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Prescribing Endomorphism Algebras. The Cotorsion-Free Case.

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0. Introduction.

In this paper we continue investigations from [3], which started in the early sixties with A. L. S. Corner's paper [1] and [2]. We want to obtain more detailed information about the category of R-modules over some class of non-zero commutative rings R. Let R have a fixed countable and multiplicatively closed subset $S = S(R)$ containing no zero-divisors such that $1 \in S$. The only demand on R will be S-cotorsion-freeness of its *R-module structure* R^+. This requirement on an R-module can be expressed as a topological condition as well as an algebraic condition; here is its topological version: The S-topology on an R-module H is generated by the set $\{sH: s \in S\}$, which is a countable basis of neighbourhoods of $0 \in H$. It is of course Hausdorff if and only if $\bigcap_{s \in S} sH = 0$ or, as we also say, if and only if H is S-reduced. The module H is S-torsion-free if its socle $H[s] = \{h \in H: sh = 0\}$ vanishes for all $s \in S$. Another notion which can also be easily explained from topology, is purity. The submodule X of H is S-pure in $\hat{H}$ if and only if $X \cap sH = sH$ for all $s \in S$. We denote by $\hat{H}$ the completion of H in the S-topology; then H is S-pure and S-dense in $\hat{H}$, i.e. $\hat{H}/H$ is S-divisible in the obvious sense. An S-reduced and S-torsion-free R-module H is S-cotorsion-free if and only if

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Hom $(\hat{R}, H) = 0$. We shall normally omit the prefix S as S is fixed throughout this paper. The notion of cotorsion-freeness originates from non-commutative groups [11] and is essential for many investigations in module theory and in particular for abelian groups, see e.g. [3, 4, 5, 13]. The class of cotorsion-free modules can be characterized in many ways; for example a module H over a Dedekind domain R with $S = R\setminus\{0\}$ is cotorsion-free if and only if H is torsion-free, reduced and has no submodules isomorphic to $\hat{R}_P$, the completion of the localisation of R at some prime ideal P. Hence a Dedekind domain R is *not* cotorsion-free if and only if R is a field or a complete discrete valuation domain. In this case, the implication of our main result is obviously totally wrong. Hence we assume that R is (as above) a cotorsion-free ring (for some fixed S).

The second standard notion needed also relates to topology. The endomorphism algebra End (H) has a natural topology in which End (H) is Hausdorff and complete. It is the *finite topology*, the analogue of weak convergence of operators in functional analysis; a basis of neighbourhoods of 0 consists of the right ideals

$$(*) \qquad \mathrm{Ann}_R(F) = \{\varphi \in \mathrm{End}_R(H) : F\varphi = 0\} ,$$

where F ranges over all *finite* subsets of H, cf. L. Fuchs [7, Vol. II, p. 221]. For *any infinite cardinal* $\varkappa$ we obtain a finer topology by admitting all subsets F of H with cardinality less than $\varkappa$ in (*), which we call the *$\varkappa$-topology*. Note that the $\aleph_0$-topology is the finite topology on End (H). The $\varkappa$-topology turns End (H) into a complete topological ring. This can be verified in the same manner as the completeness of End (H) in the finite topology, cf. [7, Vol. II, p. 221]. Apparently only the S-topology and the finite topology were useful tools for the investigation of torsion-free modules. Here, however, the $\aleph_1$-topology happens to be the natural concept.

MAIN THEOREM: Let A be an R-algebra, H a faithful right A-module and $\lambda \geqslant |H|$ a cardinal. Then the following are equivalent:

(*a*) H is a cotorsion-free R-module such that A is a pure subalgebra of End (H) which is closed in the $\aleph_1$-topology of End (H).

(*b*) There exists a cotorsion-free R-module G of cardinal $\lambda^{\aleph_0}$ such that

(1) End (G) is topologically isomorphic to A, where End (G) is equipped with the finite topology and A carries the topology induced by the $\aleph_1$-topology on End (H).

(2) G contains a (S-)dense and pure A-submodule B which is isomorphic to a direct sum of λ many copies of H.

Note that the R-algebra A acts by right multiplication on H. This allows us to regard A as a subalgebra of End (H) since H is a faithful A-module.

The implication $(b) \Rightarrow (a)$ is easily established: statement $(b)(1)$ gives the completeness of A in the $\aleph_1$-topology, because End (G) is complete in the finite topology. Thus A is closed in End (H) in the $\aleph_1$-topology. Statement $(b)(2)$ yields the remaining properties in (a): In particular H is cotorsion-free as a submodule of the cotorsion-free module G. To check the purity of A in End (H) assume

$$s\varphi = a \in s\,\mathrm{End}\,(H) \cap A\ .$$

Then φ induces an endomorphism of B acting like φ on each coordinate. This endomorphism lifts to a unique endomorphism $\hat{\varphi}$ of $\hat{B} = \hat{G}$. Now $Gs\hat{\varphi} = Ga \subseteq G \cap s\hat{B} = sG$ and the torsion-freeness of G shows $\varphi \in \mathrm{End}\,(G) = A$; i.e. A is pure in End (H).

The following sections are devoted to the proof of the converse.

Our Main Theorem leads to three corollaries, where two of them (1 and 3) include principal results in [3] and [8], respectively, and one (corollary 2) was our chief concern and starting point. In the case $H = A^+$ the topologies in question are discrete and we derive

Corollary 1: Let A be a cotorsion-free R-algebra and λ any cardinal $\geqslant |A|$. Then there exists a cotorsion-free R-module G of cardinality $\lambda^{\aleph_0}$ such that End $(G) = A$. The module G contains a direct sum of λ many copies of A as a dense and pure submodule.

In the case of cardinals with $\lambda^{\aleph_0} = \lambda$ this is (except for the topological extension mentioned in (3)) the Theorem 6.3 of [3]. For (large) strong limit cardinals λ it has been proved at first in [5] and a (different) topological extension in [6]. If S is not countable, the analogous result was proven in [10].

Corollary 2: If H is a cotorsion-free R-module and $\lambda \geqslant |H|$ is a cardinal, then there exists a cotorsion-free R-module G with

End $(G) \cong$ End (H). The module G contains a direct sum of λ many copies of H as a dense and pure submodule and has cardinal $\lambda^{\aleph_0}$.

The corollary answers a question which was recently raised by A. L. S. Corner (Oxford).

In [4] a similar result was derived under the axioms of ZFC + + $V = L$. Corollary 2 has two immediate consequences.

By induction on the class of ordinals $\mathbf{O}$ we obtain a sequence $\{H_i: i \in \mathbf{O}\}$ of cotorsion-free modules such that H_i is (isomorphic to) a submodule submodule of H_j for all ordinals $i < j$, Hom $(H_j, H_i) = 0$ and End $(H_i) =$ End (H_j). Such a sequence is called a *semi-rigid class*. We summarize one of the main results in [8], which is our

COROLLARY 3: There exist semi-rigid classes of cotorsion-free R-modules.

This links our result to torsion theories; see [8].

From Corollary 2 we also derive a new proof of a problem posed by J. T. Hallett and K. Hirsch in [14]. A proof is given in [12], however applying Corollary 2 is much more natural and needs no explanation. Just recall that the group of units of End (H) is the automorphism group Aut (H). Let $R = Z$ and assume $U \cong$ Aut H is finite for some torsion-free H. How many different groups H realize U? Without loss of generality we may assume that the given group H is cotorsion-free. We derive from Corollary 2 that there is a proper class of abelian groups H with Aut $(H) \cong U$.

Finally we would like to add some remarks concerning the proof of our main theorem. *We will not include references in the proof, however it is a composition of several ideas, some of them should be new.* We will follow the direction given in [3], however, the expert will realise that one of the fundamental arguments in [3] (which is the key algebraic lemma 4.5) is not adequate in the new situation. We will use the idea of « too large endomorphisms » and « B-bounded maps » from [10] in order to deal with this problem. We would like to emphasize that the new ideas used in section 3 could be a good starting point for further investigations. This might also be indicated by the fact that the given arguments are by no means artificial, which was a great surprise while working on this problem. Shelah's « Black Box », proven and applied in [3], has been extended to Shelah's « Stationary Black Box » [19]. While [19] employs model theoretic arguments for its proof, we will prove it by an elementary counting argument in the fashion of [3]. We will use this opportunity also to include a larger class of cardinals, so the « *vatican-problem* » in [3] is really settled. We should also remark that all the algebraic proofs could be based on a more primitive combinatoric as used in [5, 6, 16] (nevertheless derived from model theory in [17]). However, the stronger combinatorial principle—in particular the «Stationary Black Box»—first used in [10], simplifies the algebraic arguments considerably.

Using [3, 10, 12] the reader will observe that the proof given in the next sections can be extended to derive the following more general results.

(1) The restriction $|S| = \aleph_0$ is not necessary. In this case topological arguments have to be replaced by conditions on solubility of finite sets of equations; see [10].

(2) We will prove the Stationary Black Box under the restriction $\mathrm{cf}(\lambda) > \omega$. Moreover, in the Appendix we will sketch the proof of the Black Box for the case $\mathrm{cf}(\lambda) = \omega$. Our proof uses simple counting arguments and no model theory as in [19]. It is then easy to derive a Stationary Black Box with $\mathrm{cf}(\lambda) = \omega$.

