c_1)$ and denote by $I_r$ the right maximal interval of existence of $r(\cdot)$. By the assumptions, we can take $r(\cdot)$ such that $r(s) < 0$ for all $s \in I_r$ and

$$\liminf_{s \to t_r^-} r(s) < 0 , \tag{9}$$

with $t_r := \sup I_r \leqslant t_1$.

With (8) and a classical comparison theorem ([25, th. 1.4.1]), we get

$$v(s) \leqslant r(s)\,, \qquad \text{for all } s \in I_r. \tag{10}$$

Now, two possibilities arise. Either $r(\cdot)$ is defined on $[t_1 - h_1, t_1[$ and then, by (9) and (10) we have $\liminf_{s\to t_1^-} v(s) < 0$, contradicting (7), or $t_r < t_1$. In this case, $\lim_{s\to t_r^-} r(s) = -\infty$ and hence, by (10), $\lim_{s\to t_r^-} v(s) = -\infty$ too, contradicting (6).

The proof is complete. ■

REMARK 1. From the proof of the theorem it is clear that it would be sufficient to require that the function $V$ is defined on $[t-\varepsilon, t[\times \times (G \cap B(u, \varepsilon))$ assuming, instead of $(i_1)$, the more general condition

$(i_1')$ $\lim_{\substack{s\to t^- \\ G\ni x\to u}} V(s, x) = 0.$

We list now some easily verifiable hypotheses on the function $\psi$ which ensure the validity of $(c_1)$.

A first condition we propose, which obviously implies $(c_1)$, is

$(c_2)$ *for every* $0 < h,\ k < \eta$, *the problem* (4) *has a maximal solution bounded away from* 0 *on its right maximal interval of existence.*

Assumption $(c_2)$ is related to an analogous one for the comparison equation in the Cafiero uniqueness theorem ([6], [32, *p.* 188]). In particular $(c_2)$ (and hence $(c_1)$) is verified whenever $\psi$ admits the factorization

$$\psi(s, z) := -\varrho(s)\,\varphi(|z|)\,, \tag{11}$$

with

$$\varphi\colon ]0, +\infty[\ \to\ ]0, +\infty[ \qquad \text{and} \qquad \varrho\colon [t-\varepsilon, t[\ \to \mathbb{R}$$

continuous functions such that

$$\lim_{\theta\to 0^+} \int_\theta^{u_0} (1/\varphi(\xi))\, d\xi = +\infty \qquad (u_0 > 0) \tag{12}$$

and

$$(13)\qquad \int_{s_1}^{s_2} \varrho(s)\,ds \geqslant -m = \text{constant}, \textit{ for all } s_1, s_2 \in \,]t-\varepsilon, t[, \textit{ with } s_1 < s_2.$$

Indeed, let $r = r(s)$ be a maximal solution of (4) and let $\alpha \leqslant t_r \leqslant t$ be such that $r(s) < 0$ for all $s \in [t-h, \alpha[$, with $\alpha$ maximal. From (11) and (13), we easily get, for $s \in [t-h, \alpha[$,

$$\int_{|r(s)|}^{|r(t-h)|} (1/\varphi(\xi))\,d\xi = -\int_{s}^{t-h} \big(r'(\sigma)/\varphi(|r(\sigma)|)\big)\,d\sigma = -\int_{t-h}^{s} \varrho(\sigma)\,d\sigma \leqslant m\,.$$

Hence, (12) ensures that there is $\delta > 0$ such that $|r(s)| \geqslant \delta$, i.e.

$$(14)\qquad r(s) \leqslant -\delta < 0\,, \quad \text{for all } s \in [t-h, \alpha[\,.$$

Finally, (14) implies $\alpha = t_r$ and $(c_2)$ follows. ■

In particular, (11), (12) and (13) allow to take $\psi(s, z) \equiv 0$.

For other conditions related to the Cafiero uniqueness theorem, see [32].

A significant example is when $\psi$ can be continuously extended to $[t-\varepsilon, t] \times ]-\infty, 0]$, with $\psi(s, 0) = 0$ for all $s \leqslant t$. Then, condition $(c_1)$ is satisfied provided that the equation $w' = \psi(s, w)$ fulfills the backward uniqueness of the solutions to Cauchy problems. Criteria for the validity of such situation can be easily adapted from the known results of forward uniqueness, like e.g. [30], [41], [4], [3] and [16]. However, we stress the fact that assumption $(c_1)$ is more general than the request of backward uniqueness. In the case in which $\psi$ can be defined continuously on $[t-\varepsilon, t] \times ]-\infty, 0]$, the condition $(c_1)$ is trivially satisfied if $\psi(t, 0) < 0$, while it is not verified when $\psi(t, 0) > 0$.

