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Subgroups of Finite Index in Generalized T-Groups.

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1. Introduction and main results.

The class of $\boldsymbol{T}$-groups is the class of groups in which every subnormal subgroup is normal. In [1] we introduced and studied some classes of generalized $\boldsymbol{T}$-groups. We recall here the relevant definitions.

Let G be a group, m a positive integer, then:

1) $G \in \boldsymbol{T}^*$ (respectively $G \in \boldsymbol{T}_m$) if each subnormal subgroup of G has finite index (resp. index at most m) in its normal closure;

2) $G \in \boldsymbol{V}$ (respectively $G \in \boldsymbol{V}_m$) if each subnormal subgroup of G has a finite number of conjugates (resp. at most m conjugates) in G (that is $|G:N_G(H)| < \infty$, or respectively $|G:N_G(H)| \leqslant m$) for each H subnormal in G;

3) $G \in \boldsymbol{U}$ (respectively $G \in \boldsymbol{U}_m$) if $|H^G:H_G|$ is finite (resp. $|H^G:H_G| \leqslant m$) for every H sn G.

We remind that H^G and H_G denote, respectively, the normal closure and the normal core of H in G.

It is easy to check (see [1]), that $\boldsymbol{U} = \boldsymbol{V} \cap \boldsymbol{T}^*$, and it is obvious that the class of $\boldsymbol{T}$-groups is (properly) contained in $\boldsymbol{U}$.

In this paper, we consider subgroups of finite index in groups belonging to the classes above defined. More precisely, we aim to study to which extent subgroups of finite index of a group in one

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of these classes, belong to the same class. This is in the spirit of a recent paper by H. Heineken and J. Lennox [2], where the authors prove that a subgroup H of finite index in a $\boldsymbol{T}$-group G is an $\boldsymbol{U}_r$-group, the integer r depending only on the index $|G:H|$; moreover, H is a $\boldsymbol{T}$-group if it contains some term of the derived series of G.

In order to make shorter the statement of our first Theorem, we give the following definitions.

Let H be a subgroup of the group G. We say that H is almost normal (see [3, page 191]) in G if $|H^G:H|$ is finite, and almost subnormal if H has finite index in some subnormal subgroup of G; we say further that H is virtyally normal in G if $N_G(H)$ has finite index in G, and virtually subnormal if H is subnormal in a subgroup of finite index in G.

THEOREM 1. *Let G be a group; then the following are equivalent.*

- (i) $G \in \boldsymbol{T}^*$ *(that is, every subnormal subgroup of G is almost normal);*
- (ii) *every almost subnormal subgroup of G is almost normal;*
- (iii) *every virtually subnormal subgroup of G is almost normal;*
- (iv) *the relation of almost normality is transitive in G (we mean that whenever $K \leqslant H \leqslant G$, with K almost normal in H and H almost normal in G, then K is almost normal in G).*

Implication (i) $\Rightarrow$ (iii) gives immediately:

COROLLARY 1. *A subgroup of finite index in a* $\boldsymbol{T}^*$-group *is again a* $\boldsymbol{T}^*$-group.

Also, it will be easy to deduce

COROLLARY 2. *A subgroup of finite index in a* $\boldsymbol{U}$-group *is again a* $\boldsymbol{U}$-group.

An analogous result does not hold for the class $\boldsymbol{V}$, not even for the class $\bigcup_{m\in\mathbb{N}} \boldsymbol{V}_m$. We will show this by means of an example, which is essentially taken from Heineken and Lennox [2]. Turning to groups in which the index of every subnormal subgroup in its normal closure is bounded by a positive integer, we are able to prove:

THEOREM 2. *There exists a function $d: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, such that if $G \in \boldsymbol{T}_m$ and H is a subgroup of G, with $|G:H| \leqslant n$, then $H \in \boldsymbol{T}_{d(m,n)}$.*

It easily follows

COROLLARY 3. *There exists a function* $\bar{d}\colon \mathrm{N}\times\mathrm{N}\to\mathrm{N}$, *such that if* $G\in \boldsymbol{U}_m$ *and* H *is a subgroup of* G, *with* $|G:H|\leqslant n$, *then* $H\in \boldsymbol{U}_{\bar{d}(m,n)}$.

An immediate consequence is

COROLLARY 4 (Heineken and Lennox [2, Theorem B]). *There exists a function* $f\colon \mathrm{N}\to\mathrm{N}$, *such that if* H *is a subgroup of index at most* n *in a* $\boldsymbol{T}$-group, *then* $H\in \boldsymbol{U}_{f(n)}$.