(3) The implication of the main theorem can easily be changed into the existence of a rigid system $\{G_i: i \in 2^{\lambda^{\aleph_0}}\}$ where each G_i satisfies the conditions on G in the theorem and also $\mathrm{Hom}(G_i, G_j) = 0$ for $i \neq j$. The proof of this extension is similar to [3] or [12].

In order to have a transparent and straight proof these extensions have been omitted.

Once and for all we fix a sequence q_n $(n < \omega)$ of non-units in S with $q_{n+1}R \subseteq q_nR$ and $\bigcap_{n<\omega} q_nR = 0$. Further $\langle X \rangle_*$ denotes the pure submodule generated by X, X a subset of a torsion-free module.

1. Shelah's stationary black box.

Similar to [3] this section is devoted to the combinatorial methods, which are fundamental for this paper. The combinatorial results in [3; Appendix] are based on S. Shelah [18]. Here we will need a stronger theorem which is taken from S. Shelah [19]. Again, we have simplified the combinatorial construction and we will replace model theoretic arguments in [19] by more immediate and very elementary counting arguments. Some of the notions are taken from [3].

(1.1) Let λ be any infinite cardinal of cofinality cf $\lambda > \omega$. We take $T = {}^{\omega>}\lambda$ to be the tree of all functions $\tau: n \to \lambda$ $(n < \omega)$ partially ordered in the canonical way, i.e. $\sigma \leqslant \tau$ if and only if dom $(\sigma) \subseteq$ dom (τ) and $\sigma = \tau \restriction \operatorname{dom}(\sigma)$. By the length of τ we mean $l(\tau) = \operatorname{dom}(\tau)$. Clearly $|T| = \lambda$. Recall that a maximal linear ordered subset v of a tree is called a branch, i.e. in this case v is of the form $v = \{\sigma_n: n < \omega\}$ for some $\sigma_n \in T$ with dom $(\sigma_n) = n$ and $\sigma_n < \sigma_{n+1}$ $(n < \omega)$. The collection of all branches of a subtree U of T will be written as Br (U).

(1.2) We let $B = \bigoplus_{\tau \in T} H_\tau$ with $H_\tau \cong H$ for all $\tau \in T$. Every element $g \in \hat{B}$ can be represented uniquely as a convergent sum $g = \sum_{\tau \in T} g_\tau$ where $g_\tau \in \hat{H}_\tau$ and for each $n < \omega$ the set $\{\tau \in T: g_\tau \notin q_n \hat{H}_\tau\}$ is finite. Thus the *support* $[g] = \{\tau \in T: g_\tau \neq 0\}$ is at most countable. The support of a subset P of B is taken to be $[P] = \bigcup_{p \in P} [p]$. Note that $[P]$ is countable if P is countable and that the closure $\bar{P}$ of P in the S-topology has the same support as P, provided P is a pure submodule of B. The *norm* of $\tau \in T$ is defined as

$$\|\tau\| = \sup(\{\tau(n): n < \operatorname{dom}(\tau)\}) + 1\,,$$

for a subset U of T by $\|U\| = \sup\{\|\tau\|: \tau \in U\} \leqslant \lambda$. For elements or subsets of $\hat{B}$ their norms are meant to be the norms of their supports, compare with the definition of « orco » in [19]. Note that $\|P\| < \lambda$ whenever P is countable since, cf $\lambda > \omega$.

(1.3) By a *canonical submodule* of B we mean a pure submodule P of the form $P = P(X, W) = \bigoplus_{\tau \in W} X_\tau$ for some countable subset W of T and some submodule X of H generated as a pure submodule by at most countably many elements, where $X_\tau \cong X$ for all $\tau \in T$. Then the collection of all canonical submodules, denoted by $\mathcal{C}$, satisfies the following two conditions:

(*a*) given $P \in \mathcal{C}$ and a countable subset X of B there exists $P' \in \mathcal{C}$ with $P \cup X \subseteq P'$;

(*b*) $\mathcal{C}$ is closed under countable ascending unions.

(1.4) DEFINITION: A *partial trap* p_n of length $n < \omega$ is a triple $p_n = (f_n, P_n, \varphi_n)$, where $f_n: {}^{n>}\omega \to T$ is a tree-embedding (i.e. a length and order preserving injection), $P_n \in \mathfrak{C}$ and $\varphi_n \in \operatorname{Hom}(P_n, B)$. By its norm we mean $\|p_n\| = \|P_n\|$. We denote by $\mathfrak{T}_n$ $(n < \omega)$ the collection of all partial traps of length n.

Note that there are exactly $\lambda^{\aleph_0}$ many partial traps of length n. We use this fact to code them in a suitable way in the following lemma, which is similar to a lemma in [19].

(1.5) LEMMA: (*a*) There are functions $cd_n: \mathfrak{T}_0 \times \mathfrak{T}_1 \times \ldots \times \mathfrak{T}_n \to \lambda$ such that the map $d: \prod_{n<\omega} \mathfrak{T}_n \to {}^{\omega}\lambda$ given by $d((p_n)_{n<\omega}) = \big(cd_n((p_i)_{i\leqslant n})\big)_{n<\omega}$ is an injection.

(*b*) There is an injection $c: \lambda \times \lambda \times \lambda \to \lambda$ such that $c(\alpha, \beta, \gamma) > \gamma$ for all $\alpha, \beta, \gamma < \lambda$.

PROOF: (*a*) There is an injection $e: \bigcup_{n<\omega} \mathfrak{T}_n \to {}^{\omega}\lambda$, since $|\mathfrak{T}_n| = \lambda^{\aleph_0}$ $(n < \omega)$. For $i < \omega$ let $e_i: \bigcup_{n<\omega} \mathfrak{T}_n \to {}^{i+1}\lambda$ be defined by

$$e_i(p) = e(p) \restriction (i + 1) .$$

We also choose bijections $d_n: {}^{n+1}\lambda \to \lambda$ $(n < \omega)$. Then the functions cd_n given by

$$cd_n(p_i)_{i\leqslant n}) = d_n\Big((d_n e_n(p_i))_{i\leqslant n}\Big) \qquad (n < \omega)$$

are as desired; suppose $(p_n)_{n\leqslant\omega}$ and $(p'_n)_{n<\omega}$ are two distinct elements from $\prod \mathfrak{T}_n$, say $p_n \neq p'_n$. Then there is some $m \geqslant n$ such that

$$e_m(p_n) \neq e_m(p'_n)$$

as e is injective. But this implies $cd_m((p_i)_{i\leqslant m}) \neq cd_m(p'_i)_{i\leqslant m})$. Thus d is injective.

(*b*) Any decomposition of λ into λ many pairwise disjoint subsets of cardinality λ yields pairwise disjoint subsets W_γ $(\gamma < \lambda)$ of cardinality λ, whose elements are greater than γ. For each $\gamma < \lambda$ we choose injections $c_\gamma: \lambda \times \lambda \to W_\gamma$. Then the function.

$$c: \lambda \times \lambda \times \lambda \to \lambda$$

defined by $c(\alpha, \beta, \gamma) = c_\gamma(\alpha, \beta)$ is as desired.

(1.6) DEFINITION: A *trap* p is a triple $p = (f, P, \varphi)$ where $f\colon {}^{\omega>}\omega \to {}^{\omega>}\lambda = T$ is a tree embedding, $P \in \mathfrak{C}$ and φ is a homomorphism from P into its closure $\overline{P}$ in $\hat{B}$, satisfying the four conditions:

(*a*) $\operatorname{Rang} f \subseteq [P]$;

(*b*) $[P]$ is a subtree of T, i.e. $\sigma \leqslant \tau$, $\tau \in [P]$ implies $\sigma \in [P]$;

(*c*) $\operatorname{cf}(\|P\|) = \omega$;

(*d*) $\|v\| = \|P\|$ whenever $v \in \operatorname{Br}(\operatorname{Rang} f)$.

Then the norm of p is meant to be $\|P\|$. We call p a V-trap for a subset V of λ, if $\|p\| \in V$.

