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PIER CARLO CRAIGHERO

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A Remark on Abhyankar's Space Lines.

PIER CARLO CRAIGHERO (*)

SUNTO - In questa nota si prova la rettificabilità della seconda delle linee di Abhyankar, $\mathcal{C}_6: (t + t^6, t^5, t^4)$; inoltre si costruisce un esempio di linea di tipo (6, 5, 4) che risulta elementarmente rettificabile.

SUMMARY - In this paper the rectifiability of the second of Abhyankar's lines, $\mathcal{C}_6: (t + t^6, t^5, t^4)$, is proved; moreover an example is given of a (6, 5, 4)-line which is tamely rectifiable.

0. Introduction.

The notations will be as in [2] and [3]. In [2] the rectifiability of the first of Abhyankar's lines, $\mathcal{C}_5: (t + t^5, t^4, t^3)$, was proved; the automorphism $\Phi_5: \mathbb{A}_k^3 \to A_k^3$, such that $\Phi_5(\mathcal{C}_5)$ is a straight line, was explicitly found: it turns out to be the product of an automorphism Ψ of « Nagata's type », and of a tame automorphism Λ (an automorphism of $\mathbb{A}_k^3$, $\Psi: (F, G, H)$, is called of « Nagata's type » if the forms of maximum degree of F, G, H, are such that no one of them is a polynomial in the other two, and moreover every two of them are not the power of a same form; an automorphism Λ of $\mathbb{A}_k^3$ is called tame if it is the product of linear and triangular automorphisms, and it is called elementary in [2]).

About at the same time when the author got the result published in [2], $\mathcal{C}_5$ was indipendently recognized rectifiable also by A. Sathaye,

(*) Indirizzo dell'A.: Istituto di Matematica Applicata, Università, via Belzoni 7, 35100 Padova (Italy).

as himself communicated at Oberwohlfach: however up to now nothing was known about $\mathcal{C}_n$: $(t+t^n, t^{n-1}, t^{n-2})$, $n>5$. Since the interest on the subject seems to spread (see for example [5]), I thought it opportune to publish these new results.

In § 1 the rectifiability of $\mathcal{C}_6$: $(t+t^6, t^5, t^4)$ is proved; the automorphism Φ_6, which rectifies it, is in the same condition as Φ_5, and should not be tame, by the same argument.

In § 2 an example is given of a (6, 5, 4)-line $\mathcal{C}_6'$, which is actually tamely rectifiable: this shows that the fact of finding, for rectification of the line $\mathcal{C}_6$, only an automorphism such as Φ_6, non-tame by Nagata's conjecture (see [6], Part 2, pp. 41-52), is not due only to the particular kind of the triplet $(n, n-1, n-2)$ of the degrees of the polynomials defining $\mathcal{C}_6$.

In § 3 a definition of wild line is given, in such a way that $\mathcal{C}_6$ (and also $\mathcal{C}_5$) is wild, whereas $\mathcal{C}_6'$ is not wild: perhaps this could be the correct approach, pointing to the reasonably probable fact that, in order to rectify a wild line, as $\mathcal{C}_6$ (and $\mathcal{C}_5$), an automorphism which, according to Negata's conjecture, is « wild » (= non-tame) is necessary.

1. We want to show that $\mathcal{C}_6$: $(t+t^6, t^5, t^4)$ is a rectifiable line. Let us consider

$$x(t)=t+t^6\,,\qquad y(t)=t^5\,,\qquad z(t)=t^4\,.$$

We find

$$t=x(t)-t^6\,;\qquad t^6=x(t)^2z(t)-2t^{11}-t^{16}\,;$$
$$t^{11}=x(t)^3z(t)^2-3t^{16}-3t^{21}-2t^{26}\,;\qquad t^{16}=z(t)^4\,;$$
$$t^{21}=y(t)z(t)^4\,;\qquad t^{26}=y(t)^2z(t)^4\,.$$

From all this it follows

$$(*)\qquad \begin{aligned} t&=x(t)-t^6=x(t)-[x(t)^2z(t)-2t^{11}-t^{16}]=\\ &=x(t)-x(t)^2z(t)+2t^{11}+t^{16}=\\ &=x(t)-x(t)^2z(t)+2\,[x(t)^3z(t)^2-3t^{16}-3t^{21}-t^{26}]+t^{16}=\\ &=x(t)-x(t)^2z(t)+2x(t)^3z(t)^2-5t^{16}-6t^{21}-2t^{26}=\\ &=x(t)-x(t)^2z(t)+2x(t)^3z(t)^2-5z(t)^4-6y(t)z(t)^4-2y(t)^2z(t)^4\,.\end{aligned}$$