We present now an application of theorem 1 to the flow-invariance of the set $G$, in which $S = \mathrm{fr}_N\, G$ is chosen. For the sake of simplicity and in order to have a comparison with other results previously appeared in the literature, we consider only the case in which the functions $V$ are independent of the time variable, i.e. $V(s, x) = V(x)$.

COROLLARY 1. *Suppose that for each $t \in J \setminus \{a\}$ and $u \in \mathrm{fr}_N\, G$, there are constants $\varepsilon > 0$, $\alpha \geqslant 1$ and two continuous functions $\varrho\colon J \to \mathbb{R}$ and*

$V: B(u, \varepsilon) \to \mathbb{R}$, *with* $V(u) = 0$, *such that*

($j_1$) $V$ *is of class* $C^1$,

($j_2$) $G \cap B(u, \varepsilon) \subset [V < 0]$

*and*

($j_3$) $\dot{V}(s, x) = (f(s, x)|\nabla V(x)) \leqslant - \varrho(s)|V(x)|^\alpha$,
*for every* $s \in J$, *with* $t - \varepsilon < s < t$ *and* $x \in G \cap B(u, \varepsilon)$.

*Then* $G$ *is a flow-invariant set for equation* (1).

PROOF. We apply theorem 1 with $S = \mathrm{fr}_N G$, $V(s, x) = V(x)$ and $\psi(s, z) = - \varrho(s)|z|^\alpha$. Then we have only to check that condition ($c_1$) is fulfilled. To this purpose, we just observe that ($c_1$) is implied by ($c_2$) and this last assumption holds since $\psi$ decomposes as in (11), with $\varphi(\xi) = \xi^\alpha$ and $\varphi$, $\varrho$ satisfy (12) and (13) respectively (in particular (13) holds as $\varrho$ is continuous and defined on $J$). Then $\mathrm{fr}_N G$ is not reachable through $G$ and so $G$ is flow-invariant. ■

REMARK 2. A careful reading of the proof of theorem 1 shows that hypothesis ($j_1$) could be relaxed to

($j_1'$) $V$ *is differentiable on* $G \cap B(u, \varepsilon)$.

We also observe that if there is a *compact* set $K$, with $G \subset K \subset \Omega$, then corollary 1 also provides (via a classical result of continuability) the global existence in the future of all the solutions to the initial value problem (1)-(2), for each $x(t_0) = x_0 \in G$. From this point of view, corollary 1 extends and improves [28, th. 7.4], where it is assumed, in particular, that

($j_3'$) $(f(t, u)|\nabla V(u)) < 0$, *for each* $(t, u) \in J \times \mathrm{fr}\, G$.

An obvious continuity argument shows that ($j_5'$) implies ($j_3$), with $\varrho \equiv 0$, for some $\varepsilon = \varepsilon_{t,u}$.

In [15] and [28], an open bounded set $G$ for which ($j_3'$) holds with respect to a family of Liapunov-like functions $V$'s is called an *attractive bound set* and the functions are called *bounding functions*. Arguments based on the use of bound sets have been considered in the study of boundary value problems ([15], [28], [29]) and singular perturbation theory ([38], [7]); the case of $V$ not differentiable is discussed in [44]. The possibility of replacing ($j_3'$) (considered in [15], [28]) with ($j_3$) (proposed here) suggests the possibility to get improvements or new applications of the concept of bound set, in the case of boundary value problems. An example in this direction is outlined in [12].

We finally note that the assumptions in corollary 1 are, in some sense, sharp. In particular, an easy example on the real line shows that our result is false if $(j_2)$ is replaced with $G \cap B(u, \varepsilon) \subset [V \leqslant 0]$. However, by a direct proof, a variant of corollary 1 could be obtained, assuming, instead of $(j_2)$ and $(j_3)$,

$$G \cap B(u, \varepsilon) \subset [V \leqslant 0]$$

*and*

$$(f(s, x)|\nabla V(x)) < -\varrho(s)|V(x)|^\alpha,$$

$$\textit{for every } s \in J, \quad \textit{with} \quad t - \varepsilon < s < t \quad \textit{and} \quad x \in G \cap B(u, \varepsilon),$$

respectively.

For applications of corollary 1 and related results, see, for instance, [9] and [11].

## 4. Application to weakly flow-invariant sets.

A set $M \subset \Omega$ is said to be *weakly flow-invariant* (or weakly positively invariant) for equation (1) if for each $(t_0, x_0) \in J \times M$, there is a solution $x(\cdot)$ of (1), with $x(t_0) = x_0 \in M$, such that $x(t) \in M$, for every $t \in I_x$ (see [42]). Weak flow-invariance for closed sets has been widely explored in the literature, starting with M. Nagumo ([31]). We give two simple applications of corollary 1 and get weak flow-invariance for closed sets which can be obtained as closure of open flow-invariant sets. The results are achieved via the following lemma which can be easily deduced from classical facts (Kamke's lemma [21, p. 14]).