(1.7) THE STATIONARY BLACK BOX. Let V be a stationary subset of $\Omega = \{\delta < \lambda\colon \operatorname{cf}\delta = \omega\}$. Then for some ordinal $\lambda^* < (\lambda^{\aleph_0})^+$ there is a transfinite sequence of V-traps $p^\alpha = (f^\alpha, P^\alpha, \varphi^\alpha)$ $(\alpha < \lambda^*)$ such that for $\alpha, \beta < \lambda^*$,

(*a*) $\|p^\alpha\| \leqslant \|p^\beta\|$ whenever $\alpha < \beta$;

(*b*) $\operatorname{Br}(\operatorname{Rang} f^\alpha) \cap \operatorname{Br}(\operatorname{Rang} f^\beta) = \emptyset$ if $\alpha \neq \beta$;

(*c*) $\operatorname{Br}(\operatorname{Rang} f^\alpha) \cap \operatorname{Br}([P^\beta]) = \emptyset$ whenever $\beta + 2^{\aleph_0} \leqslant \alpha$;

(*d*) for any countable subset X of $\hat{B}$ and any $\varphi \in \operatorname{End}(\hat{B})$ there exists $\alpha < \lambda^*$ such that

$$X \subseteq \overline{P^\alpha}, \qquad \|X\| < \|P^\alpha\| \qquad \text{and } \varphi \restriction P^\alpha = \varphi^\alpha .$$

PROOF: We begin by describing a method that assigns to certain sequences of partial traps genuine traps. To this end let $p_i \in \mathfrak{T}_i$ $(i \leqslant n)$ be given. Then this finite sequence determines a tree embedding $C((p_i)_{i\leqslant n}) = f_{n+1}\colon {}^{n+1>}\omega \to T$ given by $f_{n+1}(\sigma) = f_n(\sigma)$, if $\sigma \in {}^{n>}\omega$, and $f_{n+1}(\sigma) = f_n(\sigma\restriction n)^\frown c\big(\sigma(n), cd_n((p_i)_{i\leqslant n}), \|p_n\|\big)$, if $\sigma \in {}^n\omega$, where c and cd_n are as in (1.5). Then clearly f_{n+1} is atree embedding with

$$\|\operatorname{Rang} f_{n+1}\| > \|p_n\| \qquad \text{and} \qquad f_{n+1} \text{ extends } f_n .$$

We call a sequence of partial traps $p_n = (f_n, P_n, \varphi_n) \in \mathfrak{T}_n$ $(n < \omega)$ *admissible* if, for every $n < \omega$,

(a1) $C((p_i)_{i\leqslant n}) = f_{n+1}$;

($a2$) $P_n \subseteq P_{n+1}$, $P_n\varphi_n \subseteq \overline{P_{n+1}}$ and $\operatorname{Rang} f_{n+1} \cup [P_n]_{\leqslant} \subseteq [P_{n+1}]$, where $[P_n]_{\leqslant}$ denotes the generated subtree of T generated by $[P_n]$,

($a3$) $\varphi_{n+1} \restriction P_n = \varphi_n$.

Each of these admissible sequences of partial traps $(p_n)_{n<\omega}$ gives rise to a genuine trap $p = (f, P, \varphi) =$ with $f = \bigcup_{n<\omega} f_n$, $P = \bigcup_{n<\omega} P_n$ and $\varphi = \bigcup_{n<\omega} \varphi_n$. This definition makes sense, since $f_n \subseteq f_{n+1}$, $P_n \subseteq P_{n+1}$ and $\varphi_n \subseteq \varphi_{n+1}$ for all $n < \omega$. P is indeed a canonical submodule by (1.3)(b) and conditions (1.6)(a), (b) are obviously satisfied. To check (1.6)(c), (d) consider any branch v of $\operatorname{Rang} f$. Given $n < \omega$ there is a unique $\tau_n \in v$ of length $l(\tau_n) = n$ and a unique $\sigma_n \in {}^n\omega$ such that $\tau_n = f(\sigma_n)$. But then

$$(1) \qquad \tau_{n+1} = f_{n+1}(\sigma_{n+1}) = f(\sigma_n)^\frown c\big(\sigma_{n+1}(n), cd_n((p_i)_{i\leqslant n}), \|p_n\|\big)$$

and $\tau_{n+1} \in \operatorname{Rang} f_{n+1} \subseteq [P_{n+1}]$ yields $\|p_{n+1}\| \geqslant \|\tau_{n+1}\| > \|p_n\|$ by (1.5)(b). Hence $\|v\| = \sup\ \{\tau_n\|: n < \omega\} = \sup\ \{\|p_n\|: n < \omega\} = \|p\|$ and of course $cf(\|p\|) = \omega$.

We call those traps admissible that are determined by admissible sequences of partial traps. Conversely, we can recover the admissible sequence $(p_n)_{n<\omega}$ of partial traps used for the construction of the admissible trap $p = (f, P, \varphi)$ from any branch v of $\operatorname{Rang} f$; namely from (1) we can compute $cd_n((p_i)_{i\leqslant n})$ for all $n < \omega$, since c is injective and by (1.5)(a) we have $(p_n)_{n<\omega} = d^{-1}\big(cd_n((p_i)_{i\leqslant n})_{n<\omega}\big)$.

We will show that the theorem holds for a suitable well-ordering of the set $\mathfrak{T}_V$ of all admissible V-traps. By what we have just shown the ranges of the tree embeddings of two distinct admissible traps have no common branch, and thus condition (b) is satisfied, no matter what the well-ordering of $\mathfrak{T}_V$ is like.

In order to take care of conditions (a) and (c) we must establish a well-ordering on $\mathfrak{T}_V$; and this coincides with [3]. First we rank the admissible V-traps according to their norm as required by (a). Moreover, if two traps $p, p' \in \mathfrak{T}_V$ have distinct norms, say $\|p'\| < \|p\|$, then every branch v of $\operatorname{Rang} f$ has norm $\|v\| = \|p\| > \|p'\|$, thus $v \not\subseteq [P']$, i.e. $\operatorname{Br}(\operatorname{Rang} f) \cap \operatorname{Br}([P']) = \emptyset$. So it will be enough to prove that the analogue of (c) holds for a suitable well-ordering of $\mathfrak{T}_V^\delta$, the set of admissible V-traps of a fixed norm δ. We construct such a well-ordering on $\mathfrak{T}_V^\delta$: Given any $p = (f, P, \varphi) \in \mathfrak{T}_V^\delta$, the support $[P]$ is countable, so $[P]$ contains at most $2^{\aleph_0}$ many branches. Together with condition (*b*) this implies that

$$D_0(p) = \{p' \in \mathfrak{T}_V^\delta : \mathrm{Br}(\mathrm{Rang}\, f') \cap \mathrm{Br}[P] \neq \emptyset\}$$

has cardinality at most $2^{\aleph_0}$. By induction, this is true for each of the sets

$$D_{n+1}(p) = \bigcup \{D_0(p') : p' \in D_n(p)\} \qquad (n < \omega)$$

as well as for their union $D(p) = \bigcup_{n<\omega} D_n(p)$. To each $p \in \mathfrak{T}_V^\delta$ we have now associated a subset $D(p) \subseteq \mathfrak{T}_V^\delta$ such that $p \in D(p)$, $|D(p)| \leqslant 2^{\aleph_0}$ and $p' \in D(p)$ implies $D_0(p') \subseteq D(p)$. The last of these three conditions means that for $p', p'' \in \mathfrak{T}_V^\delta$ from $p' \in D(p)$, $p'' \notin D(p)$ follows $p'' \notin D_0(p')$ and thus $\mathrm{Br}(\mathrm{Rang}\, f'') \cap \mathrm{Br}([P']) = \emptyset$. Put an arbitrary well-ordering on the set $\Delta = \{D(p) : p \in \mathfrak{T}_V^\delta\}$. For each $D \in \Delta$ write

$$D^* = D \setminus \bigcup \{E \in \Delta : E < D\} .$$

Then the $D^*(D \in \Delta)$ form a partition of $\mathfrak{T}_V^\delta$ in which each part has cardinality at most $2^{\aleph_0}$. Put a well-ordering of type $\leqslant 2^{\aleph_0}$ on each D^* and require that $p' < p''$ whenever $p' \in D^*$, $p'' \in E^*$ and $D < E$. However, if p', p'' belong to the same D^*, use the well-ordering of D^*. Now suppose $p' < p''$ and there are at least $2^{\aleph_0}$ many members of $\mathfrak{T}_V^\delta$ between p' and p''. Then there must be $E, D \in \Delta$ with $p' \in E^*$, $p'' \in D^*$ and $E < D$, since each part of the above partition has order type $\leqslant 2^{\aleph_0}$. Now $E = D(p)$ for some $p \in \mathfrak{T}_V^\delta$, and $p' \in E^* \subseteq D(p)$, $p'' \notin D(p)$ implies $\mathrm{Br}(\mathrm{Rang}\, f'') \cap \mathrm{Br}([P']) = \emptyset$ as we have previously noted. This is the desired analogue of (*c*).

We still have to take care of condition (*d*). So let X, φ be as in (*d*). First we show that there are numerous admissible traps $p = (f, P, \varphi \upharpoonright P)$ (not necessarily with the right norm) that catch X and φ, i.e. satisfy condition (*d*). Only later we show that this can be achieved by an admissible V-trap. We define recursively admissible sequences $(p_i)_{i<\omega}$ so that the corresponding admissible trap catches φ and X and at recursion step n the partial trap $p_n \in \mathfrak{T}_n$ can be chosen to have an arbitrarily large norm. At step 0 there is $P_0 \in \mathcal{C}$ with $X \subseteq P_0$ since X is countable. Of course $\|P_0\|$ can be chosen arbitrarily large. As there is only one embedding $f_0 : {}^{0>}\omega = \emptyset \to T$ we let $p_0 = (f_0, P_0, \varphi \upharpoonright P_0)$. Suppose $p_0, p_1, \ldots, p_n$ have already been defined. We must put $f_{n+1} = C((p_i)_{i \leqslant n})$ to satisfy (*a*1); to achieve (*a*2) we must add at most countably many elements to P_n to obtain a suitable $P_{n+1} \in \mathcal{C}$, which is possible by (1.3)(a). Again, enlarging P_{n+1} offers no problem, so P_{n+1} can be chosen to be of arbitrarily large norm. Now the partial trap $p_{n+1} \in \mathfrak{T}_{n+1}$ is given by $p_{n+1} = (f_{n+1}, P_{n+1}, \varphi \restriction P_{n+1})$. The recursion proceeds for all $n < \omega$. Any sequence constructed in this manner is admissible and the associated traps catch X and φ.

