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Conjugate π-normally embedded fitting functors

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Conjugate π-Normally Embedded Fitting Functors.

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1. Introduction.

All groups considered in this paper belong to the class $\mathcal{S}$ of all finite soluble groups. A subgroup X of G is p-normally embedded in G if each of its Sylow p-subgroups is a Sylow p-subgroup of a normal subgroup of G. A subgroup X of G is normally embedded in G if it is p-normally embedded for each prime p. If $\mathcal{F}$ is a Fischer class, then the $\mathcal{F}$-injectors of G are normally embedded (see [10]). Fitting classes whose injectors are normally embedded are called normally embedded Fitting classes. Such Fitting classes have many interesting properties (see for example [7, 11]).

In [3] the concept of Fitting functor is introduced as a map f which assigns to each $G \in \mathcal{S}$ a non-empty set $f(G)$ of subgroups of G such that

$$\{\alpha(X)\colon X \in f(G)\} = \{\alpha(G) \cap Y\colon\ Y \in f(H)\}$$

whenever α is a monomorphism of G into H with $\alpha(G) \trianglelefteq H$. Motivation for the definition of Fitting functor is provided by injectors and radicals of Fitting classes. A number of properties of Fitting functors are developed in [3, 4, 8]. A Fitting functor f is called p-normally embedded provided that $f(G)$ consists of p-normally embedded subgroups of G for each $G \in \mathcal{S}$. f is said to be normally embedded if f is

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p-normally embedded for all primes p. Normally embedded Fitting functors are classified in Satz 6.4 and Satz 7.4 of [3].

In this paper we study a generalization of the mentioned concepts which results from considering sets of primes π and Hall π-subgroups instead of prime numbers p and Sylow p-subgroups. We restrict ourselves to those Fitting functors f for which $f(G)$ is a conjugacy class of subgroups of G for all $G \in \mathcal{S}$, the so-called conjugate Fitting functors.

Let f be a conjugate Fitting functor, $G \in \mathcal{S}$, $V \in f(G)$, $V_\pi \in \mathrm{Hall}_\pi(V)$ and $V_\pi \leqq G_\pi \in \mathrm{Hall}_\pi(G)$. By a result in [8], $V_\pi \trianglelefteq G_\pi$ if and only if $V_\pi \in \mathrm{Hall}_\pi(G_{\mathfrak{L}_\pi(f)})$. This result provides motivation for studying conjugate π-normally embedded Fitting functors. In section 3 we obtain a number of properties of conjugate π-normally embedded Fitting functors. For example, f is π-normally embedded if and only if each member of the Lockett section of f is π-normally embedded. Let $\pi = \bigcup \{\pi_i : i \in I\}$. Then f is π-normally embedded if and only if f is π_i-normally embedded and $\mathfrak{L}_{\pi_i}'(f) = \mathfrak{L}_\pi(f)\,\mathcal{S}_{\pi_i'}$ for all $i \in I$. Let I be an index set and let $\{\pi(\lambda) : \lambda \in I\}$ be a partition of $\mathbb{P}$, the set of all primes. A Fitting functor g is said to be I-normally embedded if g is $\pi(\lambda)$-normally embedded for each $\lambda \in I$. I-normally embedded Fitting functors are classified. Moreover, if f is a conjugate I-normally embedded Fitting functor, then a description of f_*, the smallest member of the Lockett section of f, is obtained. This answers open question 7 of [4].

Section 4 is devoted to the study of π-normally embedded Fitting classes $\mathcal{F}$. $\mathcal{F}$ is π-normally embedded if and only if $\mathfrak{L}_\pi(\mathcal{F})$ is π-normally embedded. Further, if $\mathcal{F}$ is π-normally embedded, then $\mathcal{F}\mathcal{S}_{\pi'}$ is a dominant Fitting class.

2. Preliminaries.

A *Fitting functor* is a mapping f which assigns to each group G from $\mathcal{S}$ a non-empty set $f(G)$ of subgroups of G such that if G, H belong to $\mathcal{S}$ and $\alpha: G \to H$ is a monomorphism with $\alpha(G) \trianglelefteq H$, then

$$\{\alpha(X) : X \in f(G)\} = \{\alpha(G) \cap Y : Y \in f(H)\}\,.$$

For simplicity of notation we write $\alpha(f(G)) = \alpha(G) \cap f(H)$. A Fitting functor f is called *conjugate* provided that $f(G)$ consists of a conjugacy class of subgroups of G for all $G \in \mathfrak{S}$. A Fitting functor f is called *p-normally embedded*, p a prime number, provided that $f(G)$ consists of p-normally embedded subgroups of G for all $G \in \mathfrak{S}$. f is said to be *normally embedded* if it is p-normally embedded for each $p \in \mathbb{P}$, $\mathbb{P}$ is the set of all primes. f is said to be *pronormal* if the subgroups in $f(G)$ are pronormal in G for all $G \in \mathfrak{S}$.

If $\mathcal{F}$ is a Fitting class, then $\mathrm{Rad}_{\mathcal{F}}(G) = \{G_{\mathcal{F}}\}$ and $\mathrm{Inj}_{\mathcal{F}}(G) = \{X | X$ is an $\mathcal{F}$-injector of $G\}$ define two conjugate Fitting functors: $\mathrm{Inj}_{\mathcal{F}}$ and $\mathrm{Rad}_{\mathcal{F}}$. If $\mathcal{F} = \mathfrak{S}_\pi$, the class of all π-groups from $\mathfrak{S}$, π a set of primes, then we shall write Hall_π instead $\mathrm{Inj}_{\mathcal{F}}$. Moreover, we denote by $\mathcal{N}$ the class of all nilpotent groups from $\mathfrak{S}$, and $F(G) = G_{\mathcal{N}}$, the Fitting subgroup of G.

In the remainder of this section we present a number of known results which are used in the later two sections of this paper.

PROPOSITION 2.1 ([3]; 3.7 and 3.8). *If f is a Fitting functor and π is a set of primes, then the class $\mathfrak{L}_\pi(f)$ of groups G such that X has π'-index in G for all X in $f(G)$ is a Fitting class and $\mathfrak{L}_\pi(f)\,\mathfrak{S}_{\pi'} = \mathfrak{L}_\pi(f)$.*

PROPOSITION 2.2 ([8]; 2.3). *Let f be a conjugate Fitting functor, $G \in \mathfrak{S}$, $X \in f(G)$, $X_\pi \in \mathrm{Hall}_\pi(X)$ and $G_\pi \in \mathrm{Hall}_\pi(G)$ such that $X_\pi \leqq G_\pi$. Then the following properties are equivalent:*

(*a*) *X_π is a Hall π-subgroup of some normal subgroup of G.*

(*b*) $X_\pi \trianglelefteq G_\pi$.

(*c*) $X_\pi \leqq G_{\mathfrak{L}_\pi(f)}$.

(*d*) *X_π is a Hall π-subgroup of $G_{\mathfrak{L}_\pi(f)}$.*

New Fitting functors from previously given ones can be obtained using

PROPOSITION 2.3 ([3]; 4.1, 4.7, 4.11 and 4.15). (*a*) *Let f and g be Fitting functors and define $(f \circ g)(G) = \{X\colon X \in f(Y)$ for some $Y \in g(G)\}$, $G \in \mathfrak{S}$. Then $f \circ g$ is a Fitting functor. Moreover, if f and g are conjugate, then $f \circ g$ is conjugate.*

(*b*) *Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a family of pronormal conjugate Fitting functors and define $\left(\bigwedge_{\lambda \in \Lambda} f_\lambda\right)(G) = \left\{\bigcap_{\lambda \in \Lambda} X_\lambda\colon X_\lambda \in f_\lambda(G)\right.$, there exists a Sylow system of G reducing into X_λ for all $\left.\lambda \in \Lambda\right\}$, $G \in \mathfrak{S}$.*

Then $\bigwedge_{\lambda \in \Lambda} f_\lambda$ is a conjugate Fitting functor.

(c) *Let f and g be Fitting functors. f and g are said to commute if for each* $G \in \mathcal{S}$, $XY = YX$ *whenever* $X \in f(G)$, $Y \in g(G)$ *and there is a Sylow system of G reducing into X and Y. By the characteristic of f is meant* $\{p \in \mathbb{P}$: *there is* $G \in \mathcal{S}$ *and* $X \in f(G)$ *such that p divides* $|X|\}$.

