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A Note on the Jordan-Hölder Theorem.

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1. Introduction.

All groups considered in this note will be finite. In recent years a number of generalizations of the classic Jordan-Hölder Theorem have been done. First, Carter, Fischer and Hawkes in [1; 2, 6] proved that in a soluble group G an one-to-one correspondence like in the Jordan-Hölder theorem can be defined preserving not only G-isomorphic chief factors but even their property of being Frattini or complemented. Later, Lafuente in [4] extended this result to any (not necessarily soluble) finite group.

If P is a subgroup of G with the Cover and Avoidance Property, i.e. a subgroup which either covers or avoids any chief factor of G, one can wonder if it is possible to give a one-to-one correspondence between the chief factors avoided by P with the properties of the one in the Jordan-Hölder Theorem or in the Lafuente Theorem. Here we prove that, in general, the answer is partially affirmative. We give some sufficient conditions for a subgroup with the Cover and Avoidance Property to ensure an affirmative answer to our problem.

2. Notation and preliminaries.

A primitive group is a group G such that for some maximal subgroup U of G, $U <\cdot\ G$, $U_G = \cap \{U^g : g \in G\} = 1$. A primitive group is one of the following types:

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(1) $\mathrm{Soc}(G)$ is an abelian minimal normal subgroup of G complemented by U.

(2) $\mathrm{Soc}(G)$ is a non-abelian minimal normal subgroup of G.

(3) $\mathrm{Soc}(G)$ is the direct product of the two minimal normal subgroups of G which are both non-abelian and complemented by U.

We will denote with $\mathcal{P}_i$, $i \in \{1, 2, 3\}$, the class of all primitive groups of type i. For basic properties of the primitive groups, the reader is referred to [2, 3].

DEFINITION 2.1. We say that two isomorphic chief factors H_i/K_i $i = 1, 2$, of a group G are *G-isomorphic* if $C_G(H_1/K_1) = C_G(H_2/K_2)$. We denote then $H_1/K_1 \cong_G H_2/K_2$.

DEFINITIONS 2.2. (*a*) If H/K is a chief factor of G such that $H/K \leqslant \Phi(G/K)$ then H/K is said to be a *Frattini chief factor* of G.

If H/K is not a Frattini chief factor of G then it is *supplemented* by a maximal subgroup U in G (i.e. $G = UH$ and $K \leqslant U \cap H$). Moreover, this U can be chosen such that G/U_G is a primitive group and $\mathrm{Soc}(G/U_G) = HU_G/U_G \cong H/K$.

(*b*) [2, 3] If H/K is an abelian supplemented chief factor of G, then G/U_G is isomorphic to the semidirect product

$$[H/K](G/C_G(H/K)) \in \mathcal{P}_1$$

since $C_G(H/K) = HU_G$. If H/K is non-abelian then $U_G = C_G(H/K)$ and $G/U_G = G/C_G(H/K) \in \mathcal{P}_2$. Denote

$$[H/K] * G = \begin{cases} [H/K](G/C_G(H/K)) & \text{if } H/K \text{ is abelian} \\ G/C_G(H/K) & \text{otherwise}. \end{cases}$$

The primitive group $[H/K] * G$ is called the *monolithic primitive group associated with the chief factor H/K of the group G.*

The chief factor $\mathrm{Soc}(G/U_G) = HU_G/U_G$ is called *precrown of G associated with H/K and U.* (Notice that if H/K is non-abelian $HC_G(H/K)/C_G(H/K)$ is the unique precrown of G associated with H/K.)

(*c*) [3] If H_i/K_i, $i = 1, 2$ are supplemented chief factors of G, we say that they are *G-related* if there exist precrowns C_i/R_i associated with H_i/K_i, such that $C_1 = C_2$ and there exists a common complement U of the factors $R_i/(R_1 \cap R_2)$ in G.