Now we turn to the problem of finding an admissible V-trap. Let $Y = \{\eta \in {}^{\omega>}\mathrm{cf}\,\lambda : \eta$ is a strictly increasing sequence$\}\setminus\{\emptyset\}$ and $Y_\alpha = \{\eta \in Y : \eta(n) < \alpha$ for all $n \in \mathrm{dom}(\eta)\}$ $(\alpha < \mathrm{cf}\,\lambda)$. Clearly

$$Y = \bigcup \{Y_\alpha : \alpha < \mathrm{cf}\,\lambda\}\,.$$

Further let $(\lambda_\alpha)_{\alpha<\mathrm{cf}\,\lambda}$ be a strictly increasing ω-continuous sequence of ordinals $\lambda_\alpha < \lambda$ converging to λ. By transfinite induction on $\alpha < \mathrm{cf}\,\lambda$ we define admissible sequences $(p_{\alpha,n})_{n<\omega}$ such that the associated admissible traps p_α satisfy for each $\alpha < \mathrm{cf}\,\lambda$

(2) p_α catches X and φ;

(3) $\|p_\beta\| < \|p_\alpha\|$ for all $\beta < \alpha$

(4) $\|p_\alpha\| = \sup\{\|p_\beta\| : \beta < \alpha\}$ if $\mathrm{cf}\,(\alpha) = \omega$;

(5) $\|p_\alpha\| > \lambda_\alpha$

(6) $\|p_\alpha\| > \|p_{(\eta)}\|$ for all $\eta \in Y_\alpha$.

The $p_{(\eta)}$ $(\eta \in Y_\alpha)$ occuring in (6) are defined simultaneously: they are partial traps of length $l(\eta) - 1$ such that for all $\alpha < \mathrm{cf}\,\lambda$

(7) $p_{(\eta)}$ is an initial segment of an admissible sequence of partial traps whose associated trap catches X and φ, whenever $\eta \in Y_\alpha$.

(8) $\|p_{(\eta)}\| > \|p_{\eta(n)}\|$, if $n = l(\eta) - 1$, for all $\eta \in Y_\alpha$.

For $\alpha = 0$ we merely require p_0 to be an admissible trap catching X and φ with $\|p_0\| > \lambda_0$. Note that $Y_0 = \emptyset$. Now let $0 < \alpha < \mathrm{cf}\,\lambda$ and suppose we have found already $(p_{\beta,n})_{n<\omega}$ and $p_{(\eta)}$ $(\eta \in Y_\beta)$ for all $\beta < \alpha$ such that conditions (2)-(8) holds for all $\beta < \alpha$. First suppose $\alpha = \beta + 1$ is a successor ordinal and $\eta \in Y_\alpha$; $p_{(\eta)}$ is already defined if $\eta \in Y_\beta$. Suppose $\eta = \sigma^\frown\beta$ for some $\sigma \in Y_\beta \cup \{\emptyset\}$. Then we let $p_{(\eta)} = (f_n, P_n, \varphi \restriction P_n)$, where $n = l(\eta) - 1$, $f_n = C\big((p_{(\sigma\restriction i+1)})_{i\leqslant n-1}\big)$ if $n \geqslant 1$, and $f_n(\emptyset) = \emptyset$ if $n = 0$. Now $P_n \in \mathcal{C}$ is chosen in accordance with ($a2$) and such that $\|P_n\| > \|p_\beta\|$ and $X \subseteq \overline{P_n}$: This takes care of conditions (7) and (8) for α. Now $\sup(\{\lambda_\alpha, \|p_\beta\|\} \cup \{\|p_{(\eta)}\| : \eta \in Y_\alpha\}) < \lambda$, we let p_α be any admissible trap catching X and φ whose norm is larger than this supremum. Thus conditions (3)-(6) hold for α.

Now suppose α is a limit ordinal. First observe that the $p_{(\eta)}$, $\eta \in Y_\alpha$, have already been defined. If $\mathrm{cf}(\alpha) = \omega$ choose a strictly increasing sequence $\sigma \in {}^\omega\alpha$ with limit α. Then $\sigma\restriction n+1 \in Y_\alpha$ for all $n < \omega$ and we let $(p_{\alpha,n})_{n<\omega} = (p_{\restriction\sigma(n+1)})_{n<\omega}$. Then p_α is indeed an admissible trap catching X and φ by (7). To verify (4) observe only that $\|p_{\sigma(n+1)}\| > \|p_{(\sigma\restriction n+1)}\| > \|p_{\sigma(n)}\|$ for all $n < \omega$ by conditions (6) and (8). Again, conditions (2)-(8) hold for α. If $\mathrm{cf}(\alpha) > \omega$, we can choose any admissible trap p_α catching X and φ, merely requiring $\|p_\alpha\| > \sup\{\|p_\beta\| : \beta < \alpha\}$. Thus the recursion proceeds for all $\alpha < \mathrm{cf}\,\lambda$.

Now conditions (3), (4) and (5) imply that the function s mapping $\alpha \in \mathrm{cf}\,\lambda$ onto $s(\alpha) = \|p_\alpha\|$ is strictly increasing, ω-continuous and unbounded in λ. Therefore Rang (s) is an ω-closed unbounded subset of Ω. If $\overline{\text{Rang}\,(s)}$ denotes the closure of Rang (s) in the order topology, then Rang $(s) \cap \Omega = \overline{\text{Rang}\,(s)} \cap \Omega$ and $\overline{\text{Rang}\,(s)}$ is a closed unbounded subset of λ. Hence Rang (s) intersects the stationary set $V \subseteq \Omega$ non-trivially, i.e. there is an admissible V-trap catching X and φ. q.e.d.

2. The construction.

We begin by decomposing the stationary set Ω into two disjoint stationary subsets V and V'. We shall apply the stationary black box (1.7). So let $(f^\alpha, P^\alpha, \varphi^\alpha)$ $(\alpha < \lambda^*)$ be the transfinite sequence of V-traps from (1.7). As a technical device let ∞ denote a fixed element not in $\hat{B}$.

(2.1) Let $\mu < \lambda^*$ and assume that we have found an ascending chain of pure A-submodules G_α $(\alpha < \mu)$ of $\hat{B}$ and elements b_β $(\beta + 1 < \mu)$ of $\hat{B} \cup \{\infty\}$ such that, for $\alpha < \mu$,

(I_α) $\quad b_\beta \notin G_\alpha$ for all $\beta < \alpha$.

If $\mu = 0$, we put

(II_0) $\quad G_0 = B$

and if μ is a limit ordinal, we take

(II_μ) $\quad G_\mu = \bigcup_{\alpha<\mu} G_\alpha$ (μ limit).

If $\mu = \alpha + 1$ is a successor ordinal, we distinguish cases:

(i) It is possible to choose a branch v_α of $\operatorname{Rang} f^\alpha$, elements g_α^0, g_α^1 from $\overline{P^\alpha}$ and $G_{\alpha+1}$, b_α in such a way that $(I_{\alpha+1})$ and each of the following conditions is satisfied:

$(\mathrm{II}_{\alpha+1})$ $G_{\alpha+1} = \langle G_\alpha + g_\alpha A\rangle_*$, where $g_\alpha = g_\alpha^0 + g_\alpha^1$;

(III_α) $\|g_\alpha^0\| < \|g_\alpha^1\|$, $[g_\alpha^1] = v_\alpha$, $(g_\alpha^1)_\tau \in H_\tau$ for all $\tau \in T$, $\operatorname{Ann}_A(g_\alpha^1) = \operatorname{Ann}_A(P^\alpha)$ and $[g_\alpha^1 a]$ is infinite whenever $a \in A \setminus \operatorname{Ann}_A(P^\alpha)$;

(IV_α) either (strong version) $b_\alpha = g_\alpha \varphi^\alpha$ or (weak version) $b_\alpha = \infty$.

We use the strong version, whenever possible, and in this case we call α strong. Otherwise α will be called weak.

(ii) If (i) is not satisfied, we call α useless and take $G_{\alpha+1} = G_\alpha$, $g_\alpha = 0$, $b_\alpha = \infty$, so satisfying $(\mathrm{II}_{\alpha+1})$ in this case as well. (We will show in (2.6) that this case never arises.)

In every case (I_μ) is obviously satisfied. The recursion therefore proceeds for all $\mu < \lambda^*$ and gives rise to a pure A-submodule $G = \bigcup_{\alpha<\lambda^*} G_\alpha$ of $\hat{B}$. We will show that $\operatorname{End}(G) = A$.

For a subset U of T and $\nu < \lambda$ we let ${}_\nu U = \{\eta \in U : \|\eta\| > \nu\}$. We begin with the simple but crucial

(2.2) Recognition Lemma: Let $g \in G \setminus B$. Then there is a uniquely determined strictly decreasing sequence of ordinals

$$\alpha(0) > \alpha(1) > \dots > \alpha(k)$$

such that

(a) $\|v_{\alpha(i)}\| = \|v_{\alpha(0)}\|$ for $0 \leqslant i \leqslant k$;

(b) ${}_\nu[g] = F \cup \sum_{i=0}^{k} U_{\alpha(i)}$ (disjoint union), where

$$\nu = \sup\{\|g_{\alpha(i)}^0\| : 0 \leqslant i \leqslant k\} < \|v_{\alpha(0)}\|,$$

$U_{\alpha(i)}$ are infinite subsets of $v_{\alpha(i)}$ $(0 \leqslant i \leqslant k)$ and F is finite;

(c) $\|\tau\| \geqslant \nu$ implies $g_\tau \in H_\tau$.

