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A Simple Construction of a Parametrix for a Regular Hyperbolic Operator.

FAUSTO SEGALA (*)

0. Introduction.

In this paper we use the techniques introduced in the works [4], [5] to give a local parametrix for the operator

$$P = D_t^2 - \Phi(t, x)\, A(t, x, D_x) + B(t, x, D_t, D_x)$$

where Φ is a C^∞ function satisfiyng the following conditions:

$$\Phi \geqslant 0 \quad \text{for } t \geqslant 0\,, \tag{1}$$

$$\partial\Phi/\partial t \neq 0 \quad \text{where } \Phi = 0\,. \tag{2}$$

Moreover, $A(t, x, D_x) = \sum\limits_1^N a_{ij}(t, x) D_i D_j$ is an elliptic differential operator and B is a differential operator of order one.

In the sections 1, 2, we construct two operators $E_\pm\colon \mathcal{E}'(\mathbb{R}^N) \to \mathcal{D}'(\mathbb{R}^{N+1})$ in such a way that (locally)

$$PE_\pm \equiv 0 \quad \text{for } \Phi > 0\,,$$

and

$$\begin{bmatrix} E_+ & E_- \\ E'_+ & E'_- \end{bmatrix} \equiv I_{2N} \qquad \text{on } \Phi = 0\,,$$

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and an operator $J\colon \mathcal{E}'(\mathbb{R}^N) \to \mathcal{D}'(\mathbb{R}^{N+1})$ for which (locally)

$$PJ \equiv 0 \quad \text{for } \Phi < 0\,,$$
$$J - I_N \equiv 0 \quad \text{on } \Phi = 0\,.$$

The construction of J, E_+, E_- allows us to have a local parametrix for $Pu = f$ (see [5]).

The main application of the construction of $E_\pm$ is that we can exhibit (see sect. 3) a parametrix for the Cauchy problem

$$(3) \qquad Pu = 0\,, \quad u|_{t=0} = g_0\,, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g_1\,, \qquad t > 0\,.$$

More precisely, we can choose the initial data at $\Phi = 0$ in such a way that the solution of the Cauchy problem corresponding to these data is the solution of the Cauchy problem (3) with fixed data g_0, g_1 at $t = 0$.

When $\Phi(t, x) = t$, P is exactly the Tricomi operator and a parametrix for (3) has been constructed by Imai [2].

When $\Phi(t, x) = t + x^2$, $x \in \mathbb{R}$, a parametrix for (3) has been constructed by Yoshikawa [7], but, however, using quite different techniques from ours.

1. A change of variables.

Assume that $\Phi(0, x_0) = 0$. From (1), (2) it follows that we can write locally

$$P = D_t^2 - \big(t + b(x)\big)\tilde{A}(t, x, D_x) + \text{lower order terms}$$

where $\tilde{A}$ is elliptic and $b \geq 0$. By introducing the change of variables $(t, x) \to \big(t + b(x), x\big) = (\sigma, x)$ we can rewrite P as follows (for $|\sigma| + |x - x_0|$ small and new a_{ij})

$$(1.1) \qquad P = \left(1 - \sigma \sum_1^N a_{ij} \frac{\partial b}{\partial x_i}\frac{\partial b}{\partial x_j}\right) D_\sigma^2 - {}$$
$$- 2\sigma \sum_1^N a_{ij} \frac{\partial b}{\partial x_i} D_i D_\sigma - \sigma \sum_1^N a_{ij} D_i D_j + \text{lower order terms}\,.$$

Obviously, P is elliptic for $\sigma \in]-\delta, 0[$ and hyperbolic for $\sigma \in]0, \delta[$ and $x \in$ neighborhood of x_0.