Let $\{f_\lambda\}_{\lambda \in \Lambda}$ *be a family of pronormal conjugate pairwise commuting Fitting functors of disjoint characteristics and define* $\left(\bigvee_{\lambda \in \Lambda} f_\lambda\right)(G) = \left\{\prod_{\lambda \in \Lambda} X_\lambda \colon X_\lambda \in f_\lambda(G)\text{, there exists a Sylow system of } G \text{ reducing into } X_\lambda \text{ for all } \lambda \in \Lambda\right\}$, $G \in \mathcal{S}$.

Then $\bigvee_{\lambda \in \Lambda} f_\lambda$ *is a pronormal conjugate Fitting functor.*

(d) *Let f and g be Fitting functors with f conjugate and let π be a set of primes. Define* $\left(f \mathop{\square}_{\pi} g\right)(G) = \{T\colon \text{there exists } X \in f(G_{\mathfrak{L}_\pi(f)}),\ G_\pi \in \mathrm{Hall}_\pi(N_G(X)) \text{ such that } T/X \in g(G_\pi X/X)\}$, $G \in \mathcal{S}$. (*Note that* G_π *in this definition belongs to* $\mathrm{Hall}_\pi(G)$. *This follows from the Frattini-argument*).

Then $f \mathop{\square}_{\pi} g$ *is a Fitting functor. Moreover, if g is conjugate, then* $f \mathop{\square}_{\pi} g$ *is conjugate.*

Let f and g be Fitting functors. f is said to be *strongly contained* in g, denoted $f \ll g$, provided that for each $G \in \mathcal{S}$, the following conditions hold:

(a) If $X \in f(G)$, then there is a $Y \in g(G)$ such that $X \leq Y$, and

(b) If $W \in g(G)$, then there is a $V \in f(G)$ such that $V \leq W$. (If f and g are conjugate, then (a) and (b) are equivalent.)

A Fitting functor f is called a *Lockett functor* provided that whenever $G \in \mathcal{S}$, $X \in f(G \times G)$, then

$$X = (X \cap (G \times 1)) \times (X \cap (1 \times G)) .$$

Proposition 2.4 ([4]; 4.2, 4.4 and 4.6). *Let* $\mathfrak{F}$ *be a Fitting class and let f and g be Fitting functors. Then*

(a) *If* $\mathfrak{F}$ *is a Lockett class, then* $\mathrm{Inj}_{\mathfrak{F}}$ *and* $\mathrm{Rad}_{\mathfrak{F}}$ *are Lockett functors.*

(b) *If f and g are Lockett functors, then* $f \circ g$ *is also a Lockett functor.*

(*c*) *If f and g are Lockett functors and f is conjugate, then* $f \underset{\pi}{\square} g$ *is a Lockett functor.*

(*d*) *If f is a Lockett functor, then* $\mathfrak{L}_\pi(f)$ *is a Lockett class.*

Let f be a conjugate Fitting functor. Define f^* by $f^*(G) = \{\pi_1(T): T \in f(G \times G)\}$ for each $G \in \mathfrak{S}$. (Here π_1 is the projection of $G \times G$ onto its first coordinate).

PROPOSITION 2.5 ([4]; 6.1, 6.2, 6.3, 6.4 and 6.8). *Let f and g be conjugate Fitting functors. Then*

(*a*) f^* *is a conjugate Lockett functor.*

(*b*) *f is a Lockett functor if and only if* $f = f^*$.

(*c*) $f \ll f^*$. *If* $f \ll g$, *then* $f^* \ll g^*$.

(*d*) *Let* π *be a set of primes. Then* $\mathfrak{L}_\pi(f)^* = \mathfrak{L}_\pi(f^*)$. *If f is a Lockett functor, then* $(f \underset{\pi}{\square} g)^* = f \underset{\pi}{\square} g^*$.

We shall make use of the following lemma.

LEMMA 2.6 ([3]; 4.9). *Let* $H_1, H_2, \dots, H_n$ *be subgroups of* $G \in \mathfrak{S}$ *of pairwise relatively prime orders. Assume that* $H_i H_j = H_j H_i$ *for all* $i, j \in \{1, 2, \dots, n\}$. *Let* $N_1, N_2, \dots, N_n$ *be normal subgroups of G. Then* $(H_i \cap N_i)(H_j \cap N_j)$ *is a subgroup of G for all* $i, j \in \{1, 2, \dots, n\}$.

3. π-normally embedded Fitting functors.

This section is devoted to the study of conjugate π-normally embedded Fitting functors. A description of such functors is given in (3.3). Let f and g be conjugate π-normally embedded Fitting functors. The members of Locksec (f) are conjugate π-normally embedded Fitting functors as seen in (3.5). Further, (3.6) shows that $f \circ g$ is also such a functor.

Let I be an index set and let $\{\pi(\lambda): \lambda \in I\}$ be a partition of the primes. Conjugate I-normally embedded Fitting functors are classified in (3.14). Moreover, if f is a conjugate I-normally embedded

Fitting functor, then (3.17) gives a description of the smallest member f_* of Locksec (f). Such a description answers open question 7 of [4].

DEFINITION 3.1. Let π be a set of primes.

(*a*) A subgroup X of a group G is said to be *π-normally embedded* in G if the Hall π-subgroups of X are Hall π-subgroups of a normal subgroup of G.

(*b*) A Fitting functor f is said to be *π-normally embedded* provided that for $G \in \mathcal{S}$ and $X \in f(G)$, X is π-normally embedded in G.

As a consequence of Proposition 2.2 we obtain the following

REMARK 3.2. Let f be a conjugate Fitting functor.

(*a*) f is π-normally embedded if and only if for each $G \in \mathcal{S}$ and $X_\pi \in (\mathrm{Hall}_\pi \circ f)(G)$, then $X_\pi \in \mathrm{Hall}_\pi(G_{\mathfrak{L}_\pi(f)})$.

(*b*) f is π-normally embedded if and only if for each $G \in \mathcal{S}$ and $X_\pi \in (\mathrm{Hall}_\pi \circ f)(G)$, then $X_\pi \trianglelefteq G_\pi$, where $G_\pi \in \mathrm{Hall}_\pi(G)$ such that $X_\pi \leqq G_\pi$.

Due to (*a*) in (3.2), π-normally embedding of conjugate Fitting functors is very much related to the $\mathfrak{L}_\pi(\ )$-construction. This can be seen in the following results, a number of which are generalizations of results in [3] and [4] for the case when $\pi = \{p\}$.

PROPOSITION 3.3. *Let f be a conjugate Fitting functor and let π be a set of primes. Then*

(*a*) *If f is π-normally embedded, then* $f \ll \mathrm{Inj}_{\mathfrak{L}_\pi(f)}$.

(*b*) *f is π-normally embedded if and only if* $f \ll \mathrm{Inj}_{\mathfrak{L}_\pi(f)\mathcal{N}^\pi} \underset{\mathbb{P}}{\square} \mathrm{Hall}_{\pi'}$.

(*c*) *Let $\{\pi_i : i \in I\}$ be a collection of sets of primes such that $\pi = \bigcup_{i \in I} \pi_i$. Then f is π-normally embedded if and only if f is π_i-normally embedded and $\mathfrak{L}_{\pi_i}(f) = \mathfrak{L}_\pi(f)\,\mathcal{S}_{\pi_i'}$ for all $i \in I$.*

PROOF. (*a*) Assume that f is π-normally embedded. Let $G \in \mathcal{S}$, $X \in f(G)$ and $X_\pi \in \mathrm{Hall}_\pi(X)$. Then by part (*a*) of (3.2), $X_\pi \in \mathrm{Hall}_\pi(G_{\mathfrak{L}_\pi(f)})$. Let $X_{\pi'} \in \mathrm{Hall}_{\pi'}(X)$ and $G_{\pi'} \in \mathrm{Hall}_{\pi'}(G)$ such that $X_{\pi'} \leqq G_{\pi'}$. Since $\mathfrak{L}_\pi(f) = \mathfrak{L}_\pi(f)\,\mathcal{S}_{\pi'}$, the $\mathfrak{L}_\pi(f)$-injectors of G have π-index in G and so $G_{\pi'}$ is contained in some $\mathfrak{L}_\pi(f)$-injector of G, say V. It now follows that $X = X_\pi X_{\pi'} \leqq G_{\mathfrak{L}_\pi(f)} G_{\pi'} \leqq V$ and hence (*a*) follows.

(*b*) Let $h = \mathrm{Inj}_{\mathfrak{L}_\pi(f)\mathfrak{N}} \underset{\mathbf{P}}{\square} \mathrm{Hall}_{\pi'}$. By part (*d*) of (2.3), $h(G) = \{G_{\mathfrak{L}_\pi(f)\mathfrak{N}} G_{\pi'} \colon G_{\pi'} \in \mathrm{Hall}_{\pi'}(G)\}$.