Now we observe that it results

$$(**) \qquad y(t)^2 = x(t)z(t) - y(t)\,.$$

From $(*)$ and $(**)$ we get

$$t = x(t) - x(t)^2 z(t) + 2x(t)^3 z(t)^2 - 5z(t)^4 - 6y(t)z(t)^4 - \\ - 2[x(t)z(t) - y(t)]z(t)^4 = -5z(t)^4 + x(t)[1 - 2z(t)^5] - \\ - x(t)^2 z(t) + 2x(t)^3 z(t)^2 - 4z(t)^4 y(t)\,.$$

Now the surface of 5-th order

$$\mathcal{F} = \{-5Z^4 + X(1 - 2Z^5) - X^2 Z + 2X^3 Z^2 - 4Z^4 Y = 0\}$$

is a linear plane, that is a monoidal surface of the kind

$$\{f(X, Z) - g(X, Z)\ Y = 0\}$$

isomorphic to a plane: indeed it satisfies the conditions of Lemma 4 in [7] (of course after interchanging Y and Z in that statement, then making $u = Z$ and $v = X$). Moreover, always from [7], it follows that $\mathcal{F}$ is equivalent to a plane (see the Theorem, § 1, [7]), and by consequence we have that $\mathcal{C}_6$ is a rectifiable line, applying Prop. 3, in [2].

REMARK 1. Proceeding as above for $\mathcal{C}_6$, we find, in the case of the line $\mathcal{C}_7$: $(t + t^7, t^6, t^5)$,

$$(*') \qquad t = F(x, y, z)$$

with

$$F(X, Y, Z) = X - X^2 Z + 2X^3 Z^2 - 5X^4 Z^3 + 14Z^5 + 28YZ^5 + \\ + 20Y^2 Z^5 + 5Y^3 Z^5$$

where we write shortly x, y, z, in place of $x(t) = t + t^7$, $y(t) = t^6$, $z(t) = t^5$; now, starting again from $(**)$, which holds for every Abhyankar's line $\mathcal{C}$: $(t + t^n, t^{n-1}, t^{n-2})$, we can substitute twice $y(t)^2$ in $(*')$, but we get a monoidal surface which is no longer a linear plane: this shows that $\mathcal{C}_7$ is somewhat a crucial case in the problem of rectifying Abhyankar's lines.

Still in degree 5, there are also two other lines, namely

$$\mathcal{C}_5': (t^5, t + t^4, t^3) \qquad \text{and} \qquad \mathcal{C}_5'': (t^5, t^4, t + t^3)$$

which are particularly difficult to rectify. If these two quintics were rectifiable, then it would not be hazardous to conjecture that every quintic line is rectifiable.

2. The automorphism Φ_6 that rectifies $\mathcal{C}_6$ can be calculated following Corollary 2 in [7], which leads to the construction of an automorphism Ψ', such that $\Psi'(\mathcal{C}_6)$ is a plane, then by multiplying Ψ' by an obvious tame automorphism Λ': Ψ' turns out to be of Nagata's type, as Ψ in the case of $\mathcal{C}_5$. Being $\Lambda' \circ \Psi' = \Phi_6$, also Φ_6, as Φ_5, should not be tame, according to Nagata's conjecture (see Introduction for reference). In order to show that the difficulty of finding a possible tame automorphism that can rectify $\mathcal{C}_6$ is not due only to the particular triplet of the degrees of the polynomials defining $\mathcal{C}_6$, that is a triplet of three consecutive integers, here we give an example of a (6, 5, 4)-line which is actually tamely rectifiable.

Let us consider the following curve

$$\mathcal{C}_6': \big(x_1(t), y_1(t), z_1(t)\big)$$

where we have

$$\begin{aligned} x_1(t) &= -24t + 14t^3 + 12t^4 + 6t^5 + t^6\,, \\ y_1(t) &= 289t - 960t^2 - 872t^3 - 432t^4 - 72t^5\,, \\ z_1(t) &= 4t + 4t^2 + 4t^3 + t^4\,, \end{aligned}$$

together with the following triangular automorphisms of A

$$\Omega: \begin{pmatrix} X \\ Y + 12X + X^2 + 24Z^2 - Z^3 \\ Z \end{pmatrix},$$

$$H_1: \begin{pmatrix} X + 24Y - 14Y^3 - 12Y^4 - 6Y^5 - Y^6 \\ Y \\ Z \end{pmatrix},$$

$$H_2: \begin{pmatrix} X \\ Y \\ Z - 4Y - 4Y^2 - 4Y^3 - Y^4 \end{pmatrix}.$$

We find

$$H_2 \circ H_1 \circ \Omega(\mathcal{C}_6') = (0, t, 0)$$

so that $\mathcal{C}_6$ is a tamely rectifiable line.