Two G-isomorphic non-abelian supplemented chief factors have the same precrown and therefore are G-related. If they are abelian, then they are G-isomorphic if and only if they are G-related.

G-relatedness is an equivalence relation on the set of all supplemented chief factors of G.

For more details the reader is referred to [3].

3. CAP-subgroups.

DEFINITION 3.1. Let G be a group, M and N two normal subgroups of G, $N \leqslant M$, and P a subgroup of G. We say that.

(*a*) P *covers* M/N if $M \leqslant PN$.

(*b*) P *avoids* M/N if $P \cap M \leqslant N$.

(*c*) P is CAP-*subgroup* of G if every chief factor of G is either covered or avoided by P.

LEMMA 3.2 (Schaller [5]). Let G be a group, P a subgroup of G and N a normal subgroup of G.

(*a*) If P is a CAP-subgroup of G then NP and $N \cap P$ are CAP-subgroups of G and PN/N is a CAP-subgroup of G/N.

(*b*) If $N \leqslant P$ and P/N is a CAP-subgroup of G/N, then P is a CAP-subgroup of G.

THEOREM 3.3. Given a group G and a CAP-subgroup P of G, there exists a one-to-one correspondence between the chief factors covered by P in any two chief series of G in which corresponding factors have the same order (but they are not necessarily G-isomorphic).

PROOF. Denote

$$(1) \qquad 1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G\,,$$

$$(2) \qquad 1 = H_0 \lhd H_1 \lhd \ldots \lhd H_m = G\,,$$

two chief series of G and suppose

$$\mathcal{G} = \{G_{i_j}/G_{i_j-1}\colon j = 1, \ldots, r;\ i_j < i_{j'},\ \text{if}\ j < j'\}\,,$$

$$\mathcal{H} = \{H_{l_k}/H_{l_k-1}\colon k = 1, \ldots, s;\ l_k < l_{k'},\ \text{if}\ k < k'\}\,,$$

are the chief factors of G covered by P in (1) and (2) respectively. We prove $r = s$ by induction on $|P|$. If $P = 1$ there is nothing to prove. The normal series of P, $\{P \cap H_l : l = 0, \dots, m\}$ is by, omitting repetitions, $\{P \cap H_{l_k} : k = 0, \dots, s;\ l_0 = 0\}$ and similarly with (1): $\{P \cap G_{i_j} : j = 0, 1, \dots, r;\ i_0 = 0\}$ is a normal series of P. Consider $Q = P \cap G_{i_r - 1}$. Q is strictly contained in P and by 3.2, Q is a CAP-subgroup of G covering G_{i_j}/G_{i_j-1} for $j = 1, \dots, r-1$ and avoiding all the other chief factors in (1). By induction Q covers $r-1$ chief factors of $\mathcal{H}$. If $r - 1 = s$ we have $|Q| = |P|$ and then $Q = P$ which is not true. Hence $r \leqslant s$. Similarly taking $Q^* = P \cap H_{l_s - 1}$ we obtain $s \leqslant r$ and then $s = r$, as required.

Finally, by induction, there exists a one-to-one correspondence between the chief factors covered by Q in $\mathcal{G}$ and in $\mathcal{H}$ in which corresponding factors have the same order. The only chief factor in $\mathcal{H}$ avoided by Q is H_{i_t}/H_{i_t-1}, say; then

$$|G_{i_r}/G_{i_r-1}| = |P|/|Q| = |H_{i_t}/H_{i_t-1}|$$

and the theorem is proved.

EXAMPLE 1. If we consider $G = \langle a, b;\ c: a^9 = b^2 = c^9 = 1 = [a, c] = [b, c],\ a^b = a^{-1} > \cong D_9 \times Z_9$ and $P = \langle a^3 c^3 \rangle$ then P is a CAP-subgroup of G (G is supersoluble) and if we take the two chief series.

$$1 \lhd \langle a^3 \rangle \lhd \Phi(G) \lhd \dots \lhd G\,,$$

$$1 \lhd \langle c^3 \rangle \lhd \Phi(G) \lhd \dots \lhd G\,,$$

then P covers $\Phi(G)/\langle a^3 \rangle$ and $\Phi(G)/\langle c^3 \rangle$. The first is central and the latter eccentric and so they are not G-isomorphic.