Assume that f is π-normally embedded. Let $G \in \mathfrak{S}$, $X \in f(G)$ and $X_\pi \in \mathrm{Hall}_\pi(X)$. By part (*a*) of (3.2), $X_\pi \leqq G_{\mathfrak{L}_\pi(f)} \leqq G_{\mathfrak{L}_\pi(f)\mathfrak{N}}$. Let $X_{\pi'} \in \mathrm{Hall}_{\pi'}(X)$ and $G_{\pi'} \in \mathrm{Hall}_{\pi'}(G)$ such that $X_{\pi'} \leqq G_{\pi'}$. Then $X = X_\pi X_{\pi'} \leqq G_{\mathfrak{L}_\pi(f)\mathfrak{N}} G_{\pi'}$ and hence $f \ll h$.

Conversely, assume that $f \ll h$. Let $G \in \mathfrak{S}$, $X \in f(G)$ and $X_\pi \in \mathrm{Hall}_\pi(X)$. Let $G_{\pi'} \in \mathrm{Hall}_{\pi'}(G)$ such that $X \leqq G_{\mathfrak{L}_\pi(f)\mathfrak{N}} G_{\pi'}$. Let $K = G_{\mathfrak{L}_\pi(f)}$ and let $W/K = F(G/K)$, the Fitting subgroup of G/K. Then $X_\pi K/K \leqq W/K$ and hence $X_\pi K \trianglelefteq\trianglelefteq W$. Thus $X_\pi(X \cap K) = X \cap X_\pi K \in f(X_\pi K)$. Since f is conjugate and $|X_\pi K \colon X_\pi(X \cap K)|$ is a π'-number, we have $X_\pi K \in \mathfrak{L}_\pi(f)$ and so $X_\pi K \leqq K$. Thus $X_\pi \in \mathrm{Hall}_\pi(K)$ since $K \in \mathfrak{L}_\pi(f)$. Therefore, f is π-normally embedded.

(*c*) Assume that f is π-normally embedded and let $i \in I$. Since $\pi_i \subseteqq \pi$ it follows that $\mathfrak{L}_\pi(f) \subseteqq \mathfrak{L}_{\pi_i}(f)$. Let $G \in \mathfrak{S}$, $X \in f(G)$, $X_{\pi_i} \in \mathrm{Hall}_{\pi_i}(X)$ and $X_\pi \in \mathrm{Hall}_\pi(X)$ such that $X_{\pi_i} \leqq X_\pi$: Since f is π-normally embedded we have $X_{\pi_i} \leqq X_\pi \leqq G_{\mathfrak{L}_\pi(f)} \leqq G_{\mathfrak{L}_{\pi_i}(f)}$. This yields that $X_{\pi_i} \in \mathrm{Hall}_{\pi_i}(X \cap G_{\mathfrak{L}_{\pi_i}(f)}) \subseteqq \mathrm{Hall}_{\pi_i}(G_{\mathfrak{L}_{\pi_i}(f)})$ and hence f is π_i-normally embedded by (3.2). Moreover,

$$\mathrm{Hall}_{\pi_i} \circ \mathrm{Rad}_{\mathfrak{L}_\pi(f)} = \mathrm{Hall}_{\pi_i} \circ \mathrm{Hall}_\pi \circ \mathrm{Rad}_{\mathfrak{L}_\pi(f)} = \mathrm{Hall}_{\pi_i} \circ \mathrm{Hall}_\pi \circ f = \mathrm{Hall}_{\pi_i} \circ f\,.$$

Therefore, if $G \in \mathfrak{L}_{\pi_i}(f)$, then $\mathrm{Hall}_{\pi_i}(G) = \mathrm{Hall}_{\pi_i} \circ \mathrm{Rad}_{\mathfrak{L}_\pi(f)}(G)$ and this means that $G \in \mathfrak{L}_\pi(f)\mathfrak{S}_{\pi_i'}$. On the other hand, $\mathfrak{L}_{\pi_i}(f) = \mathfrak{L}_{\pi_i}(f)\mathfrak{S}_{\pi_i'} \supseteqq \mathfrak{L}_\pi(f)\mathfrak{S}_{\pi_i'}$ and it follows that $\mathfrak{L}_{\pi_i}(f) = \mathfrak{L}_\pi(f)\mathfrak{S}_{\pi_i'}$.

Conversely, assume that f is π_i-normally embedded and $L_{\pi_i}(f) = \mathfrak{L}_\pi(f)\mathfrak{S}_{\pi_i'}$ for all $i \in I$. Let $G \in \mathfrak{S}$, $X \in f(G)$ and $X_\pi \in \mathrm{Hall}_\pi(X)$. We note that

$$\mathrm{Hall}_{\pi_i} \circ f = \mathrm{Hall}_{\pi_i} \circ \mathrm{Rad}_{\mathfrak{L}_{\pi_i}(f)} = \mathrm{Hall}_{\pi_i} \circ \mathrm{Rad}_{\mathfrak{L}_\pi(f)\mathfrak{S}_{\pi_i}'} = \mathrm{Hall}_{\pi_i} \circ \mathrm{Rad}_{\mathfrak{L}_\pi(f)}\,.$$

Therefore, the Hall π_i-subgroups of X are contained in $G_{\mathfrak{L}_\pi(f)}$ for all $i \in I$. Since $\pi = \bigcup_{i \in I} \pi_i$ it follows that $X_\pi \leqq G_{\mathfrak{L}_\pi(f)}$. Hence $X_\pi \in \mathrm{Hall}_\pi(G_{\mathfrak{L}_\pi(f)})$ and so f is π-normally embedded.

EXAMPLES 3.4. (*a*) Let θ be a set of primes and let $\pi \subseteqq \theta$. Then Hall_θ, $\mathrm{Inj}_{\mathfrak{S}_{\pi'}\mathfrak{S}_\pi}$ and $\mathrm{Inj}_{\mathfrak{S}_\pi\mathfrak{S}_{\pi'}}$ are π-normally embedded.

(*b*) A Fitting functor f is called a *normal Fitting functor* if, for each $G \in \mathcal{S}$, $f(G)$ contains only normal subgroups. By ([3]; 7.5) f is a normal Fitting functor if and only if there is a family of Fitting classes $\{\mathfrak{X}_\lambda\}_{\lambda \in \Lambda}$ such that $f = \bigcup_{\lambda \in \Lambda} \mathrm{Rad}_{\mathfrak{X}_\lambda}$. These functors are just the **P**-normally embedded Fitting functors. Thus, if π is a set of primes and f is a normal Fitting functor, then f is π-normally embedded.

(*c*) Let p and q be distinct primes, $\pi = \{p, q\}$, $\mathcal{F} = \mathcal{S}_p \mathcal{S}_q$ and $f = \mathrm{Inj}_{\mathcal{F}}$. Let $G \in \mathcal{S}$. By Proposition 3.2 of [11] it follows that

$$f(G) = \{(O_{q'}(G) \cap G_p) G_q \colon G_p \in \mathrm{Syl}_p(G), G_q \in \mathrm{Syl}_q(G) \text{ and } G_q \leq N_G(O_{q'}(G) \cap G_p)\}\,.$$

Then f is p-normally embedded and q-normally embedded, $\mathfrak{L}_q(f) = \mathcal{S}$ and $\mathfrak{L}_p(f) = \{G\colon \mathrm{Syl}_p(G) = \mathrm{Syl}_p(O_{q'}(G))\} = \mathcal{S}_{q'} \mathcal{S}_{p'}$. $\mathfrak{L}_\pi(f) = \mathfrak{L}_p(f)$ and $\mathfrak{L}_p(f) = \mathfrak{L}_\pi(f) \mathcal{S}_{p'}$. However

$$\mathfrak{L}_\pi(f) \mathcal{S}_{q'} = \mathcal{S}_{q'} \mathcal{S}_{p'} \mathcal{S}_{q'} \neq \mathcal{S} = \mathfrak{L}_q(f)$$

and so f is not π-normally embedded by part (*c*) of (3.3).

Let f be a conjugate Fitting functor. By the *Lockett section* of f, denoted Locksec (f), is meant

$$\{g\colon g \text{ is a conjugate Fitting functor and } g^* = f^*\}\,.$$

A number of results of Locksec (f) are established in [4]. For example, f is p-normally embedded if and only if f^* is p-normally embedded ([4]; 6.5). We now generalize this result to the case of π-normally embedded Fitting functors.