LEMMA 1. *Let $(f_n)_n$ be a sequence of continuous functions, $f_n\colon J \times \Omega \to \mathbb{R}^m$ converging uniformly to $f$ on compact subsets of $J \times \Omega$. Suppose that, for each $n \in \mathbb{N}$, $M(\subset N)$ is weakly flow-invariant for*

$$(1_n) \qquad x' = f_n(t, x)\,.$$

*Then* $\mathrm{cl}_N M$ *is weakly flow-invariant for* (1).

In particular (see [42]), the closure of a weakly flow-invariant set is weakly flow-invariant and there are easy examples showing that the closure does not preserve the flow-invariance. Indeed, even in the situation described in corollary 1, $\mathrm{cl}_N G$ need not be positively invariant.

EXAMPLE 1. Let $J = \mathbb{R}$, $\Omega = \mathbb{R}^2$ and

$$f(t, x) = f(t, x_1, x_2) = (1, -x_1^2 + x_2 + 2|x_2|^{\frac{1}{2}}) ,$$

for every $t$ and $x = (x_1, x_2)$. Take $N = \Omega$ and

$$G = \{(x_1, x_2) : x_1 > -\tfrac{1}{4},\ x_2 < 0 ,\ x_2 < x_1\} .$$

Choosing, for $u = (u_1, u_2) \in \mathrm{fr}\, G = \mathrm{fr}_N\, G$,

$$V = V_u : \mathbb{R}^2 \to \mathbb{R}$$

defined by

$$V(x) = V(x_1, x_2) := \begin{cases} x_2 , & \text{for } u_1 > 0 ,\ u_2 = 0 , \\ x_2 - x_1 , & \text{for } -\tfrac{1}{4} < u_1 \leqslant 0 ,\ u_2 = u_1 , \\ -x_1 - \tfrac{1}{4} , & \text{for } u_1 = -\tfrac{1}{4} ,\ u_2 \leqslant -\tfrac{1}{4} , \end{cases}$$

we easily check that $(j_1)$, $(j_2)$ and $(j_3')$ are satisfied and hence $G$ is positively invariant and $\mathrm{cl}_N G = \mathrm{cl}\, G$ is weakly flow-invariant. But $\mathrm{cl}\, G$ is not flow-invariant as $x(t) = (t, t^2)$ is a solution of (1) on $[0, +\infty[$, with $x(0) = (0, 0) \in \mathrm{cl}\, G$ and $x(t) \notin \mathrm{cl}\, G$ for $t > 0$.

Our first application is a classical result (see, e.g. [27], [1]).

COROLLARY 2. *Let* $K = [V \leqslant 0]$, *where* $V : \Omega \to \mathbb{R}$ *is a* $C^1$ *function such that* $\nabla V(x) \neq 0$, *for each* $x \in [V = 0]$ (i.e. 0 is a regular value for $V$). *Then* $K$ *is weakly flow-invariant, provided that*

$$(k_1) \quad (f(t, u) | \nabla V(u)) \leqslant 0, \ \textit{for every} \ (t, u) \in J \times [V = 0]$$

*holds.*

PROOF. First observe that $\mathrm{int}_\Omega K = [V < 0]$, $\mathrm{fr}_\Omega (\mathrm{int}_\Omega K) = [V = 0]$ and $\mathrm{cl}_\Omega (\mathrm{int}_\Omega K) = K$, that is $K$ is a regularly closed set. Then define, on $J \times \Omega$, $f_n(t, x) := f(t, x) - (1/n) \nabla V(x)$ and observe that $(f_n)_n$

converges to $f$ uniformly on compact subsets of $J\times\Omega$. Now we are in position to apply, for each $n\in\mathbf{N}$, corollary 1 (with $(j_3')$ instead of $(j_3)$) to the equation $x'=f_n(t,x)$, with respect to $N=\Omega$, $G=\operatorname{int}_\Omega K$ and $V_u=V$ (constant with respect to $u$). Hence, lemma 1 provides the thesis. ■

Corollary 2 can be found in [1, th. 16.9], with a different proof. However, in [1], uniqueness of the solutions to Cauchy problems (in fact lipschitzianity) is assumed. Our result also improves [28, cor. 7.2] Corollary 2 may be also derived by Nagumo theorem.

For the next application, we recall that a vector $\eta\neq 0$ is said to be an *outer normal* to a convex set $K$, at a point $u\in\operatorname{fr}K$, if $K\subset\{x\in\mathbf{R}^m\colon (x-u|\eta)\leqslant 0\}$.