PROOF: Clearly $G = \langle B + \sum_{\alpha<\lambda^*} g_\alpha A\rangle_*$, thus there is $s \in S$ such that $sg = b + \sum_{i=0}^{l} g_{\alpha(i)} a_i$ for some $b \in B$, $\alpha(i) < \lambda^*$ and $a_i \in A$ $(0 \leqslant i \leqslant l)$. Since $g \notin B$ we may assume $g_{\alpha(i)} a_i \neq 0$ and further w.l.o.g. $\alpha(0) > > \alpha(1) > \ldots > \alpha(l)$. We let $k = \max\{0 \leqslant i \leqslant l\colon \|v_{\alpha(i)}\| = \|v_{\alpha(0)}\|\}$ so satisfying condition (*a*). Of course, the $\alpha(i)$'s are not useless, therefore indeed $\nu < \|v_{\alpha(0)}\|$, because $\mathrm{cf}\,(\|v_{\alpha(0)}\|) = \omega$ by (1.6)(*c*). Now by condition $(\mathrm{III}_{\alpha(i)})$

$$_\nu[g_{\alpha(i)} a_i] = {}_\nu[g^1_{\alpha(i)} a_i]$$

is an infinite subset of $v_{\alpha(i)}$. Since the intersection of two distinct branches is finite and as by (1.7)(*b*) the $v_{\alpha(n)}$'s are pairwise distinct, by (1.7)(*b*), it is easy to choose $U_{\alpha(i)}$ and F to satisfy (*b*). Condition (*c*) follows immediately from $(\mathrm{III}_{\alpha(i)})$ $(0 \leqslant i \leqslant k)$.

The uniqueness of the sequence $\alpha(i)$ $(0 \leqslant i \leqslant k)$ follows from the observation that it is determined by $[g]$: look at those branches v_α which possess infinite subsets contained in $[g]$; then those with the maximal norm give us $\alpha(0), \ldots, \alpha(k)$.

(2.3) DEFINITION: Let $\alpha < \lambda$ be any ordinal. The *constant branch* $w = w(\alpha)$ on α is the set $w = \{\sigma_n\colon n < \omega\}$, where $\sigma_n \in {}^n\lambda$ is defined by $\sigma_n(i) = \alpha$ for $i < n$.

(2.4) COROLLARY. Let $g \in G$.

(*a*) $g \in B$ if and only if $[g]$ is finite.

(*b*) For ν as in (2.2)(*b*) an infinite subset of a constant branch is never contained in $_\nu[g]$.

(*c*) $\|g\|$ is never in V'.

PROOF: (*a*) and (*b*) are obvious. To verify (*c*) note that the norm of an element from B is always a successor ordinal and that the norms of the branches occuring in (2.2) belong to V.

It will follow from the next lemma that there are no useless ordinals. It will also play an important role in the final stage of the proof of our main theorem.

(2.5) LEMMA. Let $\alpha < \lambda^*$ and $\nu < \|P^\alpha\|$. Suppose we have for each branch v of $\mathrm{Rang}\, f^\alpha$ elements $g_v \in \hat{B}$ of the form $g_v = g^0_v + g^1_v$

such that $\|g_v^0\| \leqslant \nu$, $\mathrm{Ann}_A(g_v^1) = \mathrm{Ann}_A(g_v)$, $[g_v^1] = v$ and $[g_v^1 a]$ is either empty or infinite. Then there exists a branch v of $\mathrm{Rang}\, f^\alpha$ such that

$$b_\beta \notin G_{\alpha+1}(v) \qquad (\beta < \alpha)\,,$$

where $G_{\alpha+1}(v) = \langle G_\alpha + g_v A\rangle_*$.

PROOF. Suppose not. Then for each $v \in \mathrm{Br}(\mathrm{Rang}\, f^\alpha)$ there exists $\beta = \beta(v) < \alpha$ such that $b_\beta \in G_{\alpha+1}(v)$. By (IV_β) we have $b_\beta = g_\beta \varphi^\beta \in \overline{P^\beta}$ and there exist $s = s_v \in S$, $a = a_v \in A$ with $sb_\beta - g_v a \in G_\alpha$; since $b_\beta \notin G_\alpha$ by (I_α) and since G_α is pure in $\hat{B}$ certainly $g_v a \neq 0$. But then it follows from our assumptions that ${}_v[g_v a] = {}_v[g_v^1 a]$ is an infinite subset of v. The recognition lemma applied to $sb_\beta - g_v a$ yields that an infinite subset of v is contained in $[b_\beta]$, because the branches v_γ $(\gamma < \alpha)$ have at most finitely many members with v in common. Now $b_\beta \in \overline{P^\beta}$ implies $[b_\beta] \subseteq [P^\beta]$, and as $[P^\beta]$ is a subtree of T, v is contained in $[P^\beta]$. This is possible only if $\beta < \alpha < \beta + 2^{\aleph_0}$ by (1.7)(c).

Summarizing we have found for each $v \in \mathrm{Br}(\mathrm{Rang}\, f^\alpha)$ an ordinal $\beta(v)$ and $a_v \in A$, $s_v \in S$ such that

$$\beta(v) < \alpha < \beta(v) + 2^{\aleph_0} \qquad \text{and} \qquad s_v b_{\beta(v)} - g_v a_v \in G_\alpha\,.$$

Now it is clear that there are fewer than $2^{\aleph_0}$ many distinct $\beta(v)$, thus there are two distinct branches v, w of $\mathrm{Rang}\, f^\alpha$ with $\beta(v) = \beta(w)$.

Choosing a common multiple $t = s_v t_1 = s_w t_2$ in S we find

$$g_v t_1 a_v - g_w t_2 a_w \in G_\alpha\,.$$

Arguing as before we conclude that an infinite subset of v must lie in ${}_v[g_w t_2 a_w] \subseteq w$; and this is impossible because v, w are distinct branches.

(2.6) COROLLARY. There are no useless ordinals. An ordinal $\alpha < \lambda^*$ is strong or weak according as $g_\alpha \varphi^\alpha$ lies outside or in G, resp.

PROOF. Let $\alpha < \lambda^*$ and $P^\alpha = P(X, W)$ where the pure submodule X of H is countably generated by nonzero elements $x_n \in X$ $(n < \omega)$. Decompose ω into countably many pairwise disjoint countable subsets Y_n $(n < \omega)$. For $v = \{\sigma_i \colon i < \omega\} \in \mathrm{Br}(\mathrm{Rang}\, f^\alpha)$, $l(\sigma_i) = i$, we define $g_v^1 \in \overline{P^\alpha}$ by $[g_v^1] = v$ and $g_v^1(\sigma_i) = q_{l(\sigma_i)}|x_n$ when $i \in Y_n$, where we have written $g(\tau)$ for the element of H corresponding to $g_\tau \in H_\tau \cong H$. Then $\mathrm{Ann}_A(g_v^1) = \mathrm{Ann}_A(P)$ and $[g_v^1 a]$ is infinite whenever $g_v^1 a \neq 0$, because for each $n < \omega$ multiples of x_n occur infinitely often in g_v^1. Taking $g_v^0 = 0$ the previous lemma delivers a branch v of $\mathrm{Rang}\, f^\alpha$ such that the conditions of case (i) in (2.1) are satisfied for $g_\alpha = g_\alpha^1 = g_v$.

The second assertion is obvious looking at (IV_α) in (2.1).

(2.7) Lemma: G is cotorsion-free.

Proof: We need to show that $\mathrm{Hom}(\hat{R}, G) = 0$. Suppose there is some nonzero homomorphism $\Phi\colon \hat{R} \to G$. Let $g = 1\ \Phi \in G$. Continuity of Φ implies $[\pi\Phi] \subseteq [g]$ for each $\pi \in \hat{R}$. Thus if $[g]$ were finite so were $[\pi\Phi]$ and $\mathrm{Im}\, \Phi \subseteq B$ would follow from (2.4)(a). But B is visibly cotorsion-free. If for some non-zero $\pi \in R$ the support $[\pi\Phi]$ is finite we are also done. So suppose $g \notin B$ and let $\alpha(0) > \alpha(1) > \ldots > \alpha(k)$ and ν be as in (2.2). If $\beta(0) > \beta(1) > \ldots > \beta(l)$ is the corresponding sequence for $\pi\Phi \in G \setminus B$, then either $\|v_{\beta(0)}\| < \|v_{\alpha(0)}\|$ or $\{\beta(0), \ldots, \beta(l)\} \subseteq \{\alpha(0), \ldots, \alpha(k)\}$. In both cases we have for $\tau \in T$ with $\|\tau\| > \nu$ that $(\pi\Phi)_\tau \in H_\tau$: Thus there are induced homomorphisms $\varphi_\tau\colon \hat{R} \to H_\tau$ given by $\pi\Phi_\tau = (\pi\Phi)_\tau$. Since $H_\tau \cong H$ is cotorsion-free, all of these must vanish which contradicts $\Phi \neq 0$.