Proposition 3.5. *Let f be a conjugate Fitting functor and let π be a set of primes. Then, f is π-normally embedded if and only if f^* is π-normally embedded. Thus, if f is π-normally embedded, then each member of* Locksec (f) *is π-normally embedded.*

Proof. By part (*a*) of (2.5) f^* is a conjugate Fitting functor. Since $\mathfrak{L}_\pi(f)\mathcal{N} = \mathfrak{L}_\pi(f)^*\mathcal{N}$ is a Lockett class and $\mathfrak{L}_\pi(f)^* = \mathfrak{L}_\pi(f^*)$ by part (*d*) of (2.5), it follows that

$$h = \mathrm{Inj}_{\mathfrak{L}_\pi(f)\mathcal{N}} \underset{\mathbf{P}}{\square} \mathrm{Hall}_{\pi'} = \mathrm{Inj}_{\mathfrak{L}_\pi(f^*)\mathcal{N}} \underset{\mathbf{P}}{\square} \mathrm{Hall}_{\pi'}$$

is a Lockett functor. Thus $h^* = h$ by part (*b*) of (2.5).

Assume that f is π-normally embedded. Then $f \ll h$ by part (*b*) of (3.3) and hence $f^* \ll h^* = h$ by part (*c*) of (2.5). Due to part (*b*) of (3.3) again, f^* is π-normally embedded.

Conversely, assume that f^* is π-normally embedded. Then $f^* \ll h$ by part (*b*) of (3.3). Since $f \ll f^*$ by part (*c*) of (2.5), it follows that $f \ll h$ and so f is π-normally embedded.

The next four results are concerned about the constructions in (2.3) being π-normally embedded.

PROPOSITION 3.6. *Let f and g be conjugate Fitting functors, π a set of primes and $G \in \mathfrak{S}$. If $Y \in g(G)$, $X \in f(Y)$, X is π-normally embedded in Y and Y is π-normally embedded in G, then X is π-normally embedded in G. In particular, if f and g are π-normally embedded, then $f \circ g$ is π-normally embedded.*

PROOF. Let L denote the $\mathfrak{L}_\pi(g)$-radical of G. Then $Y \cap L \in g(L)$ and, by the Frattini-argument, there exists $G_\pi \in \mathrm{Hall}_\pi(G)$ such that $G_\pi \leqq N_G(Y \cap L)$. Hence $G_\pi \cap Y \cap L \in \mathrm{Hall}_\pi(Y \cap L) \subseteqq \mathrm{Hall}_\pi(L)$ since $L \in \mathfrak{L}_\pi(g)$, and so $G_\pi \cap L \leqq Y$. Since Y is π-normally embedded in G, $\mathrm{Hall}_\pi(Y) \subseteqq \mathrm{Hall}_\pi(L)$ by (2.2). Therefore, $G_\pi \cap L \in \mathrm{Hall}_\pi(Y)$.

Let $X_\pi \in \mathrm{Hall}_\pi(X)$. Then there exists $y \in Y$ such that $X_\pi \leqq (G_\pi \cap L)^y$. Since $X \in f(Y)$ and X is π-normally embedded in Y, it follows by (2.2) that

$$X_\pi = (G_\pi \cap L)^y \cap Y_{\mathfrak{L}_\pi(f)} = (G_\pi \cap L \cap Y_{\mathfrak{L}_\pi(f)})^y = (G_\pi \cap (L \cap Y)_{\mathfrak{L}_\pi(f)})^y .$$

Since the $\mathfrak{L}_\pi(f)$-radical of $L \cap Y$ is a characteristic subgroup of $L \cap Y$ and $Y \cap L \trianglelefteq (Y \cap L)G_\pi$, it follows that G_π normalizes $(L \cap Y)_{\mathfrak{L}_\pi(f)}$. This means that $X_\pi \trianglelefteq G_\pi^y$. By part (*a*) of (2.3) $f \circ g$ is a conjugate Fitting functor. Hence by (2.2) $X_\pi \in \mathrm{Hall}_\pi(G_{\mathfrak{L}_\pi(f \circ g)})$ and so X_π is π-normally embedded. This completes the proof.

LEMMA 3.7. *Let $G \in \mathfrak{S}$, $G_\pi \in \mathrm{Hall}_\pi(G)$ and X, Y subgroups of G such that $X \cap G_\pi \in \mathrm{Hall}_\pi(X)$, $Y \cap G_\pi \in \mathrm{Hall}_\pi(Y)$ and X, Y π-normally embedded in G. Then*

(*a*) *$X \cap Y \cap G_\pi \in \mathrm{Hall}_\pi(X \cap Y)$ and $X \cap Y$ is π-normally embedded.*

(*b*) *If $XY \leqq G$, then $XY \cap G_\pi \in \mathrm{Hall}_\pi(XY)$ and XY is π-normally embedded.*

PROOF. Since X and Y are π-normally embedded in G, there exist normal subgroups M and N of G such that $X \cap G_\pi = M \cap G_\pi$ and $Y \cap G_\pi = N \cap G_\pi$.

(*a*) Let $Z \in \mathrm{Hall}_\pi(X \cap Y)$ such that $X \cap Y \cap G_\pi \leqq Z$. Then there exist $x \in X$ and $y \in Y$ such that $Z \leqq (X \cap G_\pi)^x \leqq M$ and $Z \leqq (Y \cap G_\pi)^y \leqq N$. Thus $M \cap N \cap G_\pi = X \cap Y \cap G_\pi \leqq Z \leqq M \cap N$ and Z is a π-group. Hence $X \cap Y \cap G_\pi = Z \in \mathrm{Hall}_\pi(X \cap Y) \cap \cap \mathrm{Hall}_\pi(M \cap N)$. Since $M \cap N \trianglelefteq G$, it follows that $X \cap Y$ is π-normally embedded.

(*b*) $(X \cap G_\pi)(Y \cap G_\pi) = (M \cap G_\pi)(N \cap G_\pi) = MN \cap G_\pi$ is a subgroup of $XY \cap G_\pi \leqq XY$. Thus $(X \cap G_\pi)(Y \cap G_\pi) \in \mathrm{Hall}_\pi(XY)$ and $XY \cap G_\pi = (X \cap G_\pi)(Y \cap G_\pi) \in \mathrm{Hall}_\pi(XY) \cap \mathrm{Hall}_\pi(MN)$. Therefore, XY is π-normally embedded.

As a consequence of parts (*b*) and (*c*) of (2.3) and (3.7), we obtain the following result.

PROPOSITION 3.8. *Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a family of pronormal conjugate Fitting functors, and π a set of primes.*

(*a*) *If the functors in $\{f_\lambda\}_{\lambda \in \Lambda}$ are π-normally embedded, then $\bigwedge_{\lambda \in \Lambda} f_\lambda$ is a π-normally embedded conjugate Fitting functor.*

(*b*) *If the functors in $\{f_\lambda\}_{\lambda \in \Lambda}$ are π-normally embedded functors of pairwise disjoint characteristics and pairwise commuting, then $\bigvee_{\lambda \in \Lambda} f_\lambda$ is a π-normally embedded conjugate Fitting functor.*

PROPOSITION 3.9. *Let f, g be conjugate Fitting functors and let θ, π be sets of primes. Then*

(*a*) *If $\pi \subseteqq \theta$ and g is π-normally embedded, then $f \underset{\theta}{\square} g$ is π-normally embedded.*

(*b*) *If $\pi \subseteqq \theta'$ and f is π-normally embedded, then $f \underset{\theta}{\square} g$ is π-normally embedded.*

PROOF. Let $G \in \mathfrak{S}$, $T \in (f \underset{\theta}{\square} g)(G)$. Then there exist $X \in f(G_{\mathfrak{L}_\theta(f)})$ and $G_\theta \in \mathrm{Hall}_\theta(G)$ such that $G_\theta \leqq N_G(X)$ and $T/X \in g(G_\theta X/X)$.

(*a*) Assume that $\pi \subseteqq \theta$ and g is π-normally embedded. Let $T_\pi \in \mathrm{Hall}_\pi(T)$. Then there exists $G_\pi \in \mathrm{Hall}_\pi(G)$ such that $T_\pi \leqq G_\pi \in \in \mathrm{Hall}_\pi(G_\theta X)$. Since $T_\pi X/X \in \mathrm{Hall}_\pi(T/X)$, $G_\pi X/X \in \mathrm{Hall}_\pi(G_\theta X/X)$

and g is π-normally embedded, it follows that $T_\pi X \trianglelefteq G_\pi X$. Moreover, $T_\pi = T \cap G_\pi \geq X \cap G_\pi$ so that $T_\pi = T_\pi(X \cap G_\pi) = T_\pi X \cap G_\pi \trianglelefteq G_\pi$. Because of part (*b*) of (3.2) $f \underset{\theta}{\Box} g$ is π-normally embedded.