COROLLARY 3. *Let $K\subset\Omega$ be a convex set, closed relatively to $\Omega$ and with non empty interior. Suppose that, for each $u\in\operatorname{fr}_\Omega K$, there is an outer normal $\eta_u\neq 0$ (to $K$ at $u$), such that*

$(k_2)$ $\quad (f(t,u)|\eta_u)\leqslant 0,$ *for each* $t\in J$

*holds. Then $K$ is weakly flow-invariant.*

PROOF. First note that, by convexity,

$$\operatorname{cl}_\Omega(\operatorname{int}_\Omega K)=K \qquad\text{and}\qquad \operatorname{fr}_\Omega(\operatorname{int}_\Omega K)=\operatorname{fr}_\Omega K\,.$$

Take $p\in\operatorname{int}K$ and let $\varepsilon>0$ be such that $B(p,\varepsilon)\subset K$. Then, for each $u\in\operatorname{fr}_\Omega K$, $(p+(\varepsilon/2)\eta_u-u|\eta_u)\leqslant 0$, and so

$$(p-u|\eta_u)\leqslant -(\varepsilon/2)|\eta_u|^2<0\,. \tag{15}$$

Define, on $J\times\Omega$, $f_n(t,x):=f(t,x)+(1/n)(p-x)$ and observe that $(f_n)_n$ converges to $f$ uniformly on compact subsets of $J\times\Omega$. Now we apply, for each $n\in\mathbf{N}$, corollary 1 to the equation $x'=f_n(t,x)$, with respect to $N=\Omega$, $G=\operatorname{int}_\Omega K$ and $V(x)=V_u(x):=(x-u|\eta_u)$ (see [28, p. 156]). Then assumption $(j_3')$, with respect to equation $(1_n)$, follows immediately from $(k_2)$ and (15). Hence lemma 1 gives the result. ■

Corollary 3 extends ([28, cor. 7.1.]). We notice that it seems not possible to get corollary 3 as a direct consequence of Nagumo theorem. Indeed, Nagumo theorem, in its equivalent geometric version ([2]), requires that $f(t,u)\in T_K(u)$, for each $t\in J$ and $u\in\operatorname{fr}_\Omega K$, where $T_K(u)$ is the Bouligand's contingent cone to $K$ at $u$. In terms of outer normals, such condition reads as $(f(t, u)|\eta_u) \leqslant 0$, for each $t \in J$, $u \in \mathrm{fr}_\Omega K$ and *each* outer normal $\eta_u$ (to $K$ at $u$). Therefore, corollary 3 could be obtained from Nagumo theorem only after using some continuity and density argument (like, for instance, Mazur density theorem [19]). Corollary 3 also improves [38, th. 2.1] (at least in the case of convex sets with nonempty interior), where the case of $f(\cdot,\cdot)$ locally lipschitzian is considered. A less general version of corollary 3 is implicitly used in [24, proof th. 3.2].

We finally observe that example 2 shows that a strict inequality in $(k_2)$ is not sufficient to ensure the flow-invariance of $K$.

## 5. Nonexistence results for the terminal value problem.

Let $N$, $G$ and $S$ be as in section 2 and suppose $b \notin J$.

DEFINITION 2. Let $u \notin G$. We say that $u$ is *asymptotically reachable through* $G$ if there is a solution $x(\cdot)$ of equation (1) such that $x(t_0) = x_0 \in G$, $x(t) \in G$, for all $t \in [t_0, t_x[$ and $\lim_{t \to t_x^-} x(t) = u$.

We observe that this concept is independent of that of reachability through $G$.

It is obvious that points of $S \setminus \mathrm{fr}\, G$ are not asymptotically reachable through $G$.

The above definition is significant in dealing with terminal value problems. Indeed, the TVP

$$x' = f(t, x), \quad x(t) \in G \qquad \text{for } t < b,$$
$$\lim_{t \to b^-} x(t) = u \notin G,$$

has no solution if $u$ is not asymptotically reachable through $G$. We notice that if one is interested in obtaining nonexistence results for the TVP, then it is sufficient to consider in the definition of « asymptotic reachability » only solutions $x(\cdot)$ of (1) with $t_x = b$ (i.e. solutions globally defined in the future).

A slight modification of the proof of theorem 1 gives the following

THEOREM 2. *Suppose that for each*

$$(t, u) \in ]a, b] \times (S \cap \mathrm{fr}\, G \cap \mathrm{fr}\, \Omega) \cup \{b\} \times (S \cap \mathrm{fr}\, G \cap \Omega)$$

*there are* $\varepsilon = \varepsilon_{t,u} > 0$, $\sigma = \sigma_{t,u} < t$ *and two continuous functions* $V = V_{t,u}$ *and* $\psi = \psi_{t,u}$, *with*

$$V = V(s, x) \colon ]\sigma, t[\times B(u, \varepsilon) \to \mathbb{R} ,$$

*locally lipschitzian in* $x$ (*uniformly in* $s$) *and*

$$\psi = \psi(s, x) \colon ]\sigma, t[\times ]-\infty, 0[ \to \mathbb{R} ,$$

*such that*

($i_1'$) $\lim\limits_{\substack{s\to t^- \\ G\ni x\to u}} V(s, x) = 0$,

($i_2'$) $V(s, x) < 0$, *for all* $s \in ]\sigma, t[$ *and* $x \in G \cap B(u, \varepsilon)$ ,

($i_3'$) $\dot{V}(s, x) \leqslant \psi(s, V(s, x))$, *for all* $s \in ]\sigma, t[$ *and* $x \in G \cap B(u, \varepsilon)$ .

