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On the Propagation of Solutions of a Delay Equation.

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SUMMARY - The propagation of the solutions of the delay equation $\dot{x}(t) = f(x(t)) + g(x_t)$, $x_0 = \varphi$, is related to that of the associated ordinary differential equation $\dot{x}(t) = f(x(t))$, $x(0) = \varphi(0)$.

1. Introduction.

The purpose of this paper is to relate the propagation of solutions of the delay equation

(FDE) $$\dot{x}(t) = f(x(t)) + g(x_t)\,, \qquad x = \varphi\,,$$

to that of the associated ordinary differential equation

(DE) $$\dot{x}(t) = f(x(t))\,, \qquad x(0) = \varphi(0)\,.$$

These equations are set in a Banach space X; the initial data φ is continuous, i.e. $\varphi \in C(-r, 0; X)$ where $0 < r < +\infty$ is the delay, $x_t \in C(-r, 0; X)$ is defined pointwise by $x_t(\theta) = x(t + \theta)$; f is the

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generator of a continuous semigroup of linear bounded operators in X, $T_f(t)$, and $g\colon \mathcal{C}(-r, 0; X) \to X$ is Lipschitz continuous.

The hypotheses on f and g imply that the operators in $\mathcal{C}(-r, 0; X)$

$$A\varphi = -\varphi', \qquad D_A = \{\varphi \in \mathcal{C}^1(-r, 0; X), \varphi'(0) = f(\varphi(0)) + g(\varphi)\},$$

$$B\varphi = -\varphi', \qquad D_B = \{\varphi \in \mathcal{C}^1(-r, 0; X), \varphi'(0) = f(\varphi(0))\},$$

generate two semigroups of translations in $\mathcal{C}(-r, 0; X)$, $T_A(t)$ and $T_B(t)$, and that $T_A(t)$ and $T_B(t)$ give the segments of solutions of (FDE) and (DE) respectively. Hence, in order to study the behavior of solutions of (FDE) and (DE) we are led to study the properties of the semigroups $T_A(t)$ and $T_B(t)$.

As in M. Reed [7, p. 49] and R. Grimmer and Zeman [4], we give the following definition. Let Y be a Banach space, $\{Y_t\}_{t\geqslant 0}$ be a family of subsets of Y and $\{S(t)\}_{t\geqslant 0}$ a family of operators in Y. We say that the family $S(t)$ propagates Y_t if

$$S(t-s)\, Y_s \subset Y_t$$

for all $0 \leqslant s \leqslant t$.

We prove the following result: suppose that $T_f(t)$ propagates a family $\{X_t\}$ of closed subspaces of X; then also the semigroup $T(t)$ generated in X by the solutions of (FDE) propagates the family $\{X_t\}$. This result can be compared with those of M. Reed [7] and Grimmer and Zeman [4] on the propagation of solutions of the semilinear wave equation and of an integrodifferential equation, respectively.

We in fact prove rather more. We associate with the family $\{X_t\}$ in X, a family of subsets $\{Y_t\}$ in $\mathcal{C}(-r, 0; X)$, and prove that if $T_f(t)$ propagates X_t, then $T_B(t)$ and $T_A(t)$ propagate Y_t. We also give a new version of the variation of parameters formula which clarifies the relation between the semigroups $T_A(t)$, $T_B(t)$ and $T_f(t)$.

2. The variation of parameters formula.

Let X be a Banach space with norm $|\cdot|$, and let $\mathcal{C} = \mathcal{C}(-r, 0; X)$, $0 < r < +\infty$, be the space of continuous functions in $[-r, 0]$ with values in X, endowed with the sup norm.

Suppose the operators f and g satisfy the following conditions

(H.1) $f: D_f \subset X \to X$ is the generator of a C_0-semigroup, $T_f(t)$; that is of a strongly continuous semigroup of linear bounded operators.

(H.2) $g: \mathcal{C} \to X$ is Lipschitz continuous.

