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Polyserial Modules over Valuation Domains

L. FUCHS - L. SALCE (*)

0. Introduction.

In what follows, R will denote a valuation domain, i.e. a commutative domain with 1 in which the ideals form a chain under inclusion. We assume that R is not equal to its field Q of quotients.

As is well known, finitely generated R -modules need not be direct sums of cyclic R -modules, unless R is an almost maximal valuation domain (i.e. R/I is linearly compact in the discrete topology for every ideal $I \neq 0$). The problem of describing the structure of finitely generated R -modules over arbitrary valuation domains R is extremely difficult, though considerable progress has been made recently in special cases; see the survey [SZ₄].

Here we do not wish to address ourselves directly to this problem, but intend to deal with a closely related question. A main motivation of this paper is to learn more about the submodules of finitely generated R -modules. We feel that such a study might lead to useful information about the finitely generated modules themselves. Another motivation is an interesting new class of modules which we discussed in [FS] under the name of polyserial modules and which is a natural generalization of the class of finitely generated modules.

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Actually, three different classes of R -modules will be studied here: besides the class of submodules of finitely generated R -modules and the class of polyserial R -modules, we introduce a new class (which includes both mentioned classes) whose members will be called weakly polyserial. Our purpose is to find the precise relationship between these classes by pointing out their common features and their differences. In the countable generated case, more accurate statements can be made.

All the modules under consideration are subject to certain finiteness conditions, and as a result, they admit numerical invariants. The invariants we are interested in are the length, the Malcev rank, the Fleischer rank, the Goldie dimension and its dual. One of our main concerns will be to compare these numerical invariants and to draw conclusions from the mere fact that they are finite.

Though we believe that this paper will help us understand the structure of the modules considered, we must admit that we have not succeeded in obtaining a satisfactory theory for them. In fact, some fundamental questions (like «Is a summand of a polyserial again a polyserial?») have been left unanswered.

For unexplained terminology and notation, we refer to our book [FS].

1. Preliminaries.

To start with, we collect a few lemmas which will be needed in later discussions.

The letter P will exclusively be used to denote the maximal ideal of R ; thus R/P is a field. A module which can be generated by at most n elements will be called n -generated; here n stands for a positive integer.

LEMMA 1.1. Let M be an n -generated R -module. If $X = \{x_1, \dots, x_m\}$ ($m > n$) is a generating set of M , then there is a subset $\{x_{i_1}, \dots, x_{i_n}\}$ of X which generates M .

PROOF. The cosets $\bar{x}_1, \dots, \bar{x}_m \bmod PM$ of the x_i span the R/P -vector space M/PM . By hypothesis, its dimension is $\leq n$. If $\bar{x}_{i_1}, \dots, \bar{x}_{i_n}$ span M/PM , then by Nakayama's lemma, x_{i_1}, \dots, x_{i_n} generate M . \square

Recall that the *Goldie dimension* $g(M)$ of a module M is the supremum of all cardinals n such that M contains a direct sum of n non-zero submodules.

LEMMA 1.2. Finitely generated submodules of an n -generated R -module are n -generated.

PROOF. Let S be a maximal immediate extension of R and M an n -generated R -module. Then $\hat{M} = S \otimes_R M$ is an n -generated S -module (by the way, it is the pure-injective hull of M), and as such it is the direct sum of at most n cyclic S -modules. If N is an m -generated ($m \in \mathbb{Z}$) R -submodule of M , then (by the R -flatness of S) $\hat{N} = S \otimes_R N$ is an S -submodule of \hat{M} , and is a direct sum of at most m cyclic S -modules. A simple comparison of the Goldie dimensions of \hat{M} and \hat{N} shows that $m \leq n$, i.e. N is n -generated. \square

LEMMA 1.3. Let $\{a_1, \dots, a_n\}$ be a minimal generating set of the R -module M . If N is a finitely generated submodule of M which cannot be generated by less than n elements, then M/N is a finitely presented R -module.

PROOF. The proof of Lemma 2 in Fuchs-Monari-Martinez [FM] applies to establish this claim as well. \square

For the height-ideal, see [FS, p. 157].

LEMMA 1.4. In a countably generated R -module M , the height-ideals of elements are at most countably generated.

PROOF. Write M as the union of an ascending sequence

$$0 = M_0 < M_1 < \dots < M_n < \dots$$

of finitely generated R -modules M_n . Recall that r^{-1} ($r \in R$) belongs to the height ideal of $a \in M$ if and only if $a \in rM$ which happens if and only if $a \in rM_n$ for some n . Heights of elements in finitely generated R -modules are cyclic whence the assertion should be evident. \square

Finally, a preparatory lemma of a different nature. p.d. stands for « projective dimension ».

LEMMA 1.5. Let T be a submodule of the R -module A such that p.d. $A/T \leq 1$. Every homomorphism $\varphi: T \rightarrow D$ into a divisible R -module D extends to a homomorphism $\psi: A \rightarrow D$.

PROOF. The exact sequence $0 \rightarrow T \rightarrow A \rightarrow A/T \rightarrow 0$ induces the exact sequence $\text{Hom}_R(A, D) \rightarrow \text{Hom}_R(T, D) \rightarrow \text{Ext}_R^1(A/T, D)$. The last module vanishes because p.d. $A/T \leq 1$ and D is divisible (cf. [FS, p. 126]). Hence the claim is immediate. \square

2. Weakly polyserial modules.

As pointed out in the Introduction, one of our principal goals is to investigate the submodules of finitely generated R -modules. To this end, it seems advisable to introduce a class of R -modules which is closed under taking submodules, factor modules and extensions. This is the smallest class that contains all uniserial R -modules and is closed under the mentioned operations (*uniserial* means that the submodules form a chain under inclusion).