*Then there is no point* $u \in S$ *which is asymptotically reachable through* $G$, *provided that*

($c_3$) *for every* $\sigma < \tau < t$ *and* $0 < k < \varepsilon$, *the problem*

$$w' = \psi(s, w) , \qquad w(\tau) = -k \tag{16}$$

*has a maximal solution* $r = r(s)$, *with* $\liminf\limits_{s\to t_r^-} r(s) < 0$.

Proof. Suppose, by contradiction, that there exists a point $\bar{u} \in S$ such that $\bar{u}$ is asymptotically reachable through $G$. Accordingly, let $x(\cdot)$ be a solution of equation (1) with $x(t_0) = x_0 \in G$, such that $x(s) \in G$ for $s \in [t_0, t_x[$ and $\lim\limits_{s\to t_x^-} x(s) = \bar{u}$. Observe that $\bar{u} \in S \cap \operatorname{fr} G$ and $t_x = b$ whenever $\bar{u} \in \Omega$.

Now, we just repeat the argument employed in the proof of theorem 1 (choosing $\sigma$, $\varepsilon$, $V$ and $\psi$ depending on the pair $(t_x, \bar{u})$) to achieve a contradiction. ■

In [20] and [39], it is assumed $J = [a, b[$ and $G = \Omega$ and the terminal value problem

$$x' = f(t, x) , \qquad \lim_{t\to b^-} x(t) = u \in \operatorname{fr} \Omega \tag{17}$$

is considered. In order to get a nonexistence result for the TVP (17), an obvious variant of theorem 2 can be derived, just assuming the hypotheses to hold for the sole pair $(b, u)$ (take $S = \{u\}$ and consider only the solutions $x(\cdot)$ with $t_x = b$). In particular, we have

COROLLARY 4. *Let* $u \in \operatorname{fr} \Omega$. *Suppose that there are constants* $\varepsilon > 0$, $\sigma < b$, $\alpha \geqslant 1$ *and two continuous functions*

$$\varrho\colon [\sigma, b[ \to \mathbb{R} \quad \textit{and} \quad V\colon B(u, \varepsilon) \to \mathbb{R} \quad \textit{with } V(u) = 0\,,$$

*such that*

$$V \textit{ is of class } C^1, \qquad \Omega \cap B(u, \varepsilon) \subset [V < 0]$$

*and*

$$(f(s, x)|\nabla V(x)) \leqslant -\varrho(s)|V(x)|^\alpha, \textit{ for every } s \in [\sigma, b[ \textit{ and } x \in \Omega \cap B(u, \varepsilon).$$

*Then the* TVP (17) *has no solution provided that there is a constant* $m \geqslant 0$ *such that*

$$\int_{t_1}^{t_2} \varrho(s)\, ds \geqslant -m\,, \qquad \textit{for all } \sigma \leqslant t_1 \leqslant t_2 < b\,.$$

PROOF. We apply the above described variant of theorem 2, with $V(s, x) = V(x)$ and $\psi(s, z) = -\varrho(s)|z|^\alpha$. In order to check condition $(c_3)$, it is sufficient to recall remark 1. ∎

Corollary 4 is, in some sense, complementary to analogous existence results for (17), obtained by comparison principles. See, for instance, [20] and [39].

## 6. Weakly repelling sets and weak persistence.

Let $N$, $G$, $S$ and $S^*$ be as in section 2. Furthemore, we suppose $b = \sup J \notin J$.

DEFINITION 3. We say that $S$ is *weakly repelling with respect to* $G$ if there is an open neighbourhood $A$ of $S$ such that, for each $x(\cdot)$ solution of equation (1), with $x(t) \in G$, for all $t \in [t_0, t_x[$, and $t_0 \in J$, there is $t_1 \in [t_0, t_x[$ such that $x(t_1) \notin A$.

This definition is strictly related to the concept of weak persistence previously considered in the literature (see [17], [5]). Precisely, for a set $M \subset N$, with $\operatorname{int}_N M \neq \emptyset$, we say that the system (1) is *weakly persistent* in $N$ with respect to the set $M$, if for each $(t_0, x_0) \in J \times \operatorname{int}_N M$ and $x(\cdot)$ solution of (1)-(2), we have $x(t) \in \operatorname{int}_N M$, for each $t \in [t_0, t_x[$, and $\limsup_{t \to t_x^-} \operatorname{dist}\big(x(t), \operatorname{fr}_N M\big) > 0$.