(Indeed, Heineken and Lennox proved an analogous of corollary 3 for a proper subclass of $\bigcup_{m\in\mathrm{N}} \boldsymbol{U}_m$, that they call the class of $\boldsymbol{X}$-groups).

In the hypotheses of Theorem 2, if the subgroup H contains some term of the derived series of G, then the bound d does not really depend on the index of H in G. This is the content of our next result.

THEOREM 3. *There exists a function* $b\colon \mathrm{N}\to\mathrm{N}$, *such that if* H *is a subgroup of finite index in a* $\boldsymbol{T}_m$-group G, *and* H *contains some term of the derived series of* G, *then* $H\in \boldsymbol{T}_{b(m)}$.

It will be evident from the proof, that $b(1)=1$; thus we have an alternative proof of another result of Heineken and Lennox [2, Theorem A], namely:

COROLLARY 5. *A subgroup of finite index in a* $\boldsymbol{T}$-group G, *which contains some term of the derived series of* G, *is a* $\boldsymbol{T}$-group.

2. Proofs and related results.

PROOF OF THEOREM 1. (i) $\Rightarrow$ (ii). Let H be an almost subnormal subgroup of $G\in\boldsymbol{T}^*$; then $|H_n:H|$ is finite, for some term H_n of the normal closure series of H in G. Now, $H_n \operatorname{sn} G$, whence $|H_n^G:H_n|$ is finite. But also $H_n^G=H^G$ and so $|H^G:H|=|H_n^G:H_n|\,|H_n:H|$ is finite and H is almost normal in G.

(ii) $\Rightarrow$ (iii). Let $H\leqslant L\leqslant G$, with H subnormal in L and L of finite index in G. We argue by induction on the defect n of H in L. Let firstly $H\trianglelefteq L$. Since $|G:L|$ is finite, L_G has finite index in G, and $H\cap L_G \trianglelefteq L_G \trianglelefteq G$; hence, in particular, $H\cap L_G$ is almost subnormal in G. If $W=(H\cap L_G)^G$, we have therefore $|W:H\cap L_G|<\infty$, and $W\leqslant L_G$. Now, $|HW/W|\leqslant|L/L_G|$ is finite and $N_G(HW)\geqslant L$, that is $|G:N_G(HW)|<\infty$. By Dičman Lemma (see [5, 14.5.7.]), $(HW/W)^{G/W}$ is finite. In particular, $|(HW)^G:HW|$ is finite. Because $|HW:H|=$ $=|W:H\cap L_G|$ is finite, we have that $|(HW)^G:H|$ is finite, whence $|H^G:H|$ is finite, and H is almost normal in G.

Let now $n > 1$, and $T = H^L$; then, by the case discussed above, $|T^G : T|$ is finite. Now, H is virtually subnormal in T^G and the defect of H in T is $n-1$. By inductive hypothesis and the fact that condition (ii) is obviously inherited by normal subgroups, we have that $|H^{T^G} : H|$ is finite. But H^{T^G} is subnormal in G; thus H is almost subnormal in G, and so H is almost normal in G.

(iii) $\Rightarrow$ (i). Obvious.

(i) $\Rightarrow$ (iv). Let $K \leqslant H \leqslant G$, with both $|K^H : K|$ and $|H^G : H|$ finite. Then K^H is virtually normal in H^G. Since (i) $\Rightarrow$ (iii) and (i) is clearly inherited by normal subgroups, K^H is almost normal in H^G, that is K^H has finite index in $(K^H)^{H^G} = K^{H^G}$. It follows that K has finite index in K^{H^G}. Now, K^{H^G} is subnormal in G and, since (i) $\Rightarrow$ (ii), K is almost normal in G.

(iv) $\Rightarrow$ (i). Obvious. ■

Proof of Corollary 2. Let H be a subgroup of finite index in $G \in \boldsymbol{U}$, and let S be a subnormal subgroup of H. Then S is virtually subnormal in G; thus, since $\boldsymbol{U} \subseteq \boldsymbol{T}^*$, $|S^G : S|$ is finite. Let $L = (S)_{S^G}$, then $L \operatorname{sn} G$ and, in particular, $|L : L_G|$ is finite. But $|S^G : L|$ is also finite, so $|S^G : L_G|$ is finite. Now, $L_G = S_G$ gives $|S^G : S_G| < \infty$. A fortiori, $|S^H : S_H|$ is finite. This holds for any $S \operatorname{sn} H$, thus proving that H is a $\boldsymbol{U}$-group. ■

We observe that, in general, a subgroup of a $\boldsymbol{T}$-group need not belong to $\boldsymbol{T}^*$ (nor to $\boldsymbol{V}$). Let D be a direct product of infinitely many copies of the additive group of the rationals, and $\alpha \in \operatorname{Aut}(D)$ be the inversion map on D Then the natural semidirect product $G = D \rtimes \langle \alpha \rangle$ is a $\boldsymbol{T}$-group. Let A be a free subgroup of infinite rank of D, then the subgroup $\langle A, \alpha \rangle$ of G does not belong to $\boldsymbol{V} \bigcup \boldsymbol{T}^*$.