We associate with f and g the following operators in $\mathcal{C}$.

$$A\varphi = -\varphi', \qquad D_A = \{\varphi \in \mathcal{C}^1, \varphi'(0) = f(\varphi(0)) + g(\varphi)\} \tag{1}$$

and

$$B\varphi = -\varphi', \qquad D_B = \{\varphi \in \mathcal{C}^1, \varphi'(0) = f(\varphi(0))\}. \tag{2}$$

It is well known ([2], [9]) that A and B are the generators of semigroups of translations, $T_A(t)$ and $T_B(t)$, and that $T_A(t)\varphi$ and $T_B(t)\varphi$ give the segments of solutions of the functional differential equation

$$\dot{x}(t) = f(x(t)) + g(x_t), \qquad x_0 = \varphi \tag{FDE}$$

and of the ordinary differential equation

$$\dot{x}(t) = f(x(t)), \qquad x(0) = \varphi(0), \tag{DE}$$

respectively.

The following proposition clarifies the relation between the operators f and B.

PROPOSITION 1. Let $T_f(t)$ and $T_B(t)$ be the semigroups generated respectively by f and B. Then

$$T_B(t)\varphi(\theta) = \begin{cases} \varphi(t+\theta) & t+\theta < 0 \\ T_f(t+\theta)\varphi(0) & t+\theta \geqslant 0 \end{cases} \tag{3}$$

PROOF. We prove, by induction, that

$$((I + \lambda B)^{-n}\varphi)(0) = (I - \lambda f)^{-n}(\varphi(0)). \tag{4}$$

If $n = 1$, (4) follows from

$$(I + \lambda B)^{-1}\varphi(\theta) = \exp[\theta/\lambda](I - \lambda f)^{-1}\varphi(0) + \int_\theta^0 \frac{\exp[(\theta - s)/\lambda]}{\lambda}\varphi(s)\,ds\,.$$

Suppose that (4) is true for n. Then

$$(I + \lambda B)^{-(n+1)}\varphi(\theta) = \exp[\theta/\lambda](I - \lambda f)^{-1}\big((I + \lambda B)^{-n}\varphi(0)\big) + \\ + \int_\theta^0 (I + \lambda B)^{-n}\varphi(s)\,\frac{\exp[(\theta - s)/\lambda]}{\lambda}\,\mathrm{d}s$$

and so

$$(I + \lambda B)^{-(n+1)}\varphi(0) = (I - \lambda f)^{-(n+1)}\varphi(0)\,,$$

as required. Hence, for $\lambda = t/n$,

$$\big((I + t/nB)^{-n}\varphi\big)(0) = (I - t/nf)^{-n}\big(\varphi(0)\big)\,,$$

and letting $n \to \infty$

$$T_B(t)\varphi(0) = T_f(t)\varphi(0)\,.$$

However $T_B(t)$ is a translation, and so (4) follows.

It is now easy to prove the following version of the variation of parameters formula.

TEOREMA 1. Let $T_f(t)$, $T_A(t)$ and $T_B(t)$ be the semigroups generated respectively by the operators f, A and B. Then

$$T_A(t)\varphi = T_B(t)\varphi + \int_0^{+\theta} T_f(t + \theta - s)\,g\big(T_A(s)\varphi\big)\,ds\,,$$

or, to be more precise,

$$T_A(t)\varphi(\theta) = \begin{cases} \varphi(t + \theta) & t + \theta < 0 \\ T_B(t)\varphi(\theta) + \displaystyle\int_0^{t+\theta} T_f(t + \theta - s)\,g\big(T_A(s)\varphi\big)\,ds & t + \theta \geqslant 0\,. \end{cases}$$

PROOF. We know [2] that if $x(t)$ is the solution of (FDE), then

$$x(t) = \begin{cases} T_f(t)\varphi(0) + \int_0^t T_f(t-s)g(x_s)\,ds & t \geqslant 0 . \\ \varphi(t) & t < 0 . \end{cases}$$

Hence, for $t + \theta > 0$,

$$x_t(\theta) = T_f(t+\theta)\varphi(0) + \int_0^{t+\theta} T_f(t+\theta-s)\,g(x_s)\,ds =$$

$$= T_B(t)\varphi(\theta) + \int_0^{t+\theta} T_f(t+\theta-s)\,g(x_s)\,ds .$$

And as $x_t = T_A(t)\varphi$, we have

$$T_A(t)\varphi(\theta) = T_B(t)\varphi(\theta) + \int_0^{t+\theta} T_f(t+\theta-s)\,g(T_A(s)\varphi)\,ds .$$

For results and comments on the variation of parameters formula for (FDE) see [1], [3], [5], [6].