Weak persistence was introduced by H. Freedman and P. Waltman in [14] for a particular class of equations and subsequently considered in the present generality by T. C. Gard in [17], who named it as strong flow-invariance. A definition in terms of dynamical systems in metric spaces has been used recently in [5].

It is clear that if $S$ is *compact* and weakly repelling with respect to $G$, then there is $\delta > 0$ (independent of the solutions) such that $\limsup_{t \to t_x^-} \operatorname{dist}\big(x(t), S\big) \geqslant \delta > 0$, for each $x(\cdot)$ solution of (1) such that $x(t) \in G$, for all $t \in [t_0, t_x[$. Hence, as $\operatorname{fr}_N(\operatorname{int}_N M) \subset \operatorname{fr}_N M$, we get that, if $\operatorname{fr}_N M$ is *compact*, then a sufficient condition for the weak persistence with respect to the set $M$, is that, for the choice $G = \operatorname{int}_N M$ and $S = \operatorname{fr}_N G$, $G$ is flow-invariant and $S$ is weakly repelling with respect to $G$ (indeed, in this case, we have *uniform weak persistence* with respect to $M$).

Motivated by the problems coming from the ecological applications (population dynamics, [14], [17], [18]), we suppose, through this section that

*$N$ is closed with respect to $\Omega$ and $S \subset \Omega$.*

The case $S \not\subset \Omega$ is implicitly considered in section 5 dealing with the terminal value problem and results, in this situation, can be obtained combining the arguments in theorems 2 and 3.

THEOREM 3. *Assume $S^* = S \cap \operatorname{fr}_N G$ compact and suppose that there are an open set $\Omega'$, with $S^* \subset \Omega' \subset \Omega$ and two continuous functions $V$ and $\psi$,*

$$V = V(t, x) \colon [\alpha, b[ \times \Omega' \to \mathbb{R} \quad \textit{locally lipschitzian in } x$$

*and*

$$\psi \colon [\alpha, b[ \times ]-\infty, 0[ \to \mathbb{R} \qquad (\alpha \in J)\,,$$

*such that*

$(l_1)$ $$\lim_{\substack{t\to b^- \\ x\in G,\, \mathrm{dist}(x, S^*)\to 0}} V(t, x) = 0,$$

$(l_2)$ $V(t, x) < 0,$ *for all* $(t, x) \in [\alpha, b[\times (G \cap \Omega'),$

$(l_3)$ $\dot{V}(t, x) \leqslant \psi(t, V(t, x)),$ *for all* $(t, x) \in [\alpha, b[\times (G \cap \Omega').$

*Then* $S$ *is weakly repelling with respect to* $G$ *provided that there is* $\delta > 0$ *such that*

$(r)$ *for every* $0 < k$ *and* $\alpha \leqslant \tau < b$, *the problem*

$$w' = \psi(t, w)\,, \qquad w(\tau) = -k$$

*has a maximal solution* $r = r(t)$, *with* $\liminf_{s\to t_r^-} r(t) \leqslant -\delta.$

PROOF. First we observe that it is sufficient to prove that $S^*$ is weakly repelling with respect to $G$. In fact, let $A$ be an open neighbourhood of $S^*$ such that, according to the definition, each solution of (1), remaining in $G$ for the future, escapes from $A$ at some time. Consider the set $A' := A \cap (\Omega \setminus \mathrm{cl}_N G)$. As $N$ is (relatively) closed and $S \subset \Omega$, we have that $A'$ is an open set and $A' \subset S$. Now, let $x(\cdot)$ be a solution of (1), such that $x(t) \in G$, for all $t \in [t_0, t_x[$. Then, $x(t) \notin$ $\notin \Omega \setminus \mathrm{cl}_N G$ for all $t \geqslant t_0$ and there is $t_1 \geqslant t_0$ such that $x(t_1) \notin A$. Hence $x(t_1) \notin A'$ and so we have proved that $S$ is weakly repelling with respect to $G$.

Fix an $\varepsilon$ such that $0 < \varepsilon < \delta$ and let $\varrho_0 > 0$ be such that $B[S^*, \varrho_0] \subset \Omega'$ (as $S^*$ is compact). Using $(l_1)$ we find $\beta$ and $\varrho$ such that $\alpha \leqslant \beta < b$ and $0 < \varrho \leqslant \varrho_0$, with

$$\inf\{V(t, x) : t \in [\beta, b[,\ x \in G \cap B[S^*, \varrho]\} \geqslant -\varepsilon\,. \tag{18}$$

Now, let $x(\cdot)$ be a solution of (1)-(2) such that $x(t) \in G$, for all $t \in [t_0, t_x[$. We want to prove that there is $t_1 \geqslant t_0$ such that $x(t_1) \notin B(S^*, \varrho)$.