Corollary 2 can be slightly improved, namely:

Proposition 1. *Let G be a group; the following are equivalent:*

(i) $G \in \boldsymbol{U}$;

(ii) *if $K \leqslant H \leqslant G$ and both $|K^H : K_H|$ and $|H^G : H_G|$ are finite, then $|K^G : K_G|$ is finite.*

Proof. (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii). Let $K \leqslant H \leqslant G$, with $|K^H : K_H|$ and $|H^G : H_G|$ finite, and let $G \in U$. Now, by the same argument used in the proof of Corollary 2, $|K^{H^G} : K_{H^G}|$ is finite. But K^{H^G} and K_{H^G} are both subnormal in G. Since $G \in U$, we have $|K^G : K^{H^G}| < \infty$ and $|K_{H^G} : K_G| < \infty$, and so $|K^G : K_G|$ is finite. ∎

By contrast, the class V is not closed under subgroups of finite index. Indeed, there exist groups, in which every subnormal subgroup has a bounded finite number of conjugates, that admit subgroups of finite index which are not V-groups. An example is the group constructed by Heineken and Lennox in [2]. We report a slightly simplified version of it.

EXAMPLE. Let G be the group generated by a, b, c_i, d_i ($i \in \mathbb{N}$), subject to the following relations:

$$\begin{cases} a^2 = b^3 = c_i^2 = d_i^2 = [c_i, c_j] = [d_i, d_j] = [c_i, d_j] = (ab)^2 = 1 \\ c_i^a = d_i = c_i^b, \quad d_i^b = c_i d_i, \quad \text{for all } i, j = 1, 2, \ldots \end{cases}$$

The $G = AB$, where $A = \langle c_i, d_i;\ i = 1, 2, \ldots \rangle \trianglelefteq G$ is an elementary abelian 2-group, $B = \langle a, b \rangle \simeq S_3$ and $A \cap B = 1$. If $S \operatorname{sn} G$, one easily checks that either $S \leqslant A$ or $S \geqslant \langle A, b \rangle$. Hence A normalizes every subnormal subgroup of G; since $|G/A| = |B| = 6$, this yields $G \in V_6$. Let $H = \langle A, a \rangle$; then $|G : H| = 3$ and H is nilpotent. Now, $\langle a \rangle$ is subnormal of defect 2 in H, but $N_A(\langle a \rangle) = C_A(a) = \langle c_i d_i;\ i = 1, 2, \ldots \rangle$ has infinite index in A; thus $|H : N_H(\langle a \rangle)| = \infty$, that is $H \notin V$. (The group exihibited by Heineken and Lennox in [2] shows, furthermore, that a subgroup of finite index in a V_6-group need not belong to the class of groups with a bound on the defects of their subnormal subgroups).

This example shows, in other words, that $G \in V$ does not imply that every virtually subnormal subgroup of G is virtually normal. On the other hand, we have:

PROPOSITION 2. *Let G be a group; then the following are equivalent:*

(i) *every virtually subnormal subgroup of G is virtually normal;*

(ii) *the relation of virtual normality is transitive in G.*

PROOF. (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii). Assume that the group G satisfies (i) and let $K \leqslant H \leqslant G$ be such that $|G:N_G(H)|$ and $|H:N_H(K)|$ are both finite. Let $L = (N_H(K))_H$ be the normal core of $N_H(K)$ in H; then L is subnormal in $N_G(H)$, that is L is virtually subnormal in G. By our hypothesis, $N_G(L)$ has finite index in G. Hence $M = N_G(H) \cap N_G(L)$ has finite index in G. But M acts, by conjugation, on the finite section H/L; in particular $|M:N_M(N_H(K))| = |M:N_M(N_H(K)/L)|$ is finite and, consequently, $N_G(N_H(K))$ has finite index in G. Now, $K \trianglelefteq N_H(K) \trianglelefteq N_G(N_H(K))$, and so K is virtually subnormal in G. Since G satisfies (i), K is virtually normal in G. ∎

In order to prove Theorems 2 and 3, we need some preliminary lemmas.