REMARK 2. It is very unlikely that there exists a $(5, 4, 3)$-line which can be tamely rectified: such a line is surely wild (see Definition 2, in the following paragraph), and everything seems to indicate that a wild line $\mathcal{C}$ cannot even be tamely equivalent to a line of lower total degree, this being naturally defined as the sum of the degrees of those polynomials defining $\mathcal{C}$ which are not zero.

Let us recall that a proof that a $(5, 4, 3)$-line is not tamely rectifiable would be an indirect proof of Nagata's conjecture on the existence of non-tame automorphisms of $\mathbb{A}_k^3$ (see [4], § 2).

3. After the examples considered in § 1) and § 2), we think it natural to put the following definitions.

DEFINITION 1. Given any pair of polynomials $(P(t), Q(t))$, we shall denote by Sem $(P(t), Q(t))$ the semigroup of all the degrees of $F(P(t), Q(t))$, with $F(X, Y)$ varying in the set of all the polynomials in two variables such that $F(P(t), Q(t)) \neq 0$.

DEFINITION 2. Given a line $\mathcal{C}$: $(x(t), y(t), z(t))$, we call $\mathcal{C}$ a wild line if no one of the polynomials $x(t)$, $y(t)$, $z(t)$, is a constant, and if, calling m, n, p, respectively the degrees of $x(t)$, $y(t)$, $z(t)$, we have

$$m \notin \operatorname{Sem}(y(t), z(t)), \quad n \notin \operatorname{Sem}(x(t), z(t)), \quad p \notin \operatorname{Sem}(x(t), y(t)).$$

We recall that the generators of $\operatorname{Sem}(P(t), Q(t))$ can be calculated making use of the procedure given in [1]; see also [4], § 4.

We have then the following

PROPOSITION. $\mathcal{C}_6$ *is a wild line*; $\mathcal{C}'_6$ *is not a wild line.*

PROOF. We find, for $\mathcal{C}_6$: $(t + t^6, t^5, t^4)$,

$$6 \notin \text{Sem}\,(t^5, t^4) = \langle 4, 5 \rangle\,, \qquad 5 \notin \text{Sem}\,(t + t^6, t^4) = \langle 4, 6, 7 \rangle\,,$$
$$4 \notin \text{Sem}\,(t + t^6, t^5) = \langle 5, 6 \rangle\,,$$

whence the statement about $\mathcal{C}_6$.

As for $\mathcal{C}'_6$: $(x_1(t), y_1(t), z_1(t))$, with $x_1(t)$, etc., given above in § 2), let us consider the polynomial $F(X, Z) = 12X + X^2 + 24Z^2 - Z^3$: we find

$$\deg F(x_1(t), z_1(t)) = \deg\big(12x_1(t) + x_1(t)^2 + 24z_1(t)^2 - z_1(t)^3\big) =$$
$$= \deg(-288t + 960t^2 + 872t^3 + 432t^4 + 72t^5) = 5$$

so $\deg y_1(t) = 5 \in \text{Sem}\,(x_1(t), z_1(t))$, and $\mathcal{C}'_6$ is not a wild line.

REMARK 2. Of course the fact that, according to Definition 2, $\mathcal{C}_6$ is a wild line, could account for the other fact of finding only an automorphism as Φ_6 that can rectify it.

We also note that a line $\mathcal{C}$: $(x(t), y(t), z(t))$ which is not wild, is clearly tamely equivalent to a line of lower total degree.

Finally we observe that what is remarkable about $\mathcal{C}'_6$ is the following: generally speaking, a $(6, 5, 4)$-line $\mathcal{C}$, which is not wild, is tamely equivalent only to a $(5, 4, 3)$-line, which in its turn is clearly wild, so that, for the rectification of $\mathcal{C}$, we are confronted with a new difficulty; with $\mathcal{C}'_6$ instead, we skip this impasse, coming to a tame rectification of it.

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