3. Propagation of solutions.

We want to relate the propagation of solutions of the functional differential equation (FDE) to that of the ordinary differential equation (DE), propagation in the sense of the following definition.

DEFINITION. Let Y be a Banach space, $\{Y_t\}_{t\geqslant 0}$ be a family of subsets of Y, $\{S(t)\}_{t\geqslant 0}$ a family of operators with domain and range in Y. We say that the family $S(t)$ propagates Y_t if

$$S(t-s)\,Y_s \subset Y_t$$

for all $0 \leqslant s \leqslant t$.

Note that, if in particular Y_0 is any subset of Y, $S(t)$ is a semigroup of operators in Y and we set $Y_t = S(t)\,Y_0$ then $S(t)$ propagate Y_t. We first relate families of sets propagated by the semigroups $T_f(t)$ and $T_B(t)$.

PROPOSITION 2. Suppose that $T_B(t)$ propagates the family of subsets of $\mathcal{C}$, $\{P_t\}_{t\geq 0}$. Set

$$X_t = \{\psi(0),\ \psi \in P_t\}\,; \tag{5}$$

then $T_f(t)$ propagates the family of subsets of X, $\{X_t\}_{t\geq 0}$.

PROOF. Let $x \in X_s$ and let $\varphi \in P_s$ such that $\varphi(0) = x$ then

$$T_f(t-s)x = T_f(t-s)\varphi(0) = \big(T_B(t-s)\varphi\big)(0)\,,$$

and as $T_B(t-s)\varphi \in P_t$, by (5) $T_f(t-s)x \in X_t$.

Viceversa, given a family $\{X_t\}_{t\geq 0}$ in X propagated by $T_f(t)$ it is not possible in general to find a family $\{P_t\}_{t\geq 0}$ in $\mathcal{C}$ propagated by $T_B(t)$ which satisfies (5). We must be content with condition

$$X_t \supset \{\psi(0),\ \psi \in P_t\}\,, \qquad t\geq 0\,. \tag{6}$$

Let P_0 be a subset which satisfies (6) for $t = 0$. The « smallest » family P_t propagated by $T_B(t)$ which satisfies (6) is $P_t = T_B(t)P_0$. The « largest » is the following.

Let

$$Y = \{\psi \in \mathcal{C}(-r, \infty; X),\ \psi_0 \in P_0,\ \psi(t) \in X_t \quad \text{for all } t > 0\}\,. \tag{7}$$

Define

$$Q_t = \{\varphi \in \mathcal{C},\ \varphi = \psi_t \quad \text{for some } \psi \in Y\}\,.$$

Clearly $Q_0 = P_0$, and $X_t \supset \{\varphi(0),\ \varphi \in Q_t\} \supset T_f(t)X_0$. Also

PROPOSITION 3. $T_B(t)$ propagates the family $\{Q_t\}_{t\geq 0}$.

PROOF. Let $\varphi \in Q_S$. Then $\varphi = \psi_s$ for a $\psi \in Y$. We have

$$T_B(t-s)\varphi(\theta) = T_B(t-s)\psi_s(\theta) =$$

$$= \begin{cases} T_f(t+\theta-s)\,\psi_s(0) & \text{if } t+\theta \geq s\,, \\ \psi_s(t-s+\theta) = \psi(t+\theta) & \text{if } t+\theta < s\,. \end{cases}$$

Set

$$\tilde{\psi}(t) = \begin{cases} \psi(t) & t < s \\ T_f(t-s)\psi(s) , & t \geqslant s , \end{cases}$$

clearly $\tilde{\psi}(t)$ belongs to Y, and as $T_B(t-s)\varphi = \tilde{\psi}_t$, then $T_B(t-s)\varphi \in Q_t$, q.e.d.

We have the following theorem on propagation of solutions.