LEMMA 1. *Let A be an abelian group, m a positive integer and $G \leqslant \mathrm{Aut}(A)$, such that $|H^G:H| \leqslant m$ for every $H \leqslant A$ (where $H^G = \langle H^a;\ a \in G\rangle$). Then:*

(*a*) *G fixes every subgroup of $A/A[d]$, where $d = m!$ and $A[d] = \{x \in A;\ x^d = 1\}$.*

(*b*) *If A is periodic and reduced, there exists a G-invariant subgroup N of A, such that N can be generated by $4m$ elements, and G fixes every subgroup of A/N.*

PROOF. (*a*) Let $x \in A$, then $|\langle x\rangle^G:\langle x\rangle| \leqslant m$ and $\langle x^d\rangle \leqslant (\langle x\rangle^G)^d \leqslant \langle x\rangle$. Hence $\langle x^d\rangle$ is G-invariant and thus G fixes every subgroup of $A^d = \{x^d; x \in A\}$. Now, the map $\Phi: A \to A^d$, defined by $x^\Phi = x^d$ for every $x \in A$, is a G-homomorphism. Since Φ is surjective and Ker $(\Phi) = A[d]$, we conclude that G fixes every subgroup of $A/A[d]$.

(*b*) (see [1; Lemma 2.9]). Suppose that we have already proved the assertion when A is residually finite. Let B be a basic subgroup of A (see [5; 4.3.4.]). Thus B is a direct product of cyclic groups and B^G is a finite extension of B; hence B^G is residually finite. By our assumption, there exists a G-invariant subgroup N of B^G, such that N can be generated by $4m$ elements, and G acts as a group of power automorphisms (that is, fixing every subgroup) on B^G/N. Now, since N is finite and A is reduced, A/N is also reduced; moreover, A/B^G is divisible. By Lemma 2.2 in [1], we conclude that G fixes every subgroup of A/N.

Thus, it remains to prove (*b*) when A is residually finite.

Let $1 = x_0, x_1, \ldots, x_k$ be distinct elements of A such that, for any $i = 0, 1, \ldots, k-1$:

$$(1) \qquad \langle x_0, \ldots, x_i\rangle^G \cap \langle x_{i+1}\rangle = 1\,, \quad |\langle x_0, \ldots, x_{i+1}\rangle^G : \langle x_0, \ldots, x_i\rangle^G \langle x_{i+1}\rangle| \neq 1.$$

Then, if $X = \langle x_0, \ldots, x_k\rangle$, we get $|X^G : X| > k$. In fact, proceeding by induction, we have $|X^G : X| = |\langle x_1\rangle^G : \langle x_1\rangle| \neq 1$ if $k = 1$ and, if $k > 1$, using the inductive hypothesis:

$$|X^G : X| = |X^G : \langle x_0, \ldots, x_{k-1}\rangle^G \langle x_k\rangle|\,|\langle x_0, \ldots, x_{k-1}\rangle^G \langle x_k\rangle : \langle x_0, \ldots, x_k\rangle| >$$

$$> |\langle x_0, \ldots, x_{k-1}\rangle^G \langle x_k\rangle : \langle x_0, \ldots, x_k\rangle| =$$

$$= |\langle x_0, \ldots, x_{k-1}\rangle^G : \langle x_0, \ldots, x_{k-1}\rangle(\langle x_0, \ldots, x_{k-1}\rangle^G \cap \langle x_k\rangle)| =$$

$$= |\langle x_0, \ldots, x_{k-1}\rangle^G : \langle x_0, \ldots, x_{k-1}\rangle| > k-1\,, \qquad \text{whence } |X^G : X| > k\,.$$