Suppose, by contradiction, that $x(t) \in B[S^*, \varrho]$, for all $t \in [t_0, t_x[$. Then, as $B[S^*, \varrho] \subset \Omega$ is compact, we have $t_x = b$. Consider the function $v(t) := V(t, x(t))$, with $v\colon [\gamma, b[ \to \mathbb{R}$, where $\gamma := \max\{t_0, \beta\}$. We have

$$(19)\qquad v(t) < 0\,, \qquad \text{for all } t \in [\gamma, b[\,, \qquad (\text{by } (l_2))\,,$$

$$(20)\qquad v(t) \geqslant -\varepsilon\,, \qquad \text{for all } t \in [\gamma, b[\,, \qquad (\text{by } (18))\,,$$

and

$$(21)\qquad D_+ v(t) \leqslant \psi(t, v(t))\,, \qquad \text{for all } t \in [\gamma, b[\,, \qquad (\text{by } (l_3))\,.$$

Let $r(\cdot)$ be a maximal solution, according to $(r)$, of

$$w' = \psi(t, w)\,, \qquad w(\gamma) = v(\gamma)\,.$$

By a comparison theorem, $v(t) \leqslant r(t)$, for all $t \in [\gamma, t_r[$. Hence, from (20) and (21), $\liminf_{t\to t_r^-} r(t) \geqslant \liminf_{t\to t_r^-} v(t) \geqslant -\varepsilon > -\delta$, and a contradiction is achieved with respect to condition $(r)$.

Then, for some $t_1 \geqslant t_0$, $x(t_1) \notin A := B(S^*, \varrho_0)$. Hence, $S^*$ is weakly repelling with respect to $G$.

Theorem 3 is similar to a previous result obtained by T. C. Gard ([17, th. 2]). The main differences consist in the fact that in [17] the compactness assumption is replaced by the global existence of the solutions of the comparison equation and that a type of weak persistence (namely, of « uniform » type) stronger then the one in [17], is considered here. Variants of theorem 3 can be easily derived. In particular, an extension of [17, th. 2] can be obtained.

## REFERENCES

[1] H. Amann, *Gewöhnliche Differentialgleichungen*, Walter de Gruyter, Berlin, 1983.

[2] J.-P. Aubin - A. Cellina, *Differential inclusions*, Springer-Verlag, Berlin and New York, 1984.

[3] S. R. Bernfeld - R. D. Driver - V. Lakshmikantham, *Uniqueness for ordinary differential equations*, Math. Systems Theory, **9** (1976), pp. 359-367.

[4] J. M. Bownds, *A uniqueness theorem for $y' = f(x, y)$ using a certain factorization of $f$*, J. Differential Equations, **7** (1970), pp. 227-231.

[5] G. Butler - H. I. Freedman - P. Waltman, *Uniformly persistent systems*, Proc. Amer. Math. Soc., **96** (1986), pp. 425-430.

[6] F. Cafiero, *Sui teoremi d'unicità relativi ad un'equazione differenziale ordinata del primo ordine*, Giorn. Mat. Battaglini, **78** (1948), pp. 193-215.

[7] K. W. Chang - F. A. Howes, *Nonlinear singular perturbation phenomena: theory and applications*, Springer-Verlag, Berlin and New York, 1984.

[8] M. G. Crandall, *A generalization of Peano's existence theorem and flow-invariance*, Proc. Amer. Math. Soc., **36** (1972), pp. 151-155.

[9] M. L. C. Fernandes, *Invariant sets and periodic solutions for differential systems*, Magister Ph. Thesis, I.S.A.S., Trieste, 1986.

[10] M. L. C. Fernandes - F. Zanolin, *Remarks on strongly flow-invariant sets*, J. Math. Anal. Appl., **128** (1987), pp. 176-188.

[11] M. L. C. Fernandes - F. Zanolin, *On periodic solutions, in a given set, for differential systems* (preprint).

[12] M. L. C. Fernandes - F. Zanolin, *Repelling conditions for boundary sets using Liapunov-like functions. - II: Persistence and periodic solutions* (preprint).

[13] A. Fonda, *Uniformly persistent semi-dynamical systems*, Proc. Amer. Math. Soc. (to appear).

[14] H. I. Freedman - P. Waltman, *Mathematical analysis of some three species food-chain models*, Math. Biosci., **33** (1977), pp. 257-276.

[15] R. R. Gaines - J. Mawhin, *Coincidence degree and nonlinear differential equations*, Lecture Notes in Mathematics, vol. 568, Springer-Verlag, Berlin, 1977.

[16] T. Gard, *A generalization of the Naguno uniqueness criterion*, Proc. Amer. Math. Soc., **70** (1978), pp. 166-172.