An R -module M will be called *weakly polyserial* if it has a finite chain of submodules

$$(1) \quad 0 = M_0 < M_1 < \dots < M_n = M$$

such that each factor M_i/M_{i-1} ($i = 1, \dots, n$) is uniserial. n is the *length* of (1). If, in addition, each M_i is pure in M , then M is said to be *polyserial* (see [FS, p. 189]). In this case (1) is called a *pure-composition series* for M and n is the length $l(M)$ of M (which is an invariant of M).

Finitely generated modules are polyserial; cf. [FS, p. 42]. It is easy to see that a torsion-free R -module is (weakly) polyserial exactly if it has finite rank.

LEMMA 2.1. The class of weakly polyserial R -modules is closed under taking submodules, factor modules and extensions.

PROOF. If M has a chain (1) with uniserial factors, then setting $N_i = N \cap M_i$ and $T_i = (N + M_i)/N$ for a submodule N of M , we obtain chains

$$0 = N_0 \leq N_1 \leq \dots \leq N_n = N \quad \text{and} \quad 0 = T_0 \leq T_1 \leq \dots \leq T_n = M/N$$

Here

$$N_i/N_{i-1} = (N \cap M_i)/(N \cap M_{i-1}) \cong [(N \cap M_i) + M_{i-1}]/M_{i-1}$$

and

$$T_i/T_{i-1} \cong (N + M_i)/(N + M_{i-1}) \cong M_i/[M_i \cap (N + M_{i-1})]$$

are a submodule and a factor module of M_i/M_{i-1} , respectively, hence uniserial. Thus both N and M/N are weakly polyserial. The assertion on the extensions is evident. \square

The *Fleischer rank* of a module M is the minimum rank of torsion-free R -modules having M as an epimorphic image; cf. [FS, p. 181]. (2.1) implies at once:

COROLLARY 2.2. Modules of finite Fleischer rank are weakly polyserial. \square

From (2.1) it is easy to derive:

PROPOSITION 2.3. An R -module M is (weakly) polyserial if and only if its torsion part tM is (weakly) polyserial and the torsion-free module M/tM is of finite rank.

PROOF. For weakly polyserials, this is an immediate consequence of (2.1) and the remark preceding it. For polyserials, sufficiency is pretty obvious, while necessity follows at once from the observation that if N is pure in M , then $tN = N \cap tM$ is pure in tM . \square

In view of (2.3), in our study of (weakly) polyserials we may primarily be concerned with the torsion case.

Let us point out another possible reduction in the study of polyserial modules. As a starting point, observe that a torsion uniserial module M is divisible (i.e. $rM = M$ for all $r \neq 0$ in R) exactly if it is unbounded (i.e. there is no $0 \neq r \in R$ with $rM = 0$). Consequently, a bounded (weakly) polyserial module has no divisible uniserial factors in (1). We intend to show that every polyserial R -module is a pure extension of a divisible polyserial module by a bounded polyserial module.

We begin with a lemma. $RD \text{ ext}^1(D, M)$ denotes the group of all pure extensions of M by D ; see [FS, p. 59].

LEMMA 2.4. If M is a bounded and D is a divisible R -module, then $RD \text{ ext}^1(D, M) = 0$.

PROOF. Choose $0 \neq r \in R$ such that $rM = 0$, and let E be a pure extension of M by D . Thus $M \cap rE = rM = 0$. In view of the divisibility of E/M , $M + sE = E$ for every $0 \neq s \in E$; in particular, $M + rE = E$. This yields $E = M \oplus rE$ where $rE \cong D$. \square

We can now verify

PROPOSITION 2.5. For a polyserial torsion R -module M , there exists a pure-exact sequence

$$(2) \quad 0 \rightarrow D \rightarrow M \rightarrow T \rightarrow 0$$

where D is divisible polyserial and T is bounded polyserial.

PROOF. Let (1) be a pure-composition series for M . If j is the first index for which M_j/M_{j-1} is unbounded (and thus divisible), then by (2.4) $M_j = M_{j-1} \oplus D_1$ for some divisible uniserial module D_1 . Evidently, M/D_1 is polyserial of length $n-1$; in fact, the canonical images of the M_i in M/D_1 yield a pure-composition series for M/D_1 after the deletion of M_j/D_1 . By induction, M/D_1 may be assumed to admit a pure-exact sequence like (2). Noting that D_1 is pure in M , the claim follows at once. \square

One might expect that divisible polyserials are easier to handle than polyserials in general. Unfortunately, this is not the case. As a matter of fact, there can exist very strange divisible polyserial modules. With the aid of R. Jensen's Diamond Principle, it is possible to construct, over suitable valuation domains R , indecomposable divisible polyserial R -modules of any length with all the factors in (1) isomorphic to Q/R or to any non-standard uniserial divisible R -module; see Fuchs [Fu₁].

3. The Malcev rank.

By the *Malcev rank* of an R -module M is meant the smallest cardinal m such that every finitely generated submodule of M can be generated by at most m elements. The Malcev rank $\mu(M)$ of M can be finite or \aleph_0 . We are interested in modules of finite Malcev ranks.

Evidently, the Malcev rank can not increase forming submodules or taking epimorphic images. (1.2) shows that the Malcev rank of a

finitely generated R -module is precisely the minimal cardinality of its generating sets. An obvious consequence of (1.1) is that the Malcev rank is an additive function. Evidently, the Malcev rank of a uniserial module $\neq 0$ is 1.

It is readily checked that the Malcev rank of a torsion-free R -module of finite rank n is exactly n . The following result generalizes this observation.

THEOREM 3.1. If M is weakly polyserial and (1) has uniserial factors, then the Malcev rank of M is at most n . If M is polyserial and (1) is a pure-composition series for M , then the Malcev rank of M is precisely n , i.e. $\mu(M) = l(M)$.