Thus, in our hypotheses, a subset of A satisfying conditions (1) has at most m elements. Let $y_0 = 1, y_1, \ldots, y_r$, $r < m$, be such a subset, with r maximal, and let $K = \langle y_0, \ldots, y_r\rangle^G$; then K is generated by at most $r + \log_2 m < m + \log_2 m$ elements, because $|K : \langle y_0, \ldots, y_r\rangle| \leqslant m$. Since A is residually finite, and K is finite, there exists a subgroup M of finite index in A, maximal subject to the condition: $K \cap M = 1$. Then A/M is generated by elements $z_1 M, \ldots, z_t M$ and $t < m + \log_2 m$. Let $Y = \langle z_1, \ldots, z_t\rangle$ and put $N = KY^G$; then N is generated by at most $2(m + \log_2 m) + \log_2 m = 2m + 3\log_2 m < 4m$ elements, and it is G-invariant. We now show that G fixes every subgroup of A/N; it is enough to check this for cyclic subgroups. Let C/N be a cyclic subgroup of A/N. Since $A = MY = MN$, there exists $x \in M$ such that $C = \langle x, N\rangle$. But then $K \cap \langle x\rangle \leqslant K \cap M = 1$ and, because $\{y_0, \ldots, y_r\}$ is a maximal subset of A satisfying conditions (1), we get $|\langle K, x\rangle^G : \langle K, x\rangle| = 1$; in particular, $\langle N, x\rangle^G = C^G = \langle N, x\rangle = C$. Thus G fixes every cyclic subgroup of A/N, and the Lemma is proved. ■

We remind that the group of automorphisms fixing every subgroup of a group (called the group of power automorphisms) is abelian. We will make use of this fact in the sequel. In particular, under the hypotheses of the previous Lemma, G' centralizes both $A/A[d]$ and A/N.

We observe also that the order of N, in point (b) of the Lemma, cannot be bounded by a function of m. Let n be any positive integer, $C = \langle x \rangle$ a cyclic group of order 2^{n+2} and B the direct product of $n+1$ cyclic groups of order 4; let α be an automorphism of $A = B \times C$ centralizing C and acting as the inversion map on B. If $G = \langle \alpha \rangle \leqslant \mathrm{Aut}(A)$, then $|G| = 2$ and it is easy to check that $|H^G : H| \leqslant 4$ (and indeed $|H^G : H_G| \leqslant 16$) for every $H \leqslant A$. But if N is a G-invariant subgroup of A and $|N| \leqslant 2^n$, then G does not act as a group of power automorphisms on A/N.

LEMMA 2. *There exists a function* $a : \mathbb{N} \to \mathbb{N}$, *such that, if* $G \in \boldsymbol{T}_m$ *and* G *is soluble, then* $G^{(2)}$ *has order at most* $a(m)$.

Observe that, by D. Robinson's result on $\boldsymbol{T}$-groups (see [5; 13.4.2]), we may put $a(1) = 1$.

PROOF. Let G be a soluble $\boldsymbol{T}_m$-group ($m \in \mathbb{N}$). If $m = 1$, then, by the quoted result of Robinson, G is metabelian. Hence assume $m > 1$. Let F be the Fitting radical of G; then F is nilpotent by Lemma 3.1 in [1]. Thus every subgroup of F is subnormal in G; and, in particular, $|H^F : H| \leqslant m$ for every $H \leqslant F$. By a Theorem of I. D. Macdonald [4; Theorem 5.14], $|F'| \leqslant m^{900(\log_2 m)^3} = a_1(m)$. If F_1/F' is the Fitting radical of G/F', then F_1/F' is nilpotent and so, by a well known nilpotency criterion of P. Hall (see [5; 5.2.10]), F_1 is nilpotent. Thus $F_1 = F$ and F/F' is the Fitting radical of G/F'. We may therefore assume, from now on, $F' = 1$. Observe that, G being soluble, this implies $C_G(F) = F$. Now, $|H^G : H| \leqslant m$ for every $H \leqslant F$; we are therefore in a position to apply Lemma 1(a). If $d = m!$, we get that every subgroup of $F/F[d]$ is normalized by G, and thus G' centralizes $F/F[d]$. Now, $F[d]$ is reduced and periodic; by part (b) of Lemma 1, we obtain a subgroup N of $F[d]$, normal in G, such that G normalizes every subgroup of $F[d]/N$ and N can be generated by $4m$ elements. Since $\exp(N) \leqslant d$, we have $|N| \leqslant d^{4m}$. Moreover, G' centralizes $F[d]/N$, whence $[F, G', G'] \leqslant N$. By the three subgroups lemma, $[F, G^{(2)}] \leqslant N$. Now, if $K = C_G(N)$, then the index $|G : K| \leqslant |\mathrm{Aut}(K)|$ is bounded, say $|G : K| \leqslant a_2(m)$. Futhermore, $K \cap G'$ stabilizes a finite series of F. Because G is soluble, this implies $K \cap G' \leqslant F$ and, consequently:

$$|FG' : F| = |G' : F \cap G'| \leqslant |G' : K \cap G'| = |KG' : K| \leqslant |G : K| \leqslant a_2(m)\,.$$