COROLLARY 3.4 (Jordan). All chief series of the group, G have the same length.

4. SCAP-subgroups.

DEFINITION 4.1. Let P be a CAP-subgroup of a group G. We will say that P is a *Strong* CAP-*subgroup* of G, SCAP-subgroup for short, if P satisfies

(a) Given a supplemented chief factor H/K of G avoided by P then P is contained in some maximal supplement U of H/K in G, such that $G/U_G \in \mathcal{S}_1 \cup \mathcal{S}_2$.

(*b*) If Y and M are normal subgroups of G, $M \leqslant Y$ and $Y/M \leqslant \Phi(G/M)$ then $(P \cap Y)M/M$ is a normal subgroup of G/M.

If t is a trasposition, $\langle t \rangle$ is a SCAP-subgroup of Sym(n), for every n; if G is soluble, then Hall subgroups and maximal subgroups are SCAP-subgroups. In the example of theorem 3.3, P is a CAP-subgroup satisfying (*a*) but not (*b*).

Notice that if P is a SCAP-subgroup of G and $N \lhd G$, then PN/N is a SCAP-subgroup of G/N.

Given a supplemented chief factor H/K of G avoided by the SCAP-subgroup P, then there exists always a precrown of G associated with H/K avoided by P by condition (*a*). Conversely if P avoids a precrown of G associated with H/K then P avoids H/K.

Consider now, for each SCAP-subgroup P of G and for each supplemented chief factor H/K of G, avoided by P,

$$R_P = \cap \{T\colon C/T \text{ is a precrown of } G \text{ avoided by } P \text{ associated with a chief factor } H_0/K_0,\ G\text{-related to } H/K\}.$$

If $P = 1$, we simply denote $R = R_1$.

Definition 4.2. With the above notation, the *P-crown of G associated with H/K* is the factor group C/R_P, $C = HC_G(H/K)$.

Clearly C/R_P is avoided by P. For more details on crowns see [3].

Lemma 4.3. Let P be a SCAP-Subgroup of G and H/K a supplemented chief factor of G avoided by P. Denote C/R_P the P-crown of G associated with H/K.

If H_0/K_0 is a chief factor of G such that $K_0R_P < H_0R_P \leqslant C$ then H_0/K_0 is a supplemented chief factor of G avoided by P and G-related to H/K.

Proof. Because of [3; 2.7] we have only to prove that P avoids H_0/K_0. Since $P \cap R_PH_0 \leqslant P \cap C = P \cap R_P \leqslant P \cap R_PK_0$ then equality holds and H_0R_P/K_0R_P is a chief factor of G avoided by P. If H_0/K_0 were covered by P, then $H_0 = K_0(H_0 \cap P)$ and $R_PH_0 = R_PK_0(H_0 \cap \cap P) \leqslant R_PK_0(H_0R_P \cap P)$ and hence P would cover H_0R_0/K_0R_P, a contradiction. Therefore P avoids H_0/K_0.

Lemma 4.4. Let N_i, $i = 1, 2$, two distinct supplemented minimal normal subgroups of G and P a SCAP-subgroup covering N/N_i, $(N = N_1 \times N_2)$, and avoiding N_i, $i = 1, 2$. Denote C_i/R_i the P-crown associated to N_i. Then N_1 and N_2 are G-isomorphic.