PROOF. By induction on the length n of (1). To start the induction, assume $n = 1$, i.e. $M \neq 0$ is uniserial. In this case, $\mu(M) = 1$, indeed. Let $n > 1$ and suppose the claim holds true for weakly polyserials with chains (1) of lengths $< n$. Let N denote the last but one term in (1), and let $F = \langle x_1, \dots, x_m \rangle$ ($m \geq n$) be a finitely generated submodule of M . Then $(F + N)/N$ is (finitely and so) singly generated as a submodule of M/N ; say, $x_m + N$ is a generator. Choose $r_1, \dots, r_{m-1} \in R$ such that $x_1 - r_1 x_m, \dots, x_{m-1} - r_{m-1} x_m \in N$. Induction hypothesis applied to N yields that $n - 1$ of the generators suffice to generate $\langle x_1 - r_1 x_m, \dots, x_{m-1} - r_{m-1} x_m \rangle$. These $n - 1$ along with x_m generate F , thus $\mu(M) \leq n$.

To prove the second part, suppose M polyserial and $\mu(M) \leq n - 1$. Again inducting on n , N contains a submodule G which can be generated by $n - 1$ but not by fewer elements: $G = \langle y_1, \dots, y_{n-1} \rangle$. Choose $y_n \in M/N$ and set $F = \langle G, y_n \rangle$. By (1.3), $R(y_n + N)$ is finitely presented, say, $\cong R/Rs$ for some $s \in R$. Owing to the purity of N in M , some $y_0 \in N$ satisfies $sy_0 = sy_n$. Manifestly, $G_0 = \langle G, y_0 \rangle$ requires at least $n - 1$ generators (see (1.2)), so $F_0 = \langle G, y_n \rangle = G_0 \oplus \oplus R(y_n - y_0)$ requires n generators. Therefore $\mu(M) \leq n - 1$ is impossible and $\mu(M) = n$ follows. \square

Our next purpose is to compare the Malcev rank with two numerical invariants: the Goldie dimension and its dual.

The *Goldie dimension* $g(M)$ of a module M has been defined above. It is a trivial observation that if $g(M)$ is at most countable, then necessarily $g(M) \leq \mu(M)$ for any module (over any ring).

Dual Goldie dimensions have been defined and discussed by Fleury [F1], Rangaswamy [R], Grzeszczuk and Puczyłowski [GP]. We

rephrase the definition in order to make it more suitable to our purpose.

Let M be a weakly polyserial R -module. Consider all epimorphisms

$$(3) \quad \varphi: M \rightarrow U_1 \oplus \dots \oplus U_m$$

where the U_i are non-zero R -modules, and define $\gamma(M)$ as the largest m for which such a φ exists. The next lemma shows that there is no loss of generality in restricting the U_i to uniserials.

LEMMA 3.2. Let M be a weakly polyserial R -module and K a proper submodule of M . Then there is a proper submodule H of M which contains K such that M/H is uniserial.

PROOF. Induct on the length n of the chain (1) with uniserial factors. In case $n = 1$, $H = K$ is a good choice. If $n > 1$, then let $N = M_{n-1}$ in (1), and observe that if $N + K < M$, then $H = N + K$ is as desired. If $N + K = M$, then $M/K \cong N/(N \cap K)$, and by induction hypothesis, some $H' < N$ satisfies $N \cap K \leq H'$ and N/H' is uniserial. Setting $H = H' + K$, we have that $M/H = (N + H)/H \cong N/(N \cap H) = N/H'$ is uniserial $\neq 0$. \square

We now prove:

PROPOSITION 3.3. For a weakly polyserial R -module M , both $g(M) \leq \mu(M)$ and $\gamma(M) \leq \mu(M)$.

PROOF. Only the second part requires a proof. Just observe that $\gamma(M)$ is nothing else than the Malcev rank of $U_1 \oplus \dots \oplus U_m$ (U_i uniserial $\neq 0$) for a maximal φ , and that Malcev ranks do not increase under homomorphisms. \square

For modules of finite Malcev rank, the following lemma will be required. For the definition of indicators, see [FS, p. 162].

LEMMA 3.4. Let the R -module M have finite Malcev rank m . For every $a \in M$, the indicator $i_M(a)$ can assume at most $2m + 1$ different values.

PROOF. Suppose, by way of contradiction, that $i_M(a)$ assumes more than $2m + 1$ different values for some $a \in M$. Two consecutive values of $i(a)$ can be a limit height $(J/R)^-$ and the corresponding

non-limit height J/R . In all other cases however, there is always a principal height $r^{-1}R/R$ between consecutive values of $i(a)$. Hence there exist $r_1, \dots, r_{m+1}, s_0, s_1, \dots, s_{m+1} \in R, b_0, b_1, \dots, b_{m+1} \in M$ such that

$$a = s_0 b_0, \quad r_1 a = s_1 s_0 b_1, \dots, r_{m+1} \dots r_1 a = s_{m+1} \dots s_1 s_0 b_{m+1}$$

where, for each $i \geq 1$, the value of s_i is larger than the value of r_i in the valuation of R . These equations indicate that in the submodule N generated by b_0, \dots, b_{m+1} , the indicator $i_N(a)$ assumes more than $m + 1$ different values. $\mu(M) = m$ implies N is m -generated. However, as is shown in Salce-Zanardo [SZ₁, p. 1803], indicators in m -generated modules can have at most $m + 1$ different values. \square

4. Polyserial modules of type I.

In [FS, pp. 190-191], two special kinds of polyserial modules were dealt with. For the sake of easy reference, we will call them polyserial modules of type I and type II, respectively. Let us define formally: an R -module M is called *polyserial of type I* if

$$(4) \quad M \leq U_1 \oplus \dots \oplus U_n$$

for certain uniserial R -modules $U_i \neq 0$, and *polyserial of type II* if

$$(5) \quad M = V_1 + \dots + V_n$$

with suitable uniserial modules $V_i \neq 0$. That such M 's are in fact polyserial has been proved e.g. in [FS, p. 190].