3. Strongness.

We call an element $y \in \hat{B}$ strong with respect to some $\varphi \in \mathrm{End}(\hat{B})$ if $y\varphi \notin \langle G + yA \rangle_*$. The goal of this section is to prove the following

(3.1) Proposition: For each $\varphi \in \mathrm{End}(G) \setminus A$ there is a strong element with respect to φ.

We split the proof into several lemmas according to the various cases. We begin with a simple case.

(3.2) Definition: An endomorphism φ of G is too large if for each $\alpha < \lambda$ there are sequences $(b_n)_{n<\omega}$, $(\alpha_n)_{n<\omega}$ and $(i(n))_{n<\omega}$ such that for all $n < \omega$

(a) $b_n \in B$

(b) $\alpha < \alpha_n < \|b_n\|$; $\|b_n\|, \|b_n\varphi\| < \alpha_{n+1}$;

(c) $i(n) < i(n+1) < \omega$;

(*d*) there is no $a \in A$ with

$$b_n(q_{i(n)}^2\varphi - a) \in G^{\alpha_n} + Gq_{i(n+1)} ,$$

where $G^{\alpha_n} = \{g \in G\colon \|g\| < \alpha_n\}$.

(3.3) LEMMA: If the endomorphism φ of G is too large, then there is a strong element with respect to φ.

PROOF: Let φ be too large. It is easily verified by a diagonal argument that the set C of the suprema of the ordinal sequences in the above definition is ω-closed and unbounded. Furthermore

$$C \subseteq \Omega = \{\delta < \lambda\colon \operatorname{cf} \delta = \omega\}$$

and therefore $C \cap V' = \bar{C} \cap V' \neq 0$ for the club C generated by C, because by assumption $V' \subseteq \Omega$ is stationary. Hence we have $(b_n)_{n<\omega}$, $(\alpha_n)_{n<\omega}$ and $(i(n))_{n<\omega}$ satisfying conditions (3.2)(*a*)-(*d*) above and additionally $\alpha = \sup_{n<\omega} \alpha_n \in V'$. We claim that $y = \sum_{n<\omega} q_{i(n)} b_n$ is the desired strong element. Otherwise there are $s \in S$, $a \in A$ and $g \in G$ such that $y(s\varphi - a) = g$. As both $y\varphi$ and ya have norm at most α, so has g. But by (2.4)(*c*) $\|g\| \notin V'$, therefore

$$y(s\varphi) \equiv ya + g \equiv q_{i(m)} b_m a + g_m \bmod q_{i(m+1)} \hat{B} ,$$

where $g_m = g + \sum_{n<m} q_{i(n)} b_n a \in G^{\alpha_m}$. On the other side

$$y\varphi \equiv q_{i(m)} b_m \varphi + g'_m \bmod q_{i(m+1)} \hat{B} ,$$

for $g'_m = \sum_{n<m} q_{i(n)} b_n \varphi \in G^{\alpha_m}$. Subtraction of these two equations gives

$$b_m(q_{i(m)} s\varphi - q_{i(m)} a) + sg'_m - g_m \in q_{i(m+1)} \hat{B} \cap G = q_{i(m+1)} G .$$

Now if m is large enough to force $q_{i(m)} = st$ for some $t \in S$, multiplication by t yields

$$b_m(q_{i(m)}^2 \varphi - a') \in G^{\alpha_m} + q_{i(m+1)} G ,$$

a contradiction.

(3.4) LEMMA: Let $\varphi \in \mathrm{End}(G)$ be not too large. Then there is a strong element with respect to φ or there is some $s \in S$, $a \in A$ and $\alpha < \lambda$ such that for all $b \in B$ with ${}_\alpha[b] = [b]$ we have $\|b(s\varphi - a)\| < \alpha$.

PROOF: Since φ is not too large, the construction of sequences in (3.2) has to break down at some stage, say the n-th stage. Since it is easy to take care of conditions (a), (b), (c), there are for all $b \in B$ with ${}_\alpha[b] = [b]$ and all $m > i(n)$ elements $a_{b,m} \in A$ such that

$$(*) \qquad b(s\varphi - a_{b,m}) \in G^\alpha + q_m G\,,$$

where $s = q^2_{i(n)}$. We fix b for the moment and show that $ba_{b,m}$ is a convergent sequence in the S-topology with limit $(bs\varphi)_{[b]}$, where we have written g_U for $\sum_{\tau \in U} g_\tau$ for $g \in G$, $U \subseteq T$. We may assume w.l.o.g. $\alpha \in V'$. Now (*) implies that for every $\eta \in {}_\alpha T \setminus [b]$ the η-th coordinate $(bs\varphi)_\eta$ is zero, since H is S-reduced. Thus the norm of any branch occurring in $[bs\varphi]$ is at most α and is indeed less than α since $\alpha \in V'$. Hence $bs\varphi = g^\alpha + b'$ with $g^\alpha \in G^\alpha$ and $b' \in B$, $[b'] \subseteq [b]$. Actually $b' = (bs\varphi)_{[b]}$. Now (*) reads as $b' - ba_{b,m} \in q_m B$ $(m > i(n))$, i.e. $ba_{b,m}$ converges to $(bs\varphi)_{[b]}$ as claimed.

Next we consider endomorphisms f_η of H for each $\eta \in {}_\alpha T$ defined as follows: for $h \in H$ let h^η be the element of B for which $[h^\eta] \subseteq \{\eta\}$ and $h^\eta(\eta) = h$; it is straightforward to check that $hf_\eta := (h^\eta s\varphi)(\eta)$ defines an endomorphism of H. The next step is to show that $f_\eta = f_{\eta'}$ for all $\eta, \eta' \in {}_\alpha T$. To this end fix $\eta \neq \eta'$ and consider the elements $e_h = h^\eta + h^{\eta'}$ for $h \in H$. We have shown that $e_h a_{e_h,m}$ converges to $(e_h s\varphi)_{[e_h]} = (hf_\eta)^\eta + (hf_{\eta'})^{\eta'}$. But this limit element has two identical components, since all the members of the approximating sequence have so, too. Thus $hf_\eta = hf_{\eta'}$ and $f_\eta = f_{\eta'} = f$ follows.

Now we wish to show that f is indeed multiplication by some $a \in A$. Let β be an ordinal $> \alpha$ and let $w = w(\beta)$ be the associated constant branch. For every at most countable subset F of H we choose an element $y_F \in \hat{B}$ such that $[y_F] = w$ and for each $h \in F$ suitable nonzero multiples of h occur infinitely often in y_F. If y_F is a strong element with respect to φ, we are done. So suppose y_F is not a strong element for every F. So there are $n(F) < \omega$ such that $y_F(q_{n(F)} s\varphi) \in G + y_F A$. We choose $n(F)$ to be minimal with this property and obtain $y_F(q_{n(F)} s\varphi - a_F) = g_F \in G$ for some $a_F \in A$. Now for every $\eta \in {}_\alpha T \setminus w$ we have $(g_F)_\eta = 0$ and if $\eta \in w$, then $g_F(\eta)$ is a non-zero multiple of $h(q_{n(F)} f - a_F)$. By (2.4)(b) almost all $g_F(\eta)$ $(\eta \in w)$ must be zero. But multiples of h occur infinitely often in y_F, thus $F(q_{n(F)}f - a_F) = 0$. Now for $F' \subseteq F$ we find $F'(q_{n(F)}f - a_F) = 0$ and therefore

$$y_{F'}(q_{n(F)}s\varphi - a_F) = 0$$

which implies $n(F') \leqslant n(F)$ by the minimal choice of $n(F')$. If $\sup\{n(F): F \subseteq H, |F| \leqslant \aleph_0\} = \omega$, there would be a strictly increasing sequence $n(F_1) < n(F_2) < \ldots$ which would exceed $n(F)$ after finitely many steps, where $F = \bigcup_{n<\omega} F_n$ is countable. This is impossible since $n(F) \geqslant n(F_i)$ for all $i < \omega$. Therefore

$$n = \max\{n(F): F \subseteq H, |F| \leqslant \aleph_0\} < \omega$$

and we find $s_F \in S$ with $q_n = s_F q_{n(F)}$. Then $F(q_n f - s_F a_F) = 0$ for all at most countable subsets F of H. Thus $s_F a_F$ is a net converging to $q_n f$ in the $\aleph_1$-topology of End (H). By assumption, A is a closed subalgebra of End (H) and therefore $q_n f \in A$. The purity of A in End (H) finally yields that f is multiplication by some element $a \in A$. Summarizing, we have $\|b(s\varphi - a)\| < \alpha$ for all $b \in B$ with ${}_\alpha[b] = [b]$ as desired.