PROOF. Suppose that for some $i \in \{1, 2\}$, $N_i \leqslant R_{3-i}$, If N/N_i is covered by R_{3-i} then $N_{3-i} \leqslant N \leqslant R_{3-i}$ and this is a contradiction. So, N/N_i is avoided by R_{3-i} and by 4.3, NR_{3-i}/R_{3-i} is a chief factor of G avoided by P and $R_{3-i} \cap P = NR_{3-i} \cap P$. Now $N \cap P \leqslant R_{3-i} \cap P$ and $N = N_i(P \cap N) \leqslant N_i(P \cap R_{3-i}) \leqslant R_{3-i}$, a contradiction.

Therefore $N_i \cap R_{3-i} = 1$ for any $i \in \{1, 2\}$ and hence

$$R_{3-i}N_i/R_{3-i} \cong_G N_i \quad \text{and} \quad R_{3-i} < R_{3-i}N_i \leqslant C_{3-i}.$$

By 4.3 N_i is G-related to N_{3-i}.

Suppose N_1 and N_2 are not G-isomorphic; then $T_1 \neq T_2$, where $T_i = C_G(N_i)$, $i = 1, 2$. P avoids N_i and then avoids its precrown and $P \cap T_i = P \cap N_i T_i$, $i = 1, 2$. Then $P \cap T_1 \cap T_2 = P \cap N(T_1 \cap T_2)$ and $T_1 \cap N(T_1 \cap T_2) = T_1 \cap N_2 T_2 = T_1 \cap N_1 T_1 = T_1$ since $N_1 T_1 = N_2 T_2$, and $P \cap T_1 = P \cap T_1 \cap T_2 = P \cap T_2$. Hence P avoids $T_i/T_1 \cap T_2$, $i = 1, 2$ and then avoids its precrowns $T_1 T_2/T_i$, $i = 1, 2$. Since P covers N/N_i, $N \leqslant PN_i$ and $P \cap N \leqslant P \cap T_2$. Consequently $N \leqslant PN_1 \cap T_2$ and then $N_2 \leqslant T_2$, a contradiction. Therefore $N_1 \cong_G N_2$.

THEOREM 4.5. Let P be a SCAP-subgroup of G and consider two sections

$$(1) \qquad X = N_0 \lhd N_1 \lhd \ldots \lhd N_n = Y,$$

$$(2) \qquad X = M_0 \lhd M_1 \lhd \ldots \lhd M_n = Y,$$

of two chief series of G. Denote

$$\mathcal{N}_P = \{N_i/N_{i-1}: \text{ chief factors in (1) avoided by } P\},$$

$$\mathcal{M}_P = \{M_j/M_{j-i}; \text{ chief factors in (2) avoided by } P\}.$$

Then, there exists a one-to-one correspondence σ between $\mathcal{N}_P$ and $\mathcal{M}_P$ such that

(*a*) N_i/N_{i-1} is a Frattini chief factor if and only if $(N_i/N_{i-1})^\sigma$ is a Frattini chief factor; in this case $N_i/N_{i-1} \cong_G (N_i/N_{i-1})^\sigma$.

(*b*) If N_i/N_{i-1} is supplemented, then N_i/N_{i-1} is G-isomorphic to $(N_i/N_{i-1})^\sigma$.

PROOF. WLOG we can assume that $X = 1$, $N = N_1 \neq M_1 = M$, $Y = N \times M$; by [3; 3, 2] we can assume that P covers Y/M and

Y/N and avoids N and M. So we must see if the correspondence

$$N \Leftrightarrow M, \qquad Y/N \Leftrightarrow Y/M$$

satisfies the theorem. We have three cases:

(1) $Y \cap \Phi(G) = 1$. Then all chief factors below Y are supplemented, by [3, 2, 8]. Now apply 4.4 to get that N_1 is G-isomorphic to N_2.

(2) $1 < Y \cap \Phi(G)$ and Y non-abelian. Then we can assume that N is abelian and M non-abelian. By order considerations P cannot cover Y/N and Y/M at the same time.