The class of polyserials of type I is closed under taking submodules and finite direct sums, while the class of polyserials of type II is closed under epimorphic images and finite direct sums. Finite direct sums of uniserials are both of type I and of type II, and as is shown in Fuchs [Fu₂], these are the only polyserials which are of both types.

Finite rank torsion-free R -modules are polyserials of type I (as they are contained in $Q \oplus \dots \oplus Q$). The finite direct sums of rank 1 torsion-free modules are the only torsion-free polyserials of type II.

We now concentrate on polyserials of type I and leave the study of those of type II to the next section.

LEMMA 4.1. A polyserial R -module M of type I satisfies

$$g(M) = l(M).$$

PROOF. Because of (3.1) and (3.3), it suffices to verify $g(M) \geq l(M)$. Suppose (4) with minimal n . The proof of polyseriality of M as given in [FS, p. 190] shows that then $l(M) = n$. If we had $M \cap U_j = 0$ for some j ($\leq n$), then M would be embeddable in the direct sum $\bigoplus_{i \neq j} U_i$, contradicting the minimality of n . It follows that M contains $\bigoplus_i (M \cap U_i)$, a direct sum of n non-zero modules, that is, $g(M) \geq n$. \square

The last result can be used to obtain the following characterization of polyserials of type I in the countably generated case.

THEOREM 4.2. For a countably generated R -module M and an integer $m \geq 1$, the following are equivalent:

- (i) M is polyserial of type I and of length $\leq m$;
- (ii) M has Malcev rank $\leq m$ equal to its Goldie dimension;
- (iii) every finitely generated submodule of M is a direct sum of at most m cyclic R -modules.

PROOF. (i) \Rightarrow (ii). This implication is obvious by (3.1) and (4.1).

(ii) \Rightarrow (iii). Assume M contains an essential submodule N that is a direct sum of $m = \mu(M)$ non-zero cyclic submodules. If N^* is any finitely generated submodule of M that contains N , then N^* can be generated by m (but by (1.2) not fewer than m) elements. (1.3) implies N^*/N is finitely presented, so by Fuchs-Monari-Martinez [FM, Lemma 1], N^* is a direct sum of cyclics. Finitely generated submodules of N^* are likewise direct sums of cyclics, so (iii) follows from (ii).

(iii) \Rightarrow (i). By Fuchs-Monari-Martinez [FM], a countably generated M with property (iii) can be embedded in a direct sum of uniserials. By the finiteness of the Malcev rank of M , finitely many uniserials will suffice (e.g. those containing the generators of N above). This means, M is polyserial of type I. An appeal to (4.1) concludes the proof. \square

Let us point out that the preceding theorem fails if M is uncountably generated, even if we assume to start with that M is polyserial. In fact, if M is one of the divisible polyserials of length 2 constructed in Fuchs [Fu₁], then $g(M) = \mu(M) = 2$, but M is not of type I.

In the proof of the characterization theorem of polyserials of type I, the following lemma will be needed. Recall that a submodule N of M is *cyclically pure* if N is a summand of $\langle N, x \rangle$ for every $x \in M$; see [Si].

LEMMA 4.3. Suppose D is a divisible, cyclically pure submodule of the R -module M . If M/D is countably generated and uniserial, then D is a summand of M .

PROOF. For every $x \in M/D$, we have (by the definition of cyclic purity) $\langle D, x \rangle = D \oplus Ry$ for some $y \in M$. Evidently, $(D + Ry)/Ry$ is a divisible submodule of M/Ry such that $M/(D + Ry)$ is not only countably generated uniserial, but all of its elements have principal ideal annihilators. Hence p.d. $M/(D + Ry) = 1$ [FS, p. 84], and thus $M/Ry = (D + Ry)/Ry \oplus E/Ry$ for some $E \leq M$ (see [FS, p. 126]). We infer $M = D \oplus E$. \square

We can now prove:

THEOREM 4.4. A polyserial R -module M of type I has a pure-composition series (1) such that for each $i \leq n$, M_{i-1} is cyclically pure in M_i . Conversely, if M is a countably generated R -module with a chain (1) in which M_{i-1} is cyclically pure in M_i for $i = 1, \dots, n$, then M is of type I.

PROOF. Let $M \leq U_1 \oplus \dots \oplus U_n$ (U_i uniserial) such that

$$0 = M_0 < M_1 = M \cap U_1 < \dots < M_i = \\ = M \cap (U_1 \oplus \dots \oplus U_i) < \dots < M_n = M$$

is a pure-composition series for M . Let $x = u_1 + \dots + u_{i-1} + u_i \in M_i$ ($u_j \in U_j$); then $\text{Ann}(x + M_{i-1}) = \text{Ann } u_i$. By way of contradiction, assume that, for every $y \in M_{i-1}$, $\text{Ann}(x + y) < \text{Ann } u_i$. Setting

$$x + y = v_1 + \dots + v_{i-1} + u_i \quad (v_j \in U_j),$$

let k be the maximal index j with $\text{Ann } v_j < \text{Ann } u_i$. Pick a $y_0 \in M_{i-1}$ for which this k is minimal. Let $r \in \text{Ann } u_i \setminus \text{Ann } v_k$. As $\text{Ann } v_j \geq \text{Ann } u_i$ if $j > k$ and M_k is pure in M , $r(x + y_0) \in rM_i \cap M_k = rM_k$ implies that there is a $z = w_1 + \dots + w_k \in M_k$ ($w_j \in U_j$) satisfying $r(x + y_0) = rz$. Clearly, $rw_k = rv_k \neq 0$, thus $v_k = \varepsilon w_k$ for a suitable unit $\varepsilon \in R$. Set

$$y_1 = y_0 - \varepsilon z \in M_{i-1}, \quad x + y_1 = v'_1 + \dots + v'_{i-1} + u_i \quad (v'_j \in U_j).$$