In particular, $|G^{(2)} : G^{(2)} \cap F| \leqslant a_2(m)$. But $[G^{(2)}, G^{(2)} \cap F] \leqslant N$; thus the centre of $G^{(2)}$ has index at most $a_2(m)$ in $G^{(2)}$, modulo N. It follows that $|G^{(3)}N/N|$ is bounded; indeed, Wiegold [6] has obtained

$$|G^{(3)}N/N| \leqslant a_2(m)^{\frac{1}{2}(1+\log_2 a_2(m))} = a_3(m) .$$

Thus, we may assume $G^{(3)} = 1$; so $G^{(2)} \leqslant F$ and $G'F/F$ is therefore abelian. By repeating the arguments used above, we find a $N_1 \trianglelefteq G$ such that $|N_1| \leqslant d^{4m}$, and $[F, G', G'] \leqslant N_1$: In this case $G'F/N_1$ is nilpotent. Applying again Macdonald's Theorem, we get

$|G^{(2)}N_1/N_1| \leqslant a_1(m)$. N_1 being finite, this completes the proof. ∎

The following Lemma is probably well known.

LEMMA 3. *There exists a function $d_0\colon \mathbb{N} \to \mathbb{N}$, such that, if G is a group and $|G\colon Z_r(G)| \leqslant n$ for some positive integer r, then, denoting by W the nilpotent residual of G, G/W is nilpotent and $|W| \leqslant d_0(n)$.*

PROOF. Let $L = Z_r(G)$, r a positive integer. Then, by a result of Baer (see [5; 14.5.1.]), $\gamma_{r+1}(G)$ is finite. This implies at once that W is finite and G/W is nilpotent. We have to show that the order of W is bounded. Now, it is well known that $[W, L] = 1$, thus $W \cap L \leqslant Z(W)$ and so $|W\colon Z(W)| \leqslant |W\colon W \cap L| = |WL\colon L| \leqslant n$. By an already quoted result of Wiegold [6], it is $|W'| \leqslant n^{\frac{1}{2}(1+\log_2 n)}$. Without loss of generality, we may assume, from now on, that W is abelian (observe that $LW'/W' \leqslant Z_r(G/W')$).

Let $C = C_G(W)$; then $C \geqslant LW$. In particular G/C is nilpotent, and $|G\colon C|$ divides $|G\colon L|$. For each prime p dividing $|G\colon C|$ let G_p/C be the Sylow p-subgroup of G/C, and $W_{p'}$ the p'-component of W. Since $W_{p'}$ is abelian, we have:

$$W_{p'} = [W_{p'}, G_p] \times C_{W_{p'}}(G_p) .$$

Moreover, $W_{p'} \cap L \leqslant Z_r(G)$ and so $G/C_G(W_{p'} \cap L)$ is a p'-group. Thus $W_{p'} \cap L \leqslant C_{W_{p'}}(G_p)$ and, consequently:

$$|[W_{p'}, G_p]| \leqslant |W_{p'}/W_{p'} \cap L| = |W_{p'}L/L| \leqslant n .$$

Put $R = \langle [W_{p'}, G_p];\ p \text{ divides } |G\colon C| \rangle$; then $R \trianglelefteq G$ and the order of R is at most $n^{\log_2 n}$. Now, for any prime p dividing $|G\colon C|$, $[W, {}_{s(p)}G_p] = [W_{p'}, G_p] \leqslant R$ for some $s(p) \in \mathbb{N}$.

Therefore, if $s = \max\{s(p);\ p$ dividing $|G:C|\}$, we have:

$$[W, {}_sG] = \langle [W, {}_sG_p];\ p \text{ divides } |G:C|\rangle \leqslant R\,.$$

Since G/W is nilpotent, it follows that G/R is nilpotent, and so $R = W$. In conclusion we get:

$$|W| \leqslant |W'| n^{\log_2 n} \leqslant n^{\frac{1}{2}(1+\log_2 n)} n^{\log_2 n} = n^{\frac{1}{2}(1+3\log_2 n)} = d_0(n)\,. \qquad \blacksquare$$

PROOF OF THEOREM 2. Let $G \in \boldsymbol{T}_m$ and $H \leqslant G$, with $|G:H| = n (m, n \in \mathrm{N})$. Let S be a subnormal subgroup of H; we will prove that $|S^G:S|$ is less or equal to $d(m, n)$, where d is a function from $\mathrm{N}\times\mathrm{N}$ to N, whence, in particular, $|S^H:S| \leqslant d(m, n)$.