Proof of (3.1): By (3.3) and (3.4) there are $s \in S$, $a \in A$ and $\alpha < \lambda$ such that for all $x \in B$ with ${}_\alpha[x] = [x]$ we have $\|x(s\varphi - a)\| < \alpha$, if there is no strong element for φ. Now let $w = w(\beta)$ be any constant branch for some $\beta > \alpha$. Choose any element $y \in \hat{B}$ with $[y] = w$ and such that $[ya']$ is either empty or infinite for every $a' \in A$. If y is not strong we find some $q = s \cdot t \in S$ and $a' \in A$ such that

$$y(q\varphi - a') = y(st\varphi - ta) + y(ta - a')$$

lies in G. But the first term has norm less than α, therefore $[y(ta - a')]$ is finite and a fortiori empty. Thus $y(s\varphi - a) \in G$. Suppose this holds for all constant branches $w(\beta)$ $(\beta > \alpha)$. Now if $s\varphi - a \neq 0$, then its extension to $\hat{B}$ cannot map all $\hat{B}$ into G, since by (2.7) G is cotorsion-free. So pick $x \in \hat{B}$ with $x(s\varphi - a) \notin G$. Choose a constant branch $w = w(\beta)$ for some $\beta > \|x\|$, $\|x\varphi\|$ and $y \in \hat{B}$ as above with the additional requirement

$$\mathrm{Ann}_A(y) \subseteq \mathrm{Ann}_A(x)\,.$$

This is possible because $[x]$ is countable. Now $x + y$ is a strong element for φ. If it is not, there is some $q = s \cdot t \in S$ and $a' \in A$ with

$$(x + y)(q\varphi - a') = x(q\varphi - a') + y(st\varphi - ta) - y(a' - ta) \in G.$$

Since $y(s\varphi - a) \in G$ also $x(q\varphi - a') - y(a' - ta) \in G$. But the left hand summand has norm less than β, whereas the support of the second summand is a subset of $w(\beta)$, which by (2.4)(b) has to be finite and is actually empty by the assumption on y. Now $x(q\varphi - a') = x(st\varphi - ta)$, because $\mathrm{Ann}_A(y) \subseteq \mathrm{Ann}_A(x)$, and $x(s\varphi - a) \in G$ follows, contradicting the choice of x.

So we have found strong elements with respect to φ in all cases except when $s\varphi = a$ for some $s \in S$, $a \in A$. But by the purity of A in End (H), it is clear that φ itself is multiplication by some element from A. q.e.d.

4. The main theorem.

Now we are ready to complete the proof of our

(4.1) MAIN THEOREM: Let λ be an infinite cardinal with cf $(\lambda) > \omega$ and R be a commutative ring with unit different from 0, S a countable, multiplicatively closed subset of non-zero divisors of R such that $\bigcap_{s \in S} sR = 0$. If A is an R-algebra and H a faithful right A-module of cardinality $|H| \leqslant \lambda$, then the following are equivalent:

(a) H is a cotorsion-free R-module such that A is a pure subalgebra of End (H) which is closed in the $\aleph_1$-topology of End (H).

(b) There exists a cotorsion-free R-module G of cardinal $\lambda^{\aleph_0}$ such that

(1) End (G) is topologically isomorphic to A, where End (G) is equipped with the finite topology and A carries the topology induced by the $\aleph_1$-topology on End (H).

(2) G contains an (S-)dense and pure A-submodule B which is isomorphic to a direct sum of λ many copies of H.

PROOF: Implication $(b) \Rightarrow (a)$ has been shown in section 0. For the converse let G be the module constructed in section 2. Then $(b)(2)$ is immediate. We have already shown in (2.7) that G is cotorsion-free. It is also clear that A is a subring of End (G) since A acts faithfully on G. Next we show End $(G) = A$. So suppose for a contradiction that there is some $\varphi \in \mathrm{End}\,(G)\setminus A$. We denote the extension of φ to $\hat{B}$ also by φ. By (3.1) there is a strong element $y \in \hat{B}$ with respect to φ, i.e. $y\varphi \notin \langle G + yA\rangle_*$. Now the stationary black box (1.6) provides an $\alpha < \lambda^*$ such that $y, y\varphi \in \overline{P^\alpha}$, $\|P^\alpha\| > \|y\|$, $\|y\varphi\|$ and $\varphi \restriction P^\alpha = \varphi^\alpha$. We have to show that α is strong because then $b_\alpha = g_\alpha \varphi^\alpha \notin G$ gives the desired contradiction.

Let v be any branch of Rang (f^α). We choose an element $g_v \in \overline{P_\alpha}$ with $[g_v^1] = v$, $(g_v^1)_\tau \in H_\tau$ for all $\tau \in T$, $\mathrm{Ann}_A\,(g_v^1) = \mathrm{Ann}_A\,(P^\alpha)$ and $[g_v^1 a]$ is infinite whenever $a \in A\setminus \mathrm{Ann}_A\,(P^\alpha)$. This can be done as in the proof of (2.6). We claim that there exists $\varepsilon = \varepsilon_v \in \{0, 1\}$ such that $g_v^1 + \varepsilon y$ is strong with respect to φ. If not, there are $s \in S$, $a_\varepsilon \in A$ with $(g_v^1 + \varepsilon y)(s\varphi - a_\varepsilon) \in G_\alpha$. The usual subtraction gives

$$g_v^1(a_0 - a_1) + y(s\varphi - a_1) \in G_\alpha\,.$$

As $\|y(s\varphi - a_1)\| < \|P^\alpha\|$ and the recognition lemma (2.2) implies that $[g_v^1(a_0 - a_1)]$ is finite, hence it is empty. But then $y(s\varphi - a_1) \in G_\alpha \subseteq G$ follows contradicting the choice of y. Setting $g_v^0 = \varepsilon_v y$ the assumptions of (2.5) are satisfied. It follows that for some branch v of Rang (f^α) we have $b_\beta \notin G_\alpha + g_v A\rangle^*$ $(\beta < \alpha)$. Therefore α is indeed strong and End $(G) \simeq A$ is proved.

It remains to show that the topology τ_G inherited from the finite topology on End (G) via the identification End $(G) = A$ coincides with the topology τ_H induced by the $\aleph_1$-topology of End (H) regarding A as a subalgebra of End (H). Let F be a finite subset of G. Then each $g \in F$ has at most countable support and each $g(\eta) \in \hat{H}$ $(\eta \in [g])$ can be approximated by countably many elements of H. The collection F^* of all these elements for all $g \in F$, $\eta \in [g]$ is countable and $\mathrm{Ann}_A\,(F^*) \subseteq \mathrm{Ann}_A\,(F)$ follows easily. Thus $\tau_G \subseteq \tau_H$. Now let F^* be a countable subset of H. By (1.7)(d) there is a trap p^α such that $P^\alpha = P(X, W)$ for some $W \subseteq T$ and $F^* \subseteq X$ (cf. (1.3) also). Then $\mathrm{Ann}_A\,(P^\alpha) \subseteq \mathrm{Ann}_A\,(F^*)$. But in the construction (2.1) the element $g_\alpha \in G$ was chosen to satisfy $\mathrm{Ann}_A\,(g_\alpha) = \mathrm{Ann}_A\,(P^\alpha)$. Thus $\tau_H \subseteq \tau_G$ and $\tau_H = \tau_G$ follows.

5. Appendix: The black box for $\mathrm{cf}(\lambda) = \omega$.

The given proof is similar to [19], however we will replace the model theoretic argument by a simple counting argument. In section 1 we considered the case $\mathrm{cf}(\lambda) > \omega$, hence we assume that the cardinal λ satisfies $\lambda > \omega = \mathrm{cf}(\lambda)$. Choose a strictly increasing sequence of regular cardinals

$$\lambda_n < \lambda \quad (n \in \omega \cup \{-1\}) \qquad \text{with } \lambda_{-1} = \aleph_0 < \lambda_0 \ \text{ and } \ \sup \lambda_n = \lambda .$$

Next we define a tree $T = \bigcup_{n<\omega} \prod_{k<n} \lambda_k$ and its branches $\overline{T} = \prod_{n\in\omega} \lambda_n$.

In order to introduce an ordering $<$ on $\overline{T}$, we fix a proper ultrafilter D on ω. Then

$$g < h \quad \text{if and only if } \{n \in \omega : g(n) < h(n)\} \in D .$$

In particular, we have a canonical map $\overline{T} \to \overline{T}/D$ $(f \to \bar{f})$ and $(\overline{T}, <)$ induces a total ordering on $(\overline{T}/D, <)$. Clearly

(5.1) (*a*) $(\overline{T}/D, <)$ is totally ordered

(*b*) There are a regular cardinal μ with $\lambda < \mu \leqslant \lambda^{\aleph_0}$ and a map

$$* : \mu \to \overline{T}(\xi \to \xi^*)$$

such that the composite map $\mu \to \overline{T}/D(\xi \to \xi^*)$ is an order preserving embedding.

(*c*) Moreover, for any $g \in \overline{T}$ there exists $\xi < \mu$ with $g < \xi^*$ and λ_{n-1} divides $\xi^*(n)$ for all $n \in \omega$.

The cardinal μ is used for the definition of a norm in place of $\mathrm{cf}(\lambda)$. Define $\| \ \| : \overline{T} \to \mu$ $(g \to \|g\| = \min\{\xi < \mu : g < \xi^*\})$. The norm function can be extended to countable subsets $X \subseteq \lambda$ as follows. Since $X' \in \overline{T}$ for $X' = (X_n)_{n\in\omega}$ with $X_n = \sup(X \cap \lambda_n)$, also $\|X\| = \|X'\| \in \mu$, which extends $\| \ \|$.

Similar to § 1 we define a trap. From now on we adopt (1.2), (1.3) and (1.4). In particular $\mathfrak{T}_n$ denotes all partial traps of length n.