Since $v'_j = v_j$ if $j > k$ and $v'_k = v_k - \varepsilon w_k = 0$, we have $\text{Ann } v'_j \geq \text{Ann } u_i$ for all $j \geq k$. This contradicts the minimality of y_0 , and we conclude

$$\text{Ann } (x + y_0) = \text{Ann } (x + M_{i-1}).$$

Assume M is countably generated and has a chain (1) as stated. From (1.4) we infer that each M_i/M_{i-1} is countably generated, and so is $N = M_{n-1}$. The case $n = 1$, being trivial, assume $n > 1$ and induct on m . Thus N is embeddable in $U_1 \oplus \dots \oplus U_{n-1}$ where each U_i is uniserial. The U_i may be assumed divisible as every countably generated uniserial is standard and hence embeddable in a divisible uniserial. By (4.3), the bottom row in the push-out diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \rightarrow & M/N \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & U_1 \oplus \dots \oplus U_{n-1} & \rightarrow & X & \rightarrow & M/N \rightarrow 0 \end{array}$$

splits (its cyclic purity being inherited from the top row), so M is embeddable in a direct sum of divisible uniserials. \square

5. Polyserial modules of type II.

We begin the study of polyserial modules of type II with a result dual to (4.1).

LEMMA 5.1. A polyserial R -module M of type II satisfies

$$\gamma(M) = \iota(M).$$

PROOF. In view of (3.1) and (3.3), it suffices to verify that $\gamma(M) \geq \geq l(M)$ which will be done by induction on $l(M) = m$. For $m = 1$ the claim being trivial, let $l(M) = m > 1$ and $0 = M_0 < M_1 < \dots < < M_m = M$ a pure-composition series for M ; here M is as given in (5) where n is chosen to be minimal. The proof of polyseriality of M in [FS, p. 190] shows that $m = n$ and that the V_i 's can be indexed in such a way that $M_i = V_1 + \dots + V_i$ holds for $i \leq n$. Note that the submodule $M^* = V_1 \cap (V_2 + \dots + V_n)$ is properly contained in $V_1 = M_1$ (otherwise V_1 would be superfluous in (5)). Therefore, M_1/M^* is a nonzero summand of M/M^* with complement $(V_2 + \dots + + V_n)/M^*$. The last module is evidently polyserial of type II of length $n - 1$; indeed, it has a pure-composition series isomorphic to $0 < M_2/M_1 < \dots < M/M_1$. Using the induction hypothesis, we infer

$$\begin{aligned} \gamma(M) \geq \gamma(M/M^*) &= \gamma(M_1/M^*) + \gamma((V_2 + \dots + V_n)/M^*) = \\ &= 1 + (n - 1) = n. \quad \square \end{aligned}$$

The analogue of (4.2) does not hold for polyserials of type II. As a matter of fact, it is easy to give examples of polyserials M of type I, but not of type II, with $\gamma(M) = l(M)$. For instance, an indecomposable rank 2 torsion-free module M over an almost maximal valuation domain R satisfies $\gamma(M) = l(M) = \mu(M) = 2$.

Before establishing further properties of polyserials of type II, we pause to introduce two definitions (both borrowed from abelian group theory).

A submodule of an R -module M is called *nice* if every coset $a + N$ contains an element $a + x$ ($x \in N$) of the same height in M as the coset has in M/N :

$$h_M(a + x) = h_{M/N}(a + N).$$

If N is nice and, in addition, equiheight (i.e. $h_N(x) = h_M(x)$ for each $x \in N$), then it is said to be a *balanced* submodule of M .

M is called *of standard type II* if it is a finite sum of standard uniserials. (This definition has to be distinguished from the definition of « standard polyserials » introduced in [FS].) Recall that a uniserial U is *standard* if its Fleischer rank is 1.

THEOREM 5.2. A polyserial R -module M is of standard type II if and only if it has a pure-composition series $0 = M_0 < M_1 < \dots < M_n = M$ with standard uniserial factors where for each i , M_{i-1} is balanced in M_i .

PROOF. Let $M = V_1 + \dots + V_n$ with standard uniserial modules $V_i \neq 0$ such that $M_i = V_1 + \dots + V_i$ ($0 \leq i \leq n$) form a pure-composition series of M . To show M_{i-1} nice in M_i , we prove that for any $v \in V_i \setminus M_{i-1}$, $h_{M_i}(v) = h_{M_i/M_{i-1}}(v + M_{i-1})$ holds. In view of the natural isomorphism

$$M_i/M_{i-1} = (M_{i-1} + V_i)/M_{i-1} \cong V_i/(M_{i-1} \cap V_i)$$

it is clear that the height of $v + M_{i-1}$ in M_i/M_{i-1} must be the same as the height of $v + (M_{i-1} \cap V_i)$ in $V_i/(M_{i-1} \cap V_i)$. It is evident by the uniseriality of V_i that the latter height is equal to the height of v in V_i . Now $h(v + M_{i-1}) = h_{V_i}(v) \leq h_{M_i}(v)$ proves the niceness of M_{i-1} in M_i .

As a by-product we obtain that $h_{V_i}(v) = h_{M_i}(v)$ for all $v \in V_i \setminus M_{i-1}$. The V_i were standard uniserials, so $h_{M_i}(v)$ is a nonlimit height for $v \in V_i \setminus M_{i-1}$. Because of the purity of M_i in M , $h_{M_i}(v) = h_M(v)$ follows. An easy induction verifies the equiheightness of the M_i in M .