(*b*) Assume that $\pi \subseteq \theta'$ and f is π-normally embedded. Then $\mathrm{Hall}_\pi(T) = \mathrm{Hall}_\pi(X)$. Let $X_\pi \in \mathrm{Hall}_\pi(X)$ and let $M = G_{\mathfrak{L}_\theta(f)}$. By part (*a*) of (3.2), $X_\pi \in \mathrm{Hall}_\pi(M_{\mathfrak{L}_\pi(f)})$. Therefore, $f \underset{\theta}{\Box} g$ is π-normally embedded. This completes the proof.

Let I be an index set such that

(*a*) $\mathbb{P} = \bigcup_{\lambda \in I} \pi(\lambda)$, $\pi(\lambda)$ a non-empty set of primes,

(*b*) $\pi(\lambda_1) \cap \pi(\lambda_2) = \emptyset$ whenever $\lambda_1 \neq \lambda_2$.

DEFINITION 3.10. A Fitting functor f is said to be *I-normally embedded* if f is $\pi(\lambda)$-normally embedded for each $\lambda \in I$.

REMARKS 3.11. (*a*) For $I = \mathbb{P}$ and $\pi(p) = \{p\}$ one has in (3.10) the definition of normally embedded Fitting functor.

(*b*) If f is a conjugate Fitting functor, then it follows from part (*c*) of (3.3) that f is I-normally embedded if and only if, for each $\lambda \in I$ and each $p \in \pi(\lambda)$, f is p-normally embedded and $\mathfrak{L}_p(f) = \mathfrak{L}_{\pi(\lambda)} \mathfrak{S}_{p'}$. In particular, if f is I-normally embedded, then f is normally embedded.

DEFINITION 3.12. Let $G \in \mathfrak{S}$ and, for each $\lambda \in I$, let $N(\lambda) \trianglelefteq G$. A collection of subgroups $\{H(\lambda) \colon \lambda \in I\}$ is called an *I-Sylow system associated* with $\{N(\lambda) \colon \lambda \in I\}$ if the following holds:

(*a*) $H(\lambda) \in \mathrm{Hall}_{\pi(\lambda)}(N(\lambda))$, $\lambda \in I$

(*b*) $H(\lambda_1)H(\lambda_2) = H(\lambda_2)H(\lambda_1)$, $\lambda_1, \lambda_2 \in I$.

We note that for $I = \mathbb{P}$ and $\pi(p) = \{p\}$, (3.12) is the concept of generalized Sylow system due to Fischer (see [5]).

LEMMA 3.13. *Let $G \in \mathfrak{S}$ and let $\{N(\lambda) \colon \lambda \in I\}$ be a collection of normal subgroups of G. Then*

(*a*) *There is an I-Sylow system of G associated with the normal subgroups $\{N(\lambda) \colon \lambda \in I\}$ of G.*

(*b*) *Any two such systems are conjugate.*

(c) *Let $\{H(\lambda): \lambda \in I\}$ be an I-Sylow system associated with $\{N(\lambda): \lambda \in I\}$ and let $D \trianglelefteq G$. Then $\{H(\lambda) \cap D: \lambda \in I\}$ is an I-Sylow system of D associated with the normal subgroups $\{N(\lambda) \cap D: \lambda \in I\}$ of D.*

PROOF. (a) Let Σ be a Sylow system of G, $\lambda \in I$ and $H(\lambda) = S_{\pi(\lambda)} \cap N(\lambda)$ with $S_{\pi(\lambda)}$ the Hall $\pi(\lambda)$-subgroup of G in Σ. Then $H(\lambda) \in \mathrm{Hall}_{\pi(\lambda)}(N(\lambda))$. Let $\lambda, \mu \in I$. Then $S_{\pi(\lambda)}S_{\pi(\mu)} = S_{\pi(\mu)}S_{\pi(\lambda)}$ and it follows from (2.6) that $H(\lambda)H(\mu) = H(\mu)H(\lambda)$. This shows that $\{H(\lambda): \lambda \in I\}$ is an I-Sylow system of G associated with $\{N(\lambda): \lambda \in I\}$.

(b) Let $G \in \mathfrak{S}$ and $\{H(\lambda): \lambda \in I\}$ be an I-Sylow system of G associated with the normal subgroups $\{N(\lambda): \lambda \in I\}$ of G. Since G is a finite group and $\{\pi(\lambda): \lambda \in I\}$ is a partition of $\mathbb{P}$, there is a finite set $\{\lambda_1, \dots, \lambda_n\} \subseteq I$ such that all the prime divisors of the order of G belong to $\bigcup_{i=1}^{n} \pi(\lambda_i)$. Let $H = H(\lambda_1) \dots H(\lambda_n) \leqq G$. It is clear that $H(\lambda_i) \in \mathrm{Hall}_{\pi(\lambda_i)}(H)$ for all $i \in \{1, \dots, n\}$. By a result of section 3 of P. Hall [9], $H(\lambda_i)$, $1 \leqq i \leqq n$, is part of a Sylow system of H. Therefore, there exists a Sylow system Σ of G such that $H(\lambda_i) = G_{\pi(\lambda_i)} \cap H$, $G_{\pi(\lambda_i)} \in \Sigma$ $1 \leqq i \leqq n$. Thus $H(\lambda) \leqq G_{\pi(\lambda)}$ where $G_{\pi(\lambda)} \in \Sigma$ for all $\lambda \in I$, and so $H(\lambda) = G_{\pi(\lambda)} \cap N(\lambda)$ for all $\lambda \in I$.

So we have proved that each I-Sylow system of G associated with $\{N(\lambda): \lambda \in I\}$ has the form $\{G_{\pi(\lambda)} \cap N(\lambda): \lambda \in I, G_{\pi(\lambda)} \in \Sigma\}$ for some Sylow system Σ of G. The result follows from the conjugacy of the Sylow systems of G.

(c) This follows from (2.6).

The next theorem characterizes conjugate I-normally embedded Fitting functors.

THEOREM 3.14. (a) *Let $\{\mathfrak{X}(\lambda): \lambda \in I\}$ be a family of Fitting classes. Then $f = \bigvee_{\lambda \in I} (\mathrm{Hall}_{\pi(\lambda)} \circ \mathrm{Rad}_{\mathfrak{X}(\lambda)})$ is a conjugate I-normally embedded Fitting functor and $\mathfrak{L}_{\pi(\lambda)}(f) = \mathfrak{X}(\lambda)\,\mathfrak{S}_{\pi(\lambda)}$, for each $\lambda \in I$.*

(b) *If f is a conjugate I-normally embedded Fitting functor, then $f = \bigvee_{\lambda \in I} (\mathrm{Hall}_{\pi(\lambda)} \circ \mathrm{Rad}_{\mathfrak{L}_{\pi(\lambda)}(f)})$.*

PROOF. (a) For each $G \in \mathfrak{S}$, let

$$f(G) = \left\{\prod_{\lambda \in I} H(\lambda): \{H(\lambda)\}_{\lambda \in I} \text{ is an } I\text{-Sylow system of } G \text{ associated with } \{G_{\mathfrak{X}(\lambda)}\}_{\lambda \in I}\right\}.$$

By (3.13) f is a conjugate I-normally embedded Fitting functor. It is clear that $\mathrm{Hall}_{\pi(\lambda)} \circ f = \mathrm{Hall}_{\pi(\lambda)} \circ \mathrm{Rad}_{\mathfrak{X}(\lambda)}$ and that $\mathfrak{L}_{\pi(\lambda)}(f) = \mathfrak{L}_{\pi(\lambda)}(\mathrm{Rad}_{\mathfrak{X}(\lambda)}) = \mathfrak{X}(\lambda)\,\mathfrak{S}_{\pi(\lambda)'}$. Further it follows that $f = \bigvee_{\lambda \in I} (\mathrm{Hall}_{\pi(\lambda)} \circ \mathrm{Rad}_{\mathfrak{X}(\lambda)})$.

(*b*) As f and $\bigvee (\mathrm{Hall}_{\pi(\lambda)} \circ \mathrm{Rad}_{\mathfrak{X}(\lambda)})$ are conjugate Fitting functors, the result follows from part (*a*) of (3.2).

By part (*b*) of (3.11) and Satz 7.4 of [3] we obtain the following theorem.