(3) $1 < Y \cap \Phi(G)$ and Y abelian. Suppose first that $W = P \cap \cap Y \cap \Phi(G) = 1$. Then $Y = (P \cap Y) \times (Y \cap \Phi(G))$ and we can suppose that $N \cap \Phi(G) = 1$. Then P is contained in some maximal complement of N in G, say U. Since $Y \cap \Phi(G) \leqslant Y \cap U < Y$, $Y \cap \Phi(G) = = Y \cap U$. But $Y \cap P \leqslant Y \cap U = Y \cap \Phi(G)$ and then $P \cap Y = 1$, a contradiction. Therefore $W \neq 1$, and by condition (b) in the definition of SCAP-subgroup, $1 \neq W \lhd G$ and then $Y = W \times N = W \times M$. Hence $Y/N \simeq_G Y/M$ and $N \simeq_G M$. Finally $N \leqslant \Phi(G)$ if and only if $M \leqslant \Phi(G)$.

Remark. If $P = 1$ this is Lafuente's lemma in [4]. If G is soluble (and $P = 1$) we obtain the Carter-Fischer-Hawkes lemma [1; 2, 6].

Example 2. Conditions (a) and (b) in the definition of SCAP-subgroups are necessary. Example 1 shows that we cannot remove (b). To see the same for (a), take $G = \langle a, b : a^4 = b^2 = [a, b] = 1 \rangle \simeq \simeq Z_4 \times Z_2$ and $P = \langle a^2 b \rangle$. Then P avoids $\langle a^2 \rangle$ and $\langle b \rangle$. But $\langle a^2 \rangle = = \Phi(G)$ and $\langle b \rangle$ is complemented by $\langle a \rangle$. Here P is not contained in any complement of $\langle b \rangle$ in G.

Notice that in this example P is a normal CAP-subgroup which is not a SCAP-subgroup.

Example 3. SCAP-subgroups are not the only CAP-subgroups satisfying the thesis of theorem 4.5. Consider $V = \langle a, b \rangle \simeq Z_4 \times Z_4$ and $Z = \langle z \rangle \simeq Z_3$ such that $a^z = b$, $b^z = ab$. Take two copies V_1 and V_2 of V and form $W = V_1 \times V_2$ and then the semidirect product $G = [W]Z$. Consider $P = \langle a_1^2 b_2^2, a_2^2 b_2^2 \rangle$ (indexed in the obvious way).

Any chief series of G is of one of the following forms:

$$(1_{i,j}) \qquad 1 \lhd \Phi(V_i) \lhd \Phi(G) \lhd \Phi(G) V_j \lhd F(G) \lhd G, \quad i, j \in \{1, 2\},$$

$$(2_t) \qquad 1 \lhd \Phi(V_t) \lhd V_t \lhd \Phi(G) V_t \lhd F(G) \lhd G, \quad t \in \{1, 2\},$$

P covers $\Phi(G)/\Phi(V_i)$ and $\Phi(G) V_t/V_t$ and certainly $\Phi(G)/\Phi(V_i) \cong$ $\cong_G \Phi(G) V_t/V_t$ for each $i, t \in \{1, 2\}$. So P satisfies the thesis of 4.5.

However $P \leqslant \Phi(G)$ and P is not normal in G and then P does not satisfies condition (b) of SCAP-subgroups. Since P avoids $V_i/\Phi(V_i)$ but it is not contained in any complement of $V_i/\Phi(V_i)$ in G, P does not satisfy condition (a) of SCAP-subgroups either.

EXAMPLE 4. Take $G = N \times M$ where $N \cong \text{Alt}(5) \cong M$ and consider P the diagonal subgroup $P \cong \text{Alt}(5)$. Then P is a CAP-subgroup that avoids M and N. M and N are G-related but not G-isomorphic since $C_G(M) = N \neq M = C_G(N)$. Here P is a maximal subgroup of G and $G/P_G = G \in \mathfrak{I}_3$.

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