Conversely, assume M is polyserial and has a pure-composition series $0 = M_0 < M_1 < \dots < M_n = M$ with M_{i-1} balanced in M_i ($i = 1, \dots, n$). For each i , pick an element $v_i \in M_i \setminus M_{i-1}$; without loss of generality, we can choose it such that $h_{M_i}(v_i) = h_{M_i/M_{i-1}}(v_i + M_{i-1})$. By the definition of height, there is a standard uniserial submodule V_i in M_i such that $v_i \in V_i$ and $h_{V_i}(v_i) = h_{M_i}(v_i)$. Now we claim $M = \sum_{i=1}^n V_i$. By induction we show that $M_i = V_1 + \dots + V_i$. The inclusion \supseteq being clear, suppose $i \geq 1$ and $M_{i-1} = V_1 + \dots + V_{i-1}$. The equality $h_{M_i}(v_i) = h_{M_i/M_{i-1}}(v_i + M_{i-1})$ implies that the canonical map $M_i \rightarrow M_i/M_{i-1}$ carries V_i onto M_i/M_{i-1} . In other words, we have $M_i = M_{i-1} + V_i$, indeed. \square

From the proof it follows that $h_M(v + x) = \min(h_M(v), h_M(x))$ for all $x \in M_{i-1}$, $v \in V_i \setminus M_{i-1}$. It is straightforward to check that every $a \in M$ can be written uniquely in the form

$$(6) \quad a = v_1 + \dots + v_n$$

where, for each i , either $v_i = 0$ or $v_i \in V_i \setminus M_{i-1}$. Hence we have:

PROPOSITION 5.3. For an element a of a polyserial M of standard type II,

$$h_M(a) = \min (h_{V_1}(v_1), \dots, h_{V_n}(v_n))$$

where a has form (6). \square

We can now prove:

PROPOSITION 5.4. In a polyserial R -module of standard type II, equiheight submodules are balanced and again polyserial of standard type II.

PROOF. We start off with showing that in a pure-composition series $0 = M_0 < M_1 < \dots < M_n = M$ where M_{i-1} is balanced in M_i ($i = 1, \dots, n$), all M_i ($i = 0, \dots, n$) are balanced in M . Instead of establishing the transitivity of balancedness, we refer to the proof of (5.2). Hence we conclude that if $a \in M$ is written in the form (6), then the coset $a + M_{i-1}$ cannot contain any element whose height exceeds $h(v_i + \dots + v_n)$, i.e. $h_M(v_i + \dots + v_n) = h_{M/M_{i-1}}(a + M_{i-1})$.

Next let N be any uniserial equiheight submodule of $M = V_1 + \dots + V_n$ where V_i are standard uniserials. (5.3) implies that N is likewise standard uniserial. N can be adjoined to $\{V_1, \dots, V_n\}$, and then $M = N + V_1 + \dots + V_n$. Another pure-composition series can be formed which goes through N : $0 = M'_0 < M'_1 = N < \dots < M'_n = M$ where each M'_i is the sum of N and a subset of $\{V_1, \dots, V_n\}$. By the proof of (5.2) and the preceding paragraph, N is balanced in M .

Finally, suppose N is an arbitrary equiheight submodule of M . Pick a nonzero $a \in N$ and argue with (3.4) to conclude that there is an $r \in R$ with $ra \neq 0$ such that its indicator $i_M(ra) = i_N(ra)$ is constant nonlimit.

Consequently, ra is contained in a pure standard uniserial submodule U of N . By the preceding paragraph, U is balanced in N and hence in M . Now N/U is equiheight in M/U which is again polyserial of standard type II of a smaller Malcev rank. Continuing the same way, we can find a pure-composition series for M that includes N as a member. By induction we are led to the balancedness of N in M . In addition, we obtain a pure-composition series for N with balanced members, so by (5.2) N is polyserial of type II. \square

6. Countably generated weakly polyserials.

We now focus our attention on a more detailed study of weakly polyserial modules. Satisfactory characterizations can be obtained in the countably generated case.

The following result will be needed.

LEMMA 6.1. A countably generated R -module is of finite Malcev rank if and only if it is a submodule of a polyserial module of type II.

PROOF. Let M be countably generated of finite Malcev rank m , and N a submodule with exactly m generators. From (1.3) it follows that M/N is the union of a countable ascending chain of finitely presented submodules. By [FS, p. 84], we obtain p.d. $M/N \leq 1$.

Let F be a free R -module on m letters, D its divisible (injective) hull and $\varphi: F \rightarrow N$ an epimorphism. As $D = \bigoplus_1^n Q$, the module $A = D/\text{Ker } \varphi$ is divisible polyserial of type II. Now (1.5) ensures that the isomorphism $N \rightarrow F/\text{Ker } \varphi$ (the inverse of the map induced by φ) extends to a homomorphism $\psi: M \rightarrow A$. This ψ has to be monic, since its restriction to an essential submodule N of M is injective. ψ is an embedding as desired. \square

The main result on countably generated weakly polyserials can now be established.

THEOREM 6.2. For a countably generated R -module M and integer $m \geq 1$, the following assertions are equivalent:

- (a) M has Malcev rank m ;
- (b) M is a submodule of a polyserial R -module of type II and of length m , but not one of length $m - 1$;
- (c) M has Fleischer rank m ;
- (d) M is weakly polyserial and m is the minimal length of chains (1) for M .

PROOF. (a) \Rightarrow (b). The first half of (b) follows from (6.1) and its proof, while the second half is a trivial consequence of (3.1).