(5.2) DEFINITION: A trap p is a triple $p = (f, P, \varphi)$ where $f\colon {}^{\omega>}\omega \to T$ is a tree embedding, $P \in \mathfrak{C}$ a canonical submodule and $\varphi \in \operatorname{Hom}(P, \bar{P})$ satisfying the three conditions:

(a) Rang $g \subseteq [P]$

(b) $[P]$ is a subtree of T

(c) There exists (a unique) ordinal $\xi < \mu$ of cofinality ω such that $v < \xi^*$ and $\theta^* < v$ for all $\theta < \xi$ and all branches $v \in \operatorname{Br}(\operatorname{Rang} f)$. We say that p has norm ξ, i.e. $\|p\| = \xi$.

We claim

(5.3) THE BLACK BOX FOR cf $(\lambda) = \omega$. Assertion (1.7) holds in this setting for $V = \mu$ and $\lambda > \omega = \operatorname{cf}(\lambda)$.

Inspection of (1.7) shows, that the proof of (5.3) rests on

(5.4) THE EXISTENCE OF TRAPS. For any countable subset X of B and $\Phi \in \operatorname{End} \hat{B}$ there exists a trap $p = (f, P, \varphi)$ such that $X \subseteq P$ and $\Phi \restriction P = \varphi$.

Similar to (1.7), the proof of (5.4) is based on a trivial

(5.5) CODING-LEMMA:

(1) There exist coding-functions $cd_n\colon \mathfrak{T}_0 \times \ldots \times \mathfrak{T}_n \to \lambda_n$ $(n \in \omega)$ such that

$$d\colon \prod_{n<\omega} \mathfrak{T}_n \to \prod_{n<\omega} \lambda_n((p_n)_{n\in\omega} \to \big(cd_n((p_i)_{i\geqslant n})\big)_{n\in\omega}$$

is injective.

(2) There are coding functions $c_n\colon \lambda_n \to \lambda_n$ $(n \in \omega)$ such that any $\alpha < \lambda_n$ which is divisible by λ_{n-1} and any $\gamma < \lambda_n$ satisfy

$$\big|\{\beta < \lambda_n\colon c_n(\beta) = \gamma,\ \alpha < \beta < \alpha + \lambda_{n-1}\}\big| \geqslant \aleph_0\,.$$

The proof is similar to (1.5). In case (5.5)(2) decompose $\omega = \bigcup_{r<\omega} W_r$ into countable sebsets W_r. If $\beta < \lambda_n$ and $\beta = \omega k + t$ with $t < \omega$, define $c_n(\beta) = k$ if $t \in W_r$. Then (2) holds, since $\lambda_{n-1} \geqslant \aleph_0$.

Finally we want to show (5.4) without model theoretic arguments: We say that a sequence $(p_n)_{n\in\omega}$ of partial traps $p_n = (f_n, P_n, \varphi_n)$ is permitted if

(0) $p_n \subseteq p_{n+1}$ is an extension, i.e. $f_n \subseteq f_{n+1}$, $P_n \subseteq P_{n+1}$, $\varphi_n \subseteq \varphi_{n+1}$

(1) $(p_n)_{n\in\omega}$ is partially closed, i.e.

$$P_n\varphi_n \subseteq \overline{P}_{n+1}, \text{ Rang } f_{n+1} \cup [P_n]_{\leqslant} \subseteq [P_{n+1}]$$

(2) If $\eta \in {}^{n+1}\omega$, then $c_n(f_{n+1}(\eta)(n)) = cd_n((p_i)_{i\leqslant n})$.

Let $\mathbb{P}$ denote the set of all permitted sequences of partial traps. We want to show that there exists $(p_n)_{n\in\omega} \in \mathbb{P}$ such that

$$p = \bigcup_{n\in\omega} p_n = \left(\bigcup f_n, \bigcup P_n, \bigcup \varphi_n\right)$$

is a trap with (5.4).

Clearly (5.2)(*a*), (*b*) hold and it is easy to satisfy $X \subseteq \overline{P_0}$ and $\Phi \restriction P_n = \varphi_n$ in (5.4) by the choice of p_n. However, we have to work for (5.2)(*c*):

The idea is simple: We try inductively to construct the right p with (5.2)(*c*) and use the information from each attempt at the next one. After ω attempts a diagonal process will give the right trap.

Let $Y = \left\{g \in \bigcup_{n\geqslant 0} {}^n n : g(i) \leqslant i \text{ for all } i \in \operatorname{dom}(g)\right\}$. Inductively, we construct partial traps $p_{gm} \in \mathcal{T}_m$ $(m < \omega)$ and ordinals $\xi(n)$ for each $n = \operatorname{dom}(g)$, $g \in Y$ with the following properties

(3) If $h: \omega \to \omega$ with $h(n) \leqslant n$, then $(p_{h\restriction m,m})_{m<\omega} \in \mathbb{P}$.

(4) If $m \in \operatorname{dom}(g)$, $\eta \in {}^{m+1}\omega$ and $p_{gm+1} = (f_{gm+1}, P_{gm+1}, \varphi_{gm+1})$, then $\xi(g(m))^*(m) < f_{mg+1}(\eta)(m) < \xi(g(m))^*(m) + \lambda_{m-1}$.

(5) If $P_g = \bigcup_{m<\omega} P_{gm}$, then $\xi(n) = \sup\{\|P_g\| : g \in Y, \operatorname{dom}(g) = n\}$.

(6) If $m > \operatorname{dom}(g)$ and $\eta \in {}^{m+1}\omega$, then $\sup_{k\in\operatorname{dom}(g)} \xi(k)^*(m) < f_{gm+1}(\eta)$.

If $n = 0$, we choose any $(p_{\emptyset m})_{m<\omega}$ in $\mathbb{P}$ such that $X \subseteq \overline{P}_{\emptyset m}$ and $\varphi_{\emptyset m} = \Phi \restriction P_{\emptyset m}$. Moreover $\xi(0) = \|P_\emptyset\|$.

Suppose p_{gm} $(m < \omega)$, $\xi(i)$ $(i \leqslant n)$, $\operatorname{dom} g \leqslant n$ has been defined. We will construct p_{gm} $(m < \omega)$ for $\operatorname{dom} g = n + 1$ and $\xi(n + 1)$ with (3)-(6):

Choose $p_{gm} = p = p_{g\restriction m,m}$ for $m \in \operatorname{dom}(g)$ by induction. If $m = n + 1$, we can use (5.5) to extend the three embedding $f_{g\restriction n,n}$ such that

$$f_{gm}(\eta) = f_{g\restriction n,n}(\eta\restriction n)^\frown\gamma_\eta \qquad (\eta \in {}^m\omega)$$

for some γ_η with $\xi(g(n))^*(n) < \gamma_\eta < \xi(g(n))^* + \lambda_{n-1}$ and

$$c_n(\gamma_n) = cd_n((p_{g\restriction i,i})_{i\leqslant n}).$$

Choose P_{gm} as required in (1).

It is easy to define p_{gm} for $m > \operatorname{dom} g = n + 1$ such that (1) (2) and (4) hold. Finally, pick $\xi(n + 1)$ by condition (5). Hence P contains a subset of elements $(p_{gm})_{m<\omega}$ which satisfies (3)-(6).

We want to find a function $h\colon \omega \to \omega$ with $h(n) \leqslant n$ $(n < \omega)$ such that $(p_{h\restriction n,n})_{n<\omega} \in P$ and $p = \bigcup_{n<\omega} (p_{h\restriction n,n})$ satisfies (5.2)(c). Then p is the desired trap in (5.4). If $\xi = \sup \xi(n)$, let

$$h(n) = \begin{cases} 0 & \text{if } \xi(0)^*(n) \geqslant \xi^*(n) \\ h_n & \text{if } \xi(0)^*(n) < \xi^*(n) \text{ and } h_n \leqslant n \text{ such that} \\ \xi(h_n)^*(n) = \max\{\xi(i)^*(n)\colon i \leqslant n,\ \xi(i)^*(n) < \xi^*(n)\}\,, \end{cases}$$

and apply the easy filter argument from S. Shelah [19, p. 57]: if $v \in \operatorname{Br}(\operatorname{Rang} f)$ and $W = \{n \in \omega\colon \xi(0)^*(n) < \xi^*(n)\}$, then $W \in D$ from $\xi(0)^* < \xi$, hence

$$\xi(h(n))^* < \xi^*(n)\,.$$

Also $\xi(h(n))^*(n) + \lambda_{n-1} \leqslant \xi^*(n)$, since λ_{n-1} divides $\xi(h(n))^*$ and $\xi^*(n)$. If $n \in W$, then

$$\xi(h(n))^*(n) < v(n) < \xi(h(n))^*(n) + \lambda_{n-1w} \leqslant \xi^*(n)$$

hence $v < \xi^*$.

It remains to show $\xi(i)^* < v$.

Since

$$A_m = \{n < \omega\colon n > m,\ \xi(m)^*(n) < \xi^*(n)\} \in D\,,$$

also $B_i = \bigcap_{m \leqslant i} A_m \in D$. If $n \in B_i$, then $\xi(m)^*(n) < \xi^*(n)$ for all $m \leqslant i$ and $n > i$. In particular, $\xi(h(n))^*(n) \geqslant \xi(i)^*(n)$ and

$$v(n) > \xi(h(n))^*(n) \geqslant \xi(i)^*(n)$$

implies $\xi(i)^* < v$. ■

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