THEOREM 3.15. *Let f be an I-normally embedded Fitting functor. Then f is the union of conjugate I-normally embedded Fitting functors.*

Let f be a conjugate I-normally embedded Fitting functor. By (3.5) each member of Locksec (f) is also a conjugate I-normally embedded Fitting functor. Since f is a conjugate normally embedded functor, it follows from part (*a*) of (7.7) and (7.9) of [4] that Locksec (f) has an element f_* such that $f_* \ll g$ for all $g \in$ Locksec (f). Open question 7 of [4] is to give a description of f_*. In Theorem 3.17 such a description is presented. We first establish the next routine lemma.

LEMMA 3.16. *Let f and g be conjugate I-normally embedded Fitting functors. Then $f \ll g$ if and only if $\mathfrak{L}_{\pi(\lambda)}(f) \subseteqq \mathfrak{L}_{\pi(\lambda)}(g)$ for each $\lambda \in I$.*

PROOF. Assume that $L_{\pi(\lambda)}(f) \subseteqq \mathfrak{L}_{\pi(\lambda)}(g)$ for each $\lambda \in I$. By (3.14) we conclude that $f \ll g$.

Conversely, assume that $f \ll g$. Let $\lambda \in I$ and let $G \in \mathfrak{L}_{\pi(\lambda)}(f)$. Let $V \in f(G)$ and let $V_{\pi(\lambda)} \in \mathrm{Hall}_{\pi(\lambda)}(V)$. Then $V_{\pi(\lambda)} \in \mathrm{Hall}_{\pi(\lambda)}(G)$. Since $f \ll g$, there exists $U \in g(G)$ such that $V \leqq U$ and hence $V_{\pi(\lambda)} \in \mathrm{Hall}_{\pi(\lambda)}(U)$. This means that $\mathfrak{L}_{\pi(\lambda)}(f) \subseteqq \mathfrak{L}_{\pi(\lambda)}(g)$ for each $\lambda \in I$.

THEOREM 3.17. *Let f be a conjugate I-normally embedded Fitting functor. Then $f_* = \bigvee_{\lambda \in I} (\mathrm{Hall}_{\pi(\lambda)} \circ \mathrm{Rad}_{(\mathfrak{L}_{\pi(\lambda)}(f))_*})$.*

PROOF. For each $\lambda \in I$, let $\mathfrak{X}(\lambda) = (\mathfrak{L}_{\pi(\lambda)}(f))_*$ and let $h = \bigvee_{\lambda \in I} (\mathrm{Hall}_{\pi(\lambda)} \circ \mathrm{Rad}_{\mathfrak{X}(\lambda)})$. By part (*a*) of (3.14) h is a conjugate I-normally embedded Fitting functor and $\mathfrak{L}_{\pi(\lambda)}(h) = \mathfrak{X}(\lambda)\,\mathfrak{S}_{\pi(\lambda)'}$ for each $\lambda \in I$. By part (*d*) of (2.5) we have

$$\mathfrak{L}_{\pi(\lambda)}(h^*) = \mathfrak{L}_{\pi(\lambda)}(h)^* = \mathfrak{X}(\lambda)^*\,\mathfrak{S}_{\pi(\lambda)'} = (\mathfrak{L}_{\pi(\lambda)}(f))^*\,\mathfrak{S}_{\pi(\lambda)'} = \mathfrak{L}_{\pi(\lambda)}(f^*)\,\mathfrak{S}_{\pi(\lambda)'} = \mathfrak{L}_{\pi(\lambda)}(f^*)$$

for each $\lambda \in I$. By (3.14) it follows that $h^* = f^*$ and hence $h \in \in \text{Locksec}(f)$.

Let $g \in \text{Locksec}(f)$. By part (*d*) of (2.5), we see that $(\mathfrak{L}_{\pi(\lambda)}(g))^* = = \mathfrak{L}_{\pi(\lambda)}(g^*) = \mathfrak{L}_{\pi(\lambda)}(f^*) = \mathfrak{L}_{\pi(\lambda)}(f)^*$ and hence $\mathfrak{X}(\lambda) = (\mathfrak{L}_{\pi(\lambda)}(f))_* \subseteq \mathfrak{L}_{\pi(\lambda)}(g)$ for each $\lambda \in I$. Thus $\mathfrak{L}_{\pi(\lambda)}(h) = \mathfrak{X}(\lambda)\,\mathfrak{S}_{\pi(\lambda)'} \subseteq \mathfrak{L}_{\pi(\lambda)}(g)\,\mathfrak{S}_{\pi(\lambda)'} = \mathfrak{L}_{\pi(\lambda)}(g)$ for each $\lambda \in I$. By (3.16) $h \ll g$ for all $g \in \text{Locksec}(f)$ and hence $f_* = h$. This completes the proof.

Using the description of f_* in (3.17), it follows that $f_* = f \circ \text{Rad}$ where $f = \text{Hall}_\pi$. This answers the test case in problem 7 of [4].

4. π-normally embedded Fitting classes.

Let π be a set of primes. A Fitting class $\mathcal{F}$ is said to be *π-normally embedded* provided that $\text{Inj}_{\mathcal{F}}$ is a π-normally embedded Fitting functor. In this section we generalize a number of known results for $\pi = \{p\}$ (see [7]). For example, we show in (4.2) that a Fitting class $\mathcal{F}$ is π-normally embedded if and only if $\mathfrak{L}_\pi(\mathcal{F})$ is a π-normally embedded Fitting class.

PROPOSITION 4.1. *Let $\mathcal{F}$ be a π-normally embedded Fitting class. Then*

(*a*) *If $G \in \mathfrak{S}$, then $G_{\mathfrak{L}_\pi(\mathcal{F})}G_{\pi'}$ is an $\mathfrak{L}_\pi(\mathcal{F})$-injector of G where $G_{\pi'} \in \text{Hall}_{\pi'}(G)$.*

(*b*) *$\mathcal{F}\mathfrak{S}_{\pi'}$ is a dominant Fitting class.*

PROOF. (*a*) Let V be an $\mathcal{F}$-injector of G, $V_\pi \in \text{Hall}_\pi(V)$ and $V_{\pi'} \in \text{Hall}_{\pi'}(V)$. Further, let $G_\pi \in \text{Hall}_\pi(G)$ and $G_{\pi'} \in \text{Hall}_{\pi'}(G)$ such that $V_\pi \leqq G_\pi$ and $V_{\pi'} \leqq G_{\pi'}$. Since $\text{Inj}_{\mathcal{F}}$ is π-normally embedded, $V_\pi = G_\pi \cap G_{\mathfrak{L}_\pi(\mathcal{F})}$. Therefore,

$$VG_{\pi'} = V_\pi G_{\pi'} = (G_\pi \cap G_{\mathfrak{L}_\pi(\mathcal{F})})G_{\pi'} = G_{\mathfrak{L}_\pi(\mathcal{F})}G_{\pi'}$$

is a subgroup of G. By Proposition 4.4 of [11], $G_{\mathfrak{L}_\pi(\mathcal{F})}G_{\pi'}$ is an $\mathfrak{L}_\pi(\mathcal{F})$-injector of G.

(*b*) Since $\text{Inj}_{\mathcal{F}}$ is π-normally embedded, it follows from (3.9) that $\text{Inj}_{\mathcal{F}\mathfrak{S}_{\pi'}} = \text{Inj}_{\mathcal{F}} \underset{\pi'}{\square} \text{Inj}_{\mathfrak{S}_{\pi'}}$ is π-normally embedded. Hence we may assume that $\mathcal{F} = \mathcal{F}\mathfrak{S}_{\pi'}$.