(b) \Rightarrow (c). Assume M is a submodule of a polyserial module N of type II; it is easy to see that N can be chosen to be countably generated. Hence (1.4) shows N is standard, and the uniserials V_1, \dots, V_m generating N are epic images of rank one torsion-free modules J_1, \dots, J_m , respectively. It follows that N is an epic image of the torsion-free R -module $A = J_1 \oplus \dots \oplus J_m$ of rank m , and M is an epic image of a submodule of A . We conclude that the Fleischer rank of M is at most m . Because of the second part of (b), this rank cannot be smaller than m . This proves the implication (b) \Rightarrow (c).

(c) \Rightarrow (d). If M satisfies (c), then (2.2) implies the first half of (d) with a chain (1) of length $\leq m$. If M has a chain (1) of length $< m$, then by (3.1), $\mu(M) < m$ would hold. But then (a) \Rightarrow (b) \Rightarrow (c) would imply that the Fleischer rank of M would be $< m$, contrary to hypothesis (c).

(d) \Rightarrow (a). Assuming (d) for M , from (3.1) we conclude that the Malcev rank of M is $\leq m$. But $\mu(M) < m$ would imply (in view of (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)) that M would have a chain (1) of length $< m$. This contradiction proves (d) \Rightarrow (a). \square

The preceding theorem enables us to derive relevant information about the submodules of finitely generated R -modules.

THEOREM 6.3. Submodules of finitely generated R -modules are weakly polyserial. A countably generated R -module can be embedded in a finitely generated torsion R -module with m generators if and only if it is a bounded module of Malcev rank $\leq m$.

PROOF. The first claim is obvious from (2.2). The necessity part of the second assertion is immediate.

To prove sufficiency, suppose M is a bounded R -module of Malcev rank $\leq m$. Owing to (6.2) and its proof, M is then a submodule of an epic image of a torsion-free R -module $A = J_1 \oplus \dots \oplus J_m$ where J_i is of rank 1 ($i = 1, \dots, m$). As M was bounded, $J_i \neq Q$ can be assumed. But then $J_i \leq Rq_i$ for a suitable $q_i \in Q$, so that A is a submodule of a free R -module $F = \bigoplus_{i=1}^m Rq_i$. This shows that M is a submodule of an epic image of F , i.e. M is a submodule of a finitely generated R -module. \square

From (6.2) and (6.3) it follows at once:

COROLLARY 6.4. A countably generated R -module can be embedded in a finitely generated R -module if and only if it is a bounded weakly polyserial module. \square

7. Examples.

Here we collect several examples illustrating our results.

EXAMPLE 7.1. *Polyserial modules of type I.* A general method is given for constructing polyserial modules of type I which are indecomposable of length 2.

Let $0 \neq V < U$ be two uniserial R -modules, and assume that U/V has an automorphism α such that neither α nor α^{-1} is induced by any endomorphism of U . (In particular, α is not a multiplication by an element in R .) Using the canonical map $\pi: U \rightarrow U/V$, consider the commutative diagram where M is obtained via pullback:

$$\begin{array}{ccccccc}
 & & & V & \xlongequal{\quad} & V & \\
 & & & \downarrow \eta & & \downarrow & \\
 0 & \longrightarrow & V & \xrightarrow{\varepsilon} & M & \longrightarrow & U \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \alpha\pi \\
 0 & \longrightarrow & V & \longrightarrow & U & \xrightarrow{\pi} & U/V \longrightarrow 0
 \end{array}$$

By our hypothesis on α , there is no map $U \rightarrow U$ making the arising lower triangle commute; thus the middle row is not splitting. Similarly, the middle column does not split. As a submodule of $U \oplus U$, M is polyserial of type I. From the proof of [FS, p. 190] it follows that the intersection of M with one of the U 's is pure in M . This means that either εV or ηV is pure in M .

To verify the indecomposability of M , note that if M was decomposable, then M would be the direct sum of two uniserial submodules. By [FS, p. 192], pure submodules of such a module are summands, but neither εV nor ηV is. Hence M is as stated.

EXAMPLE 7.2. *Polyserial modules of type II.* We construct dually indecomposable polyserials of type II and of length 2.

Let $0 \neq V < U$ again be two uniserials and suppose that V has an automorphism β such that neither β nor β^{-1} is induced by any endomorphism of U . Using the inclusion map $i: V \rightarrow U$, we construct a pushout diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & V & \xrightarrow{i} & U & \longrightarrow & U/V \longrightarrow 0 \\
 & & \downarrow i\beta & & \downarrow & & \parallel \\
 0 & \longrightarrow & U & \longrightarrow & N & \xrightarrow{\pi} & U/V \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & U/V & \xlongequal{\quad} & U/V & &
 \end{array}$$

By the choice of β , neither the middle row nor the middle column splits. Since N is isomorphic to $(U \oplus U)/K$ where $K = \{(v, -\beta v) : v \in V\}$, N is polyserial of type II. Again, referring to the proof of polyseriality in [FS, p. 190], we see that the image of one of two U 's is pure in N . The same argument as above in (7.1) shows N indecomposable.

EXAMPLE 7.3. *Countably generated weakly polyserial modules which are not polyserial.* Let S denote a maximal immediate extension of R , and assume that there exist units $u, v \in S \setminus R$ such that

- a) the breadth ideal $H = B(u) = \{a \in R : u \notin aS + R\}$ as well as H^{-1} is countably generated;
- b) $B(v) = I$ is not 0;
- c) $u + v$ is also a unit of S .