Let $K = H_G$, then the order of G/K divides $n!$, and $K \cap S$ is subnormal in K. Thus $K \cap S$ is subnormal in G and so, if $U = (K \cap S)^G$, we have $U \leqslant K$ and $|U:K \cap S| \leqslant m$. Now, $US \operatorname{sn} H$ and $|US:S| \leqslant m$; hence, without loss of generality, we may assume $U = 1$ and, consequently, $K \cap S = 1$.

Let $T = S^{G,m}$ be the m-th term of the normal closure series of S in G. Now, $G \in \boldsymbol{T}_m$ clearly implies that every subnormal subgroup of G has defect at most m in G. Thus T is the minimal subnormal subgroup of G containing S, and $S^T = T$. Furthermore, by Theorem 1 ((i) $\Rightarrow$ (iii)), $|T:S|$ is finite. Because $|S| = |S/K \cap S| = |KS/K|$ divides $n!$, we have that T is a finite $\boldsymbol{T}_m$-group.

Let $L = K \cap T$, then $L \trianglelefteq T$ and $L \cap S = 1$; since S is subnormal in $LS \leqslant H$, it follows that S stabilizes a finite series $[L, {}_iS]$ of L. Thus, $S/C_S(L)$ is nilpotent; SL being finite, this yields $[L, S]C_S(L)/C_S(L)$ nilpotent, and so $[L, S]$ is nilpotent. Let F be the Fitting radical of L; then $F \trianglelefteq T$ and $[L, S] \leqslant F$. Now, $F \in \boldsymbol{T}_m$ and all of its subgroups are subnormal; by Macdonald's Theorem [4; Theorem 5.14(i)], $|F'| \leqslant m^{900(\log_2 m)^3} = a_1(m)$.

Furthermore, T acts by conjugation on $\bar{F} = F/F'$ in such a way that $|\bar{B}^T:\bar{B}| \leqslant m$, for every $\bar{B} \leqslant \bar{F}$. By Lemma 1($b$), there exists an $\bar{N} = N/F' \leqslant F/F'$ generated by at most $4m$ elements, such that $N \trianglelefteq T$ and T fixes, by conjugation, every subgroup of F/N. Since $[L, S] \leqslant F$, we conclude that T normalizes every term of the series $[L, {}_iS]$ mod. N of L, $i = 1, 2, \ldots$ But every factor of this series is centralized by S (because $S \cap L = 1$), and so, from $S^T = T$, we infer that T itself centralizes every such factor. Thus $L/N \leqslant Z_s(T/N)$, where s is the defect of S in SL, and $Z_s(T/N)$ is the s-th term of the upper central series of T/N.

Similarly, by point (*a*) in Lemma 1, T acts as a group of power automorphisms on F/V, where $V/F' = (F/F')[m!]$. Arguing as before, we obtain $L/V \leqslant Z_s(T/V)$. Thus, if $M = N \cap V$, we get: $L/M \leqslant Z_s(T/M)$. Now, M/F' is an abelian group whose exponent divides $m!$, and it is generated by $4m$ elements or less, whence M/F' has order at most $(m!)^{4m}$. Since $|F'| \leqslant a_1(m)$, we have $|M| \leqslant d_1(m)$, where $d_1(m) = a_1(m)(m!)^{4m}$. Moreover L/M is contained in the hypercentre of T/M and:

$$|T:L| = |T:T \cap K| = |TK:K| \leqslant |G:K| \leqslant n!\,.$$

If W/M is the nilpotent residual of T/M, it follows from Lemma 3: $|W/M| \leqslant d_0(n!)$ and, consequently, $|W| \leqslant d_0(n!)\, d_1(m)$. Since T/W is nilpotent, SW is subnormal in $T = S^T$; thus $SW = T$, yielding:

$$|T| = |SW| \leqslant |S|\,|W| \leqslant n!\, d_0(n!)\, d_1(m)\,.$$

Finally, $T = S^{G,m}$ is subnormal in G, whence $|T^G:T| = |S^G:T| \leqslant m$.