Let $G \in \mathfrak{S}$ and $H \leqq G$ such that $G_{\mathcal{F}} \leqq H \in \mathcal{F}$. We show that H is a subgroup of an $\mathcal{F}$-injector of G. Let $F/G_{\mathcal{F}}$ be the Fitting subgroup of $G/G_{\mathcal{F}}$. Since $\mathcal{F}\mathcal{S}_{\pi'} = \mathcal{F}$, and $F/G_{\mathcal{F}} \in \mathcal{N}$, we have $F/G_{\mathcal{F}} \in \mathcal{S}_\pi$. Moreover $H \cap F/G_{\mathcal{F}} \trianglelefteq\trianglelefteq G/G_{\mathcal{F}}$ and so $H \cap F \trianglelefteq\trianglelefteq G$. $H \cap F \trianglelefteq H$ and so $H \cap F \in \mathcal{F}$. Therefore $H \cap F = G_{\mathcal{F}}$ which is an $\mathcal{F}$-injector of F. By Lemma 4 of [6], H is an $\mathcal{F}$-injector of HF. Let $P \in \mathrm{Hall}_\pi(HF)$ and $H_\pi \in \mathrm{Hall}_\pi(H)$ such that $H_\pi \leqq P$. By part (b) of (3.2), we have $H_\pi \trianglelefteq P$ and so $H_\pi G_{\mathcal{F}}/G_{\mathcal{F}} \trianglelefteq PG_{\mathcal{F}}/G_{\mathcal{F}}$. Since $PG_{\mathcal{F}}/G_{\mathcal{F}} \in \mathrm{Hall}_\pi(HF/G_{\mathcal{F}})$ and $F/G_{\mathcal{F}} \in \mathcal{S}_\pi$, $F/G_{\mathcal{F}} \trianglelefteq PG_{\mathcal{F}}/G_{\mathcal{F}}$. This means that

$$[H_\pi G_{\mathcal{F}}/G_{\mathcal{F}}, F/G_{\mathcal{F}}] \leqq (H_\pi G_{\mathcal{F}} \cap F)/G_{\mathcal{F}} \leqq (H \cap F)/G_{\mathcal{F}} = G_{\mathcal{F}}/G_{\mathcal{F}} .$$

and hence $H_\pi G_{\mathcal{F}}/G_{\mathcal{F}}$ centralizes $F/G_{\mathcal{F}}$. Therefore, $H_\pi \leqq F \cap H = G_{\mathcal{F}}$ and it follows that $H \leqq G_{\mathcal{F}} G_{\pi'}$ for some $G_{\pi'} \in \mathrm{Hall}_{\pi'}(G)$. Since $\mathcal{F}\mathcal{S}_{\pi'} = \mathcal{F}$, $\mathfrak{L}_\pi(\mathcal{F}) = \mathcal{F}$ by Proposition 3.1 of [11]. By (a) $G_{\mathcal{F}} G_{\pi'}$ is an $\mathcal{F}$-injector of G and so the proof is complete.

THEOREM 4.2. *Let $\mathcal{F}$ be a Fitting class and π a set of primes. Then $\mathcal{F}$ is π-normally embedded if and only if $\mathfrak{L}_\pi(\mathcal{F})$ is π-normally embedded.*

PROOF. Assume that $\mathcal{F}$ is π-normally embedded. Then, by part (a) of (4.1), $\mathrm{Inj}_{\mathfrak{L}_\pi(\mathcal{F})}(G) = \{G_{\mathfrak{L}_\pi(\mathcal{F})} G_{\pi'} : G_{\pi'} \in \mathrm{Hall}_{\pi'}(G)\}$ and so $\mathfrak{L}_\pi(\mathcal{F})$ is π-normally embedded.

Conversely, assume that $\mathfrak{L}_\pi(\mathcal{F})$ is π-normally embedded. By part (b) of (4.1) $\mathfrak{L}_\pi(\mathcal{F})\mathcal{S}_{\pi'} = \mathfrak{L}_\pi(\mathcal{F})$ is dominant. Let V be an $\mathcal{F}$-injector of G. Since V is an $\mathcal{F}$-injector of $G_{\mathfrak{L}_\pi(\mathcal{F})} V$, it follows that $G_{\mathfrak{L}_\pi(\mathcal{F})} V \in \mathfrak{L}_\pi(\mathcal{F})$. Hence $\mathcal{F} \ll \mathfrak{L}_\pi(\mathcal{F})$ since $\mathfrak{L}_\pi(\mathcal{F})$ is dominant. This means that $\mathrm{Hall}_\pi \circ \mathrm{Inj}_{\mathcal{F}} = \mathrm{Hall}_\pi \circ \mathrm{Inj}_{\mathfrak{L}_\pi(\mathcal{F})}$, and since $\mathfrak{L}_\pi(\mathcal{F})$ is π-normally embedded and $\mathfrak{L}_\pi(\mathfrak{L}_\pi(\mathcal{F})) = \mathfrak{L}_\pi(\mathcal{F})$, we have

$$\mathrm{Hall}_\pi \circ \mathrm{Inj}_{\mathcal{F}} = \mathrm{Hall}_\pi \circ \mathrm{Inj}_{\mathfrak{L}_\pi(\mathcal{F})} = \mathrm{Hall}_\pi \circ \mathrm{Rad}_{\mathfrak{L}_\pi(\mathcal{F})} .$$

Therefore, $\mathcal{F}$ is π-normally embedded.

The next proposition gives three necessary conditions for $\mathcal{F}$ to be π-normally embedded. Note that, in the case $\pi = \{p\}$, they are all satisfied for every $\mathcal{F}$.

PROPOSITION 4.3. *Let $\mathcal{F}$ be a Fitting class, π a set of primes and consider the following properties*

(a) *$\mathcal{F}$ is π-normally embedded.*

(b) *$\mathfrak{L}_p(\mathcal{F}) = \mathfrak{L}_\pi(\mathcal{F})\mathcal{S}_{p'}$ for all $p \in \pi$.*

(*c*) *The groups in* $\mathcal{F}\mathcal{S}_\pi$ *have normal* $\mathcal{F}$*-injectors.*

(*d*) $\mathcal{F} \subseteq \mathcal{S}_{\pi'}$ *or* $\mathcal{S}_\pi \subseteq \mathcal{F}^*$.

Then (*a*) *implies* (*b*), (*b*) *implies* (*c*) *and* (*c*) *implies* (*d*).

PROOF. (*a*) $\Rightarrow$ (*b*). This is due to part (*c*) of (3.3).

(*b*) $\Rightarrow$ (*c*). Suppose for a contradiction that G is a group of minimal order such that $G \in \mathcal{F}\mathcal{S}_\pi$ and an $\mathcal{F}$-injector of G is not a normal subgroup of G. Let us consider Theorem 1.1 of [1] for $\mathfrak{X} = \mathcal{F}$ and $\mathfrak{Y} = \mathcal{S}$. The subgroups S in the proof of this theorem contain $G_\mathcal{F}$ and hence $S/G_\mathcal{F} \in \mathcal{S}_\pi$. Therefore, the arguments on the minimality of G are valid here and it follows that $G = MV$ where M is the unique maximal normal subgroup of G, $V \in \mathrm{Inj}_\mathcal{F}(G)$, $M \cap V = G_\mathcal{F}$, $M/G_\mathcal{F}$ is a non-trivial q-group and $|G:M| = p$ where p and q are distinct prime numbers. Since $G \in \mathcal{F}\mathcal{S}_\pi$, we have $p, q \in \pi$ and $G_{\mathfrak{L}_\pi(\mathcal{F})} \in \mathcal{F}\mathcal{S}_\pi \cap \mathfrak{L}_\pi(\mathcal{F}) = \mathcal{F}$. Thus $G_{\mathfrak{L}_\pi(\mathcal{F})} = G_\mathcal{F}$ and so $G \notin \mathfrak{L}_\pi(\mathcal{F})\mathcal{S}_{p'}$, $p \in \pi$. But V has q-index in G and consequently $G \in \mathfrak{L}_p(\mathcal{F})$, contradiction.

(*c*) $\Rightarrow$ (*d*). Assume that the groups in $\mathcal{F}\mathcal{S}_\pi$ have normal $\mathcal{F}$-injectors. In particular, the groups in $\mathcal{S}_\pi$ have normal $\mathcal{F}$-injectors. Since $\mathrm{Inj}_{\mathcal{F}\cap\mathcal{S}_\pi} = \mathrm{Inj}_\mathcal{F} \circ \mathrm{Hall}_\pi$, we have that $\mathcal{F} \cap \mathcal{S}_\pi$ is strictly normal in $\mathcal{S}_\pi$. By Theorem 4.7 of [2], it follows that $\mathcal{F} \cap \mathcal{S}_\pi = \{1\}$ or $(\mathcal{F} \cap \mathcal{S}_\pi)^* = \mathcal{S}_\pi$. This means that $\mathcal{F} \subseteq \mathcal{S}_{\pi'}$ or $\mathcal{S}_\pi \subseteq \mathcal{F}^*$.

In the next example it is shown that (*d*) does not imply (*c*).

EXAMPLE 4.4. Let $\pi = \{2, 3\}$ and let $\mathcal{F} = \mathcal{S}_\pi \mathcal{S}_{3'}$. Let $G = C_5 \wr (C_3 \wr C_2)$ where C_p is the cyclic group of order p. Then $O_\pi(G) = 1$, $G \in \mathcal{S}_\pi \mathcal{S}_{3'} \mathcal{S}_\pi = \mathcal{F}\mathcal{S}_\pi$ and $\mathrm{Inj}_\mathcal{F}(G) = \mathrm{Hall}_{3'}(G)$. Thus G does not have normal $\mathcal{F}$-injectors and $\mathcal{S}_\pi \subseteq \mathcal{F}$.