Pick any $0 \neq t \in I$ such that $tH : I < H$, and set $J = tR : I = \{x \in Q : xI < tR\}$. In view of [SZ₂, Theorem 6], the triple (tR, J, I) is what is called a compatible triple (notice that $I \not\cong tR^\# = P$, because R/I is Hausdorff in the R/I -topology). For every $s \in J \setminus tR$ there is a unit $v_s \in R$ such that $v - v_s \in s^{-1}tS$; thus if $J > s'R > sR > tR$ then $v_s - v_{s'} \in s^{-1}tR$. For each $r \in R \setminus H$ there is a unit $u_r \in R$ such that $u - u_r \in rS$; thus if $rR > r'R > H$, then $u_r - u_{r'} \in rR$. Note that c) implies that for all $s \in J \setminus tR$ and $r \in R \setminus H$, $u_r + v_s$ is a unit of R as is evident from

$$u_r + v_s = (u_r - u) + u + (v_s - v) + v \equiv u + v \pmod{PS}.$$

Let now $V = Qx \oplus Qy$ be a two-dimensional vector space over Q . Consider the R -submodule of V

$$A = \langle x, r^{-1}(x + u_r y) : r \in R \setminus H \rangle .$$

In [SZ₃] it is shown that A is a rank 2 indecomposable torsion-free R -module such that Rx is a basic submodule in A and A/Rx is isomorphic to H^{-1} (under $y + Rx \leftrightarrow 1$). Clearly, A is a countably generated R -module.

Let F be a submodule of $Rx \oplus Ry < A$ defined by

$$F = \langle tx, sx - sv_s y : s \in J \setminus tR \rangle .$$

From [SZ₂] it follows that $(Rx \oplus Ry)/F (< V/F)$ is an indecomposable 2-generated torsion R -module in which the cyclic submodule $(Rx + F)/F$ is pure.

We claim that $M = A/F$ is weakly polyserial of length 2. Indeed, it is not uniserial, since it contains the noncyclic submodule $(Rx \oplus Ry)/F$, while the exact sequence

$$0 \rightarrow (Rx + F)/F \rightarrow M \rightarrow A/(Rx + F) \rightarrow 0$$

shows that it is an extension of a uniserial by a quotient of the uniserial $A/Rx \cong H^{-1}$. Manifestly, M is likewise countably generated.

Next we show that for all $s \in J \setminus tR$ we have

$$h_M(sy + F) = (s^{-1}H^{-1}/R)^- ,$$

i.e. the elements $sy + F$ have limit heights in M .

First, we verify the inequality \geq . As $v_s + u_r$ is a unit for each $r \in R \setminus H$, we have

$$h_M(sy + F) = h_M(s(v_s + u_r)y + F)$$

and

$$s(v_s + u_r)y + F = sx + su_r y + F = sr[r^{-1}(x + u_r y)] + F$$

whence we deduce $h_M(sy + F) \geq (s^{-1}H^{-1}/R)^-$.

Next we assume $h_M(sy + F) \geq s^{-1}H^{-1}/R$ and show that this leads to a contradiction.

Case 1. $h_M(sy + F) > s^{-1}H^{-1}/R$. Thus there is a $z \in A$ such that $asz + F = sy + F$ for an $a \in R$ with $a^{-1} \notin H^{-1}$. From $\text{Ann}(sy + F) = s^{-1}tR$ we obtain $\text{Ann}(z + F) = atR$. As every element of A/F has annihilator containing $tR: H^{-1}$ while $atR < tR: H^{-1}$, we conclude that Case 1 is impossible.

Case 2. $h_M(sy + F) = s^{-1}H^{-1}/R$ for some $s \in \mathcal{J} \setminus tR$. This must be true (in view of the impossibility of Case 1) for all $s_0 \in sR \setminus tR$. Hence the indicator of $sy + F$ is constant up to its annihilator, so $sy + F$ is contained in a pure uniserial submodule U of M . Necessarily U is standard, since M is countably generated (cf. (1.14)). It is clear that U is pure in the polyserial module $K = (H^{-1}x \oplus H^{-1}y)/F$ of type II, thus $K = U + U'$ for a suitable uniserial U' . Hence $M = U + V$ with $V = M \cap U'$, i.e. M itself is polyserial of type II. Since $Jx \oplus Jy > F$, $A/(Jx \oplus Jy)$ is polyserial of type II (being a quotient of M). But it is also of type I, because it is contained in $(H^{-1}x \oplus H^{-1}y)/(Jx \oplus Jy)$. Thus $A/(Jx \oplus Jy)$ is a direct sum of uniserials, see [Fu₂].

To show that this is impossible, observe that this module is obtainable as a pullback diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Rx/Jx & \longrightarrow & A/(Jx \oplus Jy) & \longrightarrow & H^{-1}x/Jx \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \alpha\pi \\
 0 & \longrightarrow & Rx/Jx & \longrightarrow & H^{-1}x/Jx & \xrightarrow{\pi} & H^{-1}x/Rx \longrightarrow 0
 \end{array}$$

where α is an automorphism of $H^{-1}x/Rx$ induced by the unit u of S not in R . Furthermore, α cannot be lifted to an endomorphism of $H^{-1}x/Jx$ (so the top row cannot split), because $\text{Ann } H^{-1}x/Jx = tH: I < H$. An appeal to (7.1) proves the impossibility of Case 2.

We can now show that M can not be polyserial. If $0 \neq V$ is a pure (necessarily standard) uniserial submodule of M , then $(Ry + F)/F$ essential in M implies $V \cap [(Ry + F)/F] \neq 0$. Therefore, there is a nonzero $sy + F$ which has height $s^{-1}H^{-1}/R$. But this was ruled out by Case 2.

It is easy to find examples of valuation domains satisfying the hypotheses in this example.

(7.3) shows that a module of finite Malcev rank need not be cohesive. It also shows that neither a quotient of a polyserial of type I nor a submodule of a polyserial of type II need be polyserial. In addition, we note that (7.3) is an example of a noncohesive module without pure uniserial submodules.

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