Reminding that we assumed at the beginning $K \cap S = 1$, we find:

$$|S^G:S| \leqslant m^2 n!\, d_0(n!)\, d_1(m) = d(m, n)\,. \quad ■$$

Proof of Corollary 3. Let $G \in \boldsymbol{U}_m$, H a subgroup of G of index n, and S a subnormal subgroup of H. Since $\boldsymbol{U}_m \subseteq \boldsymbol{T}_m$, the proof of Theorem 2 gives $|S^G:S| \leqslant d(m, n)$; whence $|S^G:(S)_{S^G}| \leqslant d(m, n)!$. But $K = (S)_{S^G}$ is subnormal in G, and so $|K:K_G| \leqslant m$. Since, clearly, $K_G \leqslant S_G$, we get $|S^G:S_G| \leqslant \bar{d}(m, n)$, where $\bar{d}(m, n) = d(m, n)!\, m$, and thus $H \in \boldsymbol{U}_{\bar{d}(m,n)}$. ■

Observe that, in Theorem 2 and Corollary 3, since the classes $\boldsymbol{T}_m$ and $\boldsymbol{U}_m$ are closed under normal subgroups, nothing changes if we assume $|H^G:H| \leqslant n$ instead of $|G:H| \leqslant n$.

Proof of Theorem 3. Let G be a $\boldsymbol{T}_m$-group, and let H be a subgroup of finite index in G, containing some term of the derived series of G. Let S be a subnormal subgroup of H. If $K = H_G$, then G/K is a finite soluble group, and $S \cap K$ is subnormal in G. As in the proof of Theorem 2, since $|(S \cap K)^G:S \cap K| \leqslant m$, we may assume $S \cap K = 1$.

Let $T = S^{G,m}$ be the *m-th* term of the normal closure series of S in G. Like in the proof of Theorem 2, $S^T = T$ and T is finite.

Let $L = K \cap T$; then $L \trianglelefteq T$ and S sn $SL \leqslant H$. Now S stabilizes the series $[L, {}_iS]$ ($i \in \mathbb{N}$) of L. Since SL is finite, we have that $[L, S]$ is nilpotent; whence S centralizes the factor L/F, where F is the Fitting radical of L. Because $S^T = T$, we obtain that L/F is a central factor of T. In particular, L is soluble. Since $T/L = T/T \cap K \simeq \simeq TK/K \leqslant G/K$ is also soluble, we conclude that T is a (finite) soluble $\boldsymbol{T}_m$-group. By Lemma 2, $|T^{(2)}| \leqslant a(m)$.

We now use a bar to denote subgroups of T modulo $T^{(2)}$. Let $\overline{W}$ be the nilpotent residual of $\overline{T} = T/T^{(2)}$. Since $S^T = T$, we have $\overline{S}\,\overline{W} = \overline{T}$. Moreover, $\overline{W}$ is abelian, and so:

$$(1) \qquad [\overline{S}, \overline{W}] = [\overline{T}, \overline{W}] = \overline{W}$$

and, because $C_{\overline{S}}(\overline{W}) \geqslant \overline{S} \cap \overline{W}$, $\overline{S}/C_{\overline{S}}(\overline{W})$ is nilpotent. This fact, together with (1) and the finiteness of $\overline{T}$, implies:

$$N_{\overline{W}}(\overline{S}) = C_{\overline{W}}(\overline{S}/\overline{S} \cap \overline{W}) = \overline{S} \cap \overline{W}$$

and, consequently: $N_{\overline{T}}(\overline{S}) = \overline{S}$. Thus $N_T(ST^{(2)}) = ST^{(2)}$ and

$$(2) \qquad N_T(S) \leqslant ST^{(2)} .$$

But S sn $H \cap T$ implies $ST^{(2)}$ sn $(H \cap T)T^{(2)}$ and so, by (2), we get $ST^{(2)} = (H \cap T)\,T^{(2)}$, which yields $H \cap T \leqslant ST^{(2)}$. Consequently:

$$|H \cap T : S| \leqslant |ST^{(2)} : S| \leqslant |T^{(2)}| \leqslant a(m) .$$

Now, $S^H \leqslant S^G = T^G$, that is $S^H \leqslant H \cap T^G$. Therefore we find

$$|S^H : H \cap T| \leqslant |H \cap T^G : H \cap T| \leqslant T^G : T| \leqslant m$$

because T is subnormal in G. Together with (2), this gives:

$$|S^H : S| = |S^H : H \cap T|\; |H \cap T : S| \leqslant m a(m) .$$

Finally, recalling that we assumed $S \cap K = 1$, we obtain:

$$|S^H : S| \leqslant m^2 a(m) .$$

This holds for every subnormal subgroup of H, and so $H \in T_{b(m)}$, where $b(m) = m^2 a(m)$. It is at once evident that $b(1) = 1$, and this completes the proof of the Theorem and its Corollary. ∎

Needless to say, $b(m)$ may actually be strictly greater than m. For instance, the group $GL(2, 3)$ is easily seen to be a soluble T_2-group, while its Sylow 2-subgroups belong to $T_4 \setminus T_2$.

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