THEOREM 2. Let $\{X_t\}_{t\geqslant 0}$ be a family of closed linear subspaces of X, let P_0 be a subset of $\mathcal{C}$ such that $X_0 \supset \{\varphi(0), \varphi \in P_0\}$ and let $\{Q_t\}_{t\geqslant 0}$ be the family of subsets of $\mathcal{C}$ defined by (7).

Suppose that $T_f(t)$ propagates X_t and that

$$\text{(H.3)} \qquad g\colon Q_t \to X_t , \qquad \text{for all } t \geqslant 0 .$$

Then $T_A(t)$ propagated Q_t. Hence, in particular, if $x(t, \varphi)$ is the solution of (FDE) and $\varphi \in P_0$, then $x(t, \varphi) \in X_t$, for all $t \geqslant 0$.

PROOF. Let $\varphi \in P_0$. As in [8], we define a sequence of functions $u^{(n)}(t)$, where

$$u^{(0)}(t) = \begin{cases} \varphi(t) , & -r \leqslant t \leqslant 0 \\ T_f(t)\varphi(0) & t > 0 \end{cases}$$

$$u^{(n)}(t) = \begin{cases} \varphi(t) & -r \leqslant t \leqslant 0 \\ T_f(t)\varphi(0) + \int_0^t T_f(t-s) g(u_s^{(n-1)})\, ds . \end{cases}$$

We known that $\lim_{n\to\infty} u^{(n)}(t) = u(t)$ exists uniformly in $[-r, T]$, for $T > 0$, $u(t)$ is continuous and it is the unique solution of (FDE); and that $u_t = T_A(t)\varphi$, $t \geqslant 0$.

We prove by induction that $u^{(n)}(t) \in Y$, for all n. From the definition, we have that $u^{(0)}(t)$ is continuous, $u_0^{(0)} = \varphi \in P_0$; also, as $T_f(t)$ propagates X_t, $u^{(0)}(t) \in X_t$, for all t; it follows that $u^{(0)} \in Y$. Suppose that $u^{(n-1)} \in Y$. Then $u_s^{(n-1)} \in Q_s$, and so $T_f(t-s)g(u_s^{(n-1)}) \in X_t$ and as X_t is closed and linear, also $\int_0^t T_f(t-s) g(u_s^{(n-1)})\, ds \in X_t$. It follows that $u^{(n)}(t) \in X_t$ for $t \geqslant 0$, and as $u_0^{(n)} = \varphi \in P_0$ and $u^{(n)}$ is continuous, $u^{(n)} \in Y$.

As X_t is closed, $u(t) = \lim_{n\to\infty} u^{(n)}(t) \in X_t$. It follows that $u \in Y$; hence $T_A(t)\varphi \in Q_t$.

More generally, let $\varphi \in Q_s$ and let $\psi \in Y$ be such that $\varphi = \psi_s$. Set

$$v^{(0)}(t) = \begin{cases} \psi(t), & -r \leqslant t \leqslant s \\ T_f(t-s)\psi(s), & t > s \end{cases}$$

and

$$v^{(n)}(t) = \begin{cases} \psi(t), & -r \leqslant t \leqslant s \\ T_f(t-s)\psi(s) + \int_0^{t-s} T_f(t-s-\tau) g(v_\tau^{(n-1)})\, d\tau, & t > s \end{cases}$$

then, again by induction, we prove that $v^{(n)} \in Y$ and hence that, if $v(t) = \lim_{n\to\infty} v^{(n)}(t)$, $v \in Y$. But

$$v(t) = \begin{cases} \psi(t), & -r \leqslant t \leqslant s \\ (T_A(t-s)\psi_s)(0), & t > s, \end{cases}$$

and $T_A(t)$ is a translation, and so

$$(T_A(t-s)\psi_s)(\theta) = \begin{cases} \psi_s(t-s+\theta) = \psi(t+\theta), & t+\theta \leqslant s \\ (T_A(t-s+\theta)\psi_s)(0) & t+\theta > s, \end{cases}$$

So $T_A(t-s)\varphi = T_A(t-s)\psi_s = v_t \in Q_t$. And $T_A(t)$ propagates Q_t.

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