[17] T. C. Gard, *Strongly flow-invariant sets*, Appl. Analysis, **10** (1980), pp. 285-293.

[18] T. C. Gard - T. G. Hallam, *Persistence in food webs-1. Lotka Volterra food chains*, Bull. Math. Biol., **41** (1979), pp. 877-891.

[19] P. M. Gruber, *Aspects of convexity and its applications*, Expo. Math., **2** (1984), pp. 47-83.

[20] T. G. Hallam, *A comparison principle for terminal value problems in ordinary differential equations*, Trans. Amer. Math. Soc., **169** (1972), pp. 49-57.

[21] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

[22] J. Hofbauer, *A general cooperation theorem for hypercycles*, Monatsh. Math., **91** (1981), pp. 233-240.

[23] V. Hutson, *A theorem on average Liapunov functions*, Monatsh. Math., **98** (1984), pp. 267-275.

[24] M. A. Krasnosel'skii, *The operator of translation along trajectories of of differential equations*, Amer. Math. Soc., Providence, R.I., 1968.

[25] V. LAKSHMIKANTHAM - S. LEELA, *Differential and integral inequalities*, vol. I, Academic Press, New York, 1969.

[26] J. P. LA SALLE, *The stability of dynamical systems*, Reg. Conf. Ser. in Math., SIAM, Philadelp ia, 1976.

[27] J. MASSERA, *Contributions to stability theory*, Ann. Math., **64** (1956), pp. 182-206.

[28] J. MAWHIN, *Functional analysis and boundary value problems*, in « Studies in ordinary differential equations », vol. 14 (J. K. Hale, ed.), The Math. Assoc. of America, U.S.A., 1977.

[29] J. MAWHIN, *Topological degree methods in nonlinear boundary value problems*, Reg. Conf. Ser. in Math., CBMS no. 40, Amer. Math. Soc., Providence, R.I., 1979.

[30] M. NAGUMO, *Eine hinreichende Bedingung für die Unität der Lösung von Differentialgleichungen erster Ordnung*, Japan J. Math., **3** (1926), pp. 107-112.

[31] M. NAGUMO, *Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen*, Proc. Phys.-Math. Soc. Japan, **24** (1942), pp. 551-559.

[32] L. C. PICCININI - G. STAMPACCHIA - G. VIDOSSICH, *Ordinary differential equations in $R^n$, problems and methods*, Springer-Verlag, Berlin and New York, 1984.

[33] R. M. REDHEFFER - W. WALTER, *Flow-invariant sets and differential inequalities in normed spaces*, Appl. Analysis, **5** (1975), pp. 149-161.

[34] R. REISSIG - G. SANSONE - R. CONTI, *Qualitative Theorie nichtlinearer Differentialgleichungen*, Cremonese, Roma, 1963.

[35] N. ROUCHE - P. HABETS - M. LALOY, *Stability theory by Liapunov's Direct Method*, Springer-Verlag, Berlin and New York, 1977.

[36] L. SALVADORI, *Famiglie ad un parametro di funzioni di Liapunov, nello studio della stabilità*, Symposia Math., **6** (1971), pp. 309-330.

[37] P. SCHUSTER - K. SIGMUND - R. WOLFF, *Dynamical systems under constant organization. - III: Cooperative and competitive behavior of hypercycles*, J. Differential Equations, **32** (1979), pp. 357-368.

[38] M. TURINICI, *A singular perturbation result for a system of ordinary differential equations*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **27** (1983), pp. 273-282.

[39] G. VIDOSSICH, *Solutions of Hallam's problem on the terminal comparison principle for ordinary differential inequalities*, Trans. Amer. Math. Soc., **220** (1976), pp. 115-132.

[40] P. VOLKMANN, *Über die positive Invaranz einer abgeschlossenen Teilmenge eines Banachschen Raumes bezüglich der Differentialgleichung $u' = f(t, u)$*, J. reine angew. Math., **285** (1976), pp. 59-65.

[41] D. V. V. WEND, *Existence and uniqueness of solutions of ordinary differential equations*, Proc. Amer. Math. Soc., **23** (1969), pp. 27-33.

[42] J. A. YORKE, *Invariance for ordinary differential equations*, Math. Systems Theory, **1** (1967), pp. 353-372.

[43] T. Yoshizawa, *Stability theory by Liapunov's second method*, The Math. Soc. of Japan, Tokyo, 1966.

[44] F. Zanolin, *Bound sets, periodic solutions and flow-invariance for ordinary differential equations in $R^n$ some remarks*, in « Colloquium on Topological Methods in BPVs for ODEs », ISAS, Rend. Ist. Mat. Univ. Trieste, **19** (1987), pp. 76-92.

Manoscritto pervenuto in redazione il 21 agosto 1987.