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Characters and the Generalized Fitting Subgroup.

Gabriel Navarro (*)

Summary - In this short paper, we prove that the generalized Fitting subgroup of a finite group G equals the intersection of the inertia subgroups of the irreducible characters of the chief factors of G.

1. A group G is said to be quasinilpotent if given any chief factor X of G, every automorphism of X induced by an element of G is inner.

It is well known that the finite quasinilpotent groups form a Fitting formation.

For any group G, the generalized Fitting subgroup $F^*(G)$ is the set of all elements x of G which induce an inner automorphism on every chief factor of G.

If we denote by N^* the class of finite quasinilpotent groups and G is a finite group, in [1] it is proved that the N^*-radical of G equals $F^*(G)$.

Every group in this article is finite and every character is complex.

The main object of our paper will be the following.

Theorem A. Let G be a group. Then $F^*(G) = \cap \{I_G(\chi): \chi \in \in \operatorname{Irr}(K/L)\}$, where K/L runs over the chief factors of G.

In order to prove theorem A, we use the following result which appears in [2]. This theorem relies on the simple group classification.

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THEOREM B (G. Seitz, W. Feit, 1984). Let G be a simple group and let $\alpha \in \mathrm{Aut}(G)$. Suppose tat $\alpha(C) = C$ for all conjugacy classes C of G. Then $\alpha \in \mathrm{Inn}(G)$.

It is not difficult to see that theorem A and theorem B are equivalent.

2. LEMMA1 . A group G is quasinilpotent if and only if every irreducible character of any chief factor of G is G-invariant.

PROOF. Suppose that G is quasinilpotent. Let K/L be a chief factor of G and $\chi \in \mathrm{Irr}(K/L)$. If $g \in G$, there exists $k \in K$ satisfying $gxg^{-1}L = kxk^{-1}L$ for all $x \in K$. Then, $\chi^g = \chi^k = \chi$.

Let K/L be a chief factor of G, $g \in G$ and suppose that the irreducible characters of K/L are G-invariant. If $\chi \in \mathrm{Irr}(K/L)$ and $x \in G$, it is clear that $\ker(\chi^x) = (\ker \chi)^x$. Consequently, $\ker \chi \trianglelefteq G$. But $L \leqslant \ker \chi \leqslant K$, and since K/L is a chief factor of G, we have $\ker \chi = L$ if $\chi \neq 1_K$. Then K/L is simple.

By Brauer's lemma on character tables [6.32; 3], we have that g fixes all conjugacy classes in K/L. Theorem B gives now that g is inner.

LEMMA 2. Suppose that $G = G_0 > G_1 > \ldots > G_n = 1$, where $G_i \trianglelefteq G$ and G_{i-1}/G_i is either abelian or the direct product of non-abelian simple groups ($i = 1, \ldots, n$). If $x \in G$, fixes the irreducible characters of G_{i-1}/G_i for each $i = 1, \ldots, n$, then x fixes the irreducible characters of any chief factor of G.

PROOF. Let K/L be a chief factor of G such that $G_{i-1} \geqslant K > L \geqslant G_i$.

Write $H = G_{i-1}/G_i$, and let $\mu \in \mathrm{Irr}(K/L)$. If H is abelian, $K/L \leqslant G_{i-1}/L$ an abelian group. By [5.5; 3], $\mu = \chi_K$ for some

$$\chi \in \mathrm{Irr}(G_{i-1}/L) \subseteq \mathrm{Irr}(G_{i-1}/G_i) .$$

Then $\chi^x = \chi$ and consequently $\mu^x = \mu$.

Suppose that H is non-abelian.

By hypothesis, $H = S_1 \times \ldots \times S_r$ where $S_1, \ldots, S_r$ are non-abelian simple groups. Hence, $K/G_i = S_{j_1} \times \ldots \times S_{j_s}$ for certain indices $j_1, \ldots, j_s$. Then $H = K/G_i \times S/G_i$.

Let $\mu \in \mathrm{Irr}(K/L) \subseteq \mathrm{Irr}(K/G_i)$. Since $[K, S] \leqslant G_i$, we have that μ is H-invariant. Let $\beta \in \mathrm{Irr}(H|\mu)$. Then $\beta_{K/G_i} = u\mu$. Hence, $\beta_K = u\mu$. By hypothesis, $\beta^x = \beta$ and then $\mu^x = \mu$.

In the general case, if K/L is a chief factor of G, K/L is G-isomorphic to some other chief factor K_1/L_1, where $G_{i-1} \geqslant K_1 > L_1 \geqslant G_i$ for certain i. If φ is the G-isomorphism, each $\mu_1 \in \mathrm{Irr}(K_1/L_1)$ is the image under φ of certain $\mu \in \mathrm{Irr}(K/L)$. Since $\mu_1^x = \mu_1$, we conclude that $\mu^x = \mu$.

PROOF OF THEOREM A. Let $I(G) = \{g \in G: \mu^g = \mu,\ \forall \mu \in \mathrm{Irr}(K/L),\ \forall K/L$ chief factor of $G\}$, a characteristic subgroup of G.

We have to prove that $F^*(G) = I(G)$.

If $g \in F^*(G)$, g^{-1} induces an inner automorphism on any chief factor of G. If $\mu \in \mathrm{Irr}(K/L)$, where K/L is a chief factor of G, then $gxg^{-1}L = kxk^{-1}L$ for certain $k \in K$ and for all $x \in K$. Then $\mu^g = \mu^k = \mu$. Consequently, $F^*(G) \leqslant I(G)$.

We consider now a chief series of G of the form

$$G = G_0 > \ldots > G_m = I(G) > G_{m+1} > \ldots > G_n = 1\,.$$

Thus G_{i-1}/G_i is the direct product of isomorphic simple groups. If $x \in I(G)$, $\mu^x = \mu$, $\forall \mu \in \mathrm{Irr}(G_{i-1}/G_i)$ for each $i = m+1, \ldots, n$. By Lemma 2, $\mu^x = \mu$ for all $\mu \in \mathrm{Irr}(K/L)$, for all K/L chief factor of $I(G)$.

By Lemma 1, we have that $I(G)$ is quasinilpotent. Consequently, $I(G) \leqslant F^*(G)$.

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