The next result is used to establish another equivalent property to (2.2) in the case $f = \mathrm{Inj}_\mathcal{F}$, $\mathcal{F}$ a Fitting class.

LEMMA 4.5. *Let* $\mathcal{F}$ *be a Fitting class and* π *a set of primes. Then* $\mathrm{Rad}_\mathcal{F} \circ \mathrm{Inj}_{\mathcal{F}\mathcal{S}_\pi} = \mathrm{Inj}_\mathcal{F} \circ \mathrm{Rad}_{\mathfrak{L}_\pi(\mathcal{F})}$.

PROOF. Let us write $f = \mathrm{Rad}_\mathcal{F} \circ \mathrm{Inj}_{\mathcal{F}\mathcal{S}_\pi}$ and $g = \mathrm{Inj}_\mathcal{F} \circ \mathrm{Rad}_{\mathfrak{L}_\pi(\mathcal{F})}$: Let $G \in \mathcal{S}$ and $H_\mathcal{F} \in f(G)$ where $H \in \mathrm{Inj}_{\mathcal{F}\mathcal{S}_\pi}(G)$. By Proposition 3.2 of [11] there exist $W \in g(G)$ and $G_\pi \in \mathrm{Hall}_\pi(G)$ such that $G_\pi \leq N_G(W)$ and $H = WG_\pi$. Since $W \trianglelefteq H$, it follows that $W \leq H_\mathcal{F}$ and so $H_\mathcal{F} = WG_\pi \cap H_\mathcal{F} = W(G_\pi \cap H_\mathcal{F})$. Hence we have that $H_\mathcal{F} \cap G_\pi \in \mathrm{Hall}_\pi(H_\mathcal{F})$, $H_\mathcal{F} \cap G_\pi \trianglelefteq G_\pi$ and $H_\mathcal{F} \in f(G)$ and so, by (2.2), $H_\mathcal{F} \cap G_\pi \leq G_{\mathfrak{L}_\pi(f)}$. Moreover, by part (*b*) of Proposition 4.4 of [3],

$$\mathfrak{L}_\pi(f) = \mathfrak{Y}(\mathcal{F}\mathfrak{S}_\pi, \mathcal{F}\mathfrak{S}_{\pi'}) \cap \mathfrak{L}_\pi(\mathcal{F}\mathfrak{S}_\pi) = \mathfrak{Y}(\mathcal{F}\mathfrak{S}_\pi, \mathcal{F}) = \mathfrak{L}_\pi(\mathcal{F}) .$$

Therefore, $W \leqq H_{\mathcal{F}} = W(H_{\mathcal{F}} \cap G_\pi) \leqq G_{\mathfrak{L}_\pi(\mathcal{F})}$ and since W is an $\mathcal{F}$-injector of $G_{\mathfrak{L}_\pi(\mathcal{F})}$, it follows that $W = H_{\mathcal{F}} \in f(G) \cap g(G)$. Since f and g are conjugate Fitting functors, the result follows.

Let V be an $\mathcal{F}$-injector of G. Then $V \cap G_{\mathfrak{L}_\pi(\mathcal{F})}$ is an $\mathcal{F}$-injector of $G_{\mathfrak{L}_\pi(\mathcal{F})}$ and, by the Frattini-argument, the Hall π-subgroups of $N_G(V \cap G_{\mathfrak{L}_\pi(\mathcal{F})})$ are Hall π-subgroups of G. Since $V \leqq N_G(V \cap G_{\mathfrak{L}_\pi(\mathcal{F})})$, if $V_\pi \in \mathrm{Hall}_\pi(V)$, then there exists $G_\pi \in \mathrm{Hall}_\pi(G)$ such that $V_\pi \leqq G_\pi$ and $G_\pi \leqq N_G(V \cap G_{\mathfrak{L}_\pi(\mathcal{F})})$. Under these circumstances we have

PROPOSITION 4.6. *The following are equivalent*

(*a*) *V is π-normally embedded in G*

(*b*) *$V_\pi \trianglelefteq\trianglelefteq G_\pi$ and $V_\pi(V \cap G_{\mathfrak{L}_\pi(\mathcal{F})}) \in \mathcal{F}$.*

PROOF. Assume that V is π-normally embedded in G and let L denote the $\mathfrak{L}_\pi(\mathcal{F})$-radical of G. Then by (2.2) $V_\pi \trianglelefteq G_\pi$ and $V_\pi \leqq L$, so $V_\pi(V \cap L) = V \cap L \in \mathcal{F}$.

Conversely, let $V_\pi \trianglelefteq\trianglelefteq G_\pi$ and $V_\pi(V \cap L) \in \mathcal{F}$. Then $V_\pi(V \cap L) \trianglelefteq\trianglelefteq G_\pi(V \cap L)$ which is an $\mathcal{F}\mathfrak{S}_\pi$-injector of G by Proposition (3.2) of [11]. Hence the $\mathcal{F}$-radical of $G_\pi(V \cap L)$ contains $V_\pi(V \cap L)$. By (4.5), $V_\pi(V \cap L) \leqslant L$ and so $V_\pi \leqq L$. From (2.2) we conclude that V is π-normally embedded.

Let $\mathcal{F}$ be a Fitting class and π a set of primes. $\mathcal{F}$ is said to satisfy *condition* α provided that for all $G \in \mathfrak{S}$, $V_\pi \in \mathrm{Hall}_\pi \circ \mathrm{Inj}_{\mathcal{F}}(G)$, there exists $G_\pi \in \mathrm{Hall}_\pi(G)$ such that $V_\pi \trianglelefteq\trianglelefteq G_\pi$ and $V_\pi G_{\mathcal{F}} \in \mathcal{F}$.

COROLLARY 4.7. *Let π be a set of primes and let $\mathcal{F}$ be a Fitting class satisfying condition α. Then $\mathcal{F}$ is π-normally embedded.*

PROOF. Assume that $\mathcal{F}$ satisfies condition α and let G be of minimal order such that V_π is not normal in G_π for some $V_\pi \leqq G_\pi$, $G_\pi \in \mathrm{Hall}_\pi(G)$, $V_\pi \in \mathrm{Hall}_\pi(V)$, and $V \in \mathrm{Inj}_{\mathcal{F}}(G)$. Let L denote the $\mathfrak{L}_\pi(\mathcal{F})$-radical of G. V is an $\mathcal{F}$-injector of $N_G(V \cap L)$ and $N_G(V \cap L)$ has π'-index in G. Therefore, by minimality of G, $G = N_G(V \cap L)$ and hence $G_{\mathcal{F}} = V \cap L$. This contradicts the hypothesis of (4.6) and consequently $\mathcal{F}$ is π-normally embedded.

REFERENCES

[1] J. C. Beidleman - B. Brewster, *Strict normality in Fitting classes I*, J. Algebra, **51** (1978), pp. 211-217.

[2] J. C. Beidleman - B. Brewster, *Strict normality in Fitting classes III*, Comm. in Algebra, **10** (7) (1982), pp. 741-766.

[3] J. C. Beidleman - B. Brewster - P. Hauck, *Fittingfunktoren in endlichen auflösbaren Gruppen I*, Math. Z., **182** (1983), pp. 359-384.

[4] J. C. Beidleman - B. Brewster - P. Hauck, *Fitting functors in finite solvable groups II*, Math. Proc. Camb. Phil. Soc., **101** (1987), pp. 37-55.

[5] G. Chambers, *p-normally embedded subgroups of finite soluble groups*, J. Algebra, **16** (1970), pp. 442-455.

[6] R. S. Dark, *Some examples in the theory of injectors of finite soluble groups*, Math. Z., **127** (1972), pp. 145-156.

[7] K. Doerk - M. Porta, *Über Vertauschbarkeit, normale Einbettung und Dominanz bei Fittingklassen endlicher auflösbarer Gruppen*, Arch. Math., **35** (1980), pp. 319-327.

[8] M. P. Gállego, *The radical of the Fitting class defined by a Fitting functor and a set of primes*, Arch. Math., **48** (1987), pp. 36-39.

[9] P. Hall, *On the Sylow systems of a soluble group*, Proc. London Math. Soc., **43** (1937), pp. 316-323.

[10] B. Hartley, *On Fischer's dualization of formation theory*, Proc. London Math. Soc., **19** (3) (1969), pp. 193-207.

[11] F. P. Lockett, *On the theory of Fitting classes of finite soluble groups*, Math. Z., **131** (1973), pp. 103-115.

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