For $\sigma \geqslant 0$ we can write the principal symbol of P as follows

$$(1.2) \qquad p(\sigma, x, \tau, \xi) = \left(1 - \sigma \sum a_{ij} \frac{\partial b}{\partial x_i} \frac{\partial b}{\partial x_j}\right) \cdot$$

$$\cdot [\tau - (\sigma\mu(\sigma, x, \xi) - \sqrt{\sigma B(\sigma, x, \xi) + \sigma^2 C(\sigma, x, \xi)})] \cdot$$

$$\cdot [\tau - (\sigma\mu(\sigma, x, \xi) + \sqrt{\sigma B(\sigma, x, \xi) + \sigma^2 C(\sigma, x, \xi)})]$$

where $\mu(\sigma, x, \xi)$ is homogeneous of degree 1, $B(\sigma, x, \xi)$ and $C(\sigma, x, \xi)$ are homogeneous of degree 2 and $B(0, x, \xi) = A(0, x, \xi)$.

2. Hyperbolic region.

Eiconal equation associated to the operator P, from (1.2) is the following

$$(2.1) \qquad \frac{\partial \varphi_\pm}{\partial \sigma} = \sigma\mu\left(\sigma, x, \frac{\partial \varphi_\pm}{\partial x}\right) \pm \sqrt{\sigma B\left(\sigma, x, \frac{\partial \varphi_\pm}{\partial x}\right) + \sigma^2 C\left(\sigma, x, \frac{\partial \varphi_\pm}{\partial x}\right)},$$

$$\varphi_\pm(0, x, \xi) = \langle x, \xi \rangle .$$

By changing σ with s^2 and by writing $\hat{\varphi}_\pm(s, x, \xi) = \varphi_\pm(s^2, x, \xi)$, we have

$$\frac{\partial \hat{\varphi}_\pm}{\partial s} = 2s^3\mu\left(s^2, x, \frac{\partial \hat{\varphi}_\pm}{\partial x}\right) \pm 2s^2 \sqrt{B\left(s^2, x, \frac{\partial \hat{\varphi}_\pm}{\partial x}\right) + s^2 C\left(s^2, x, \frac{\partial \hat{\varphi}_\pm}{\partial x}\right)}$$

that is we can write

$$(2.2) \qquad \frac{\partial \hat{\varphi}_\pm}{\partial s} = 2s^3\hat{\mu}\left(s, x, \frac{\partial \hat{\varphi}_+}{\partial x}\right) \pm 2s^2\lambda\left(s, x, \frac{\partial \hat{\varphi}_\pm}{\partial x}\right), \qquad \hat{\varphi}_\pm(0, x, \xi) = \langle x, \xi \rangle$$

and $\lambda(s, x, \eta)$ is regular for small s.

Then, by arguing in a standard way, by means of an application of the theory of Hamilton-Jacobi (see for exemple [2]), we can solve (2.2) and the solution $\hat{\varphi}_\pm$ can be written for small s as

$$\hat{\varphi}_\pm(s, x, \xi) = < x, \xi > + s^4\alpha(s^2, x, \xi) \pm \tfrac{2}{3} s^3(\beta(s^2, x, \xi))^{3/2}$$

where α is homogeneous of degree 1, β is homogeneous of degsee $\frac{2}{3}$ and $\beta(0, x, \xi) \geqslant$ positive constant $\times |\xi|^{2/3}$. In conclusion the solutions $\varphi_\pm(\sigma, x, \xi)$ of (2.1), for small σ are of the form

$$\varphi_\pm(\sigma, x, \xi) = \theta(\sigma, x, \xi) \pm \tfrac{2}{3}\varrho(\sigma, x, \xi)^{2/3}$$

where θ is homogeneous of degree 1 and

$$\theta(\sigma, s, \xi) = < x, \xi > + o(\sigma) \tag{2.3}$$

and ϱ is homogeneous of degree $\frac{2}{3}$ and

$$\varrho(0, x, \xi) = 0\,, \qquad \varrho'_\sigma(0, x, \xi) > 0\,. \tag{2.4}$$

Now we try to solve the transport equation

$$P(\exp[i\varphi_\pm] k_\pm) = 0$$

where formally $k_\pm(\sigma, x, \xi) = \varrho(\sigma, x, \xi)^{-1/4}\hat{k}_\pm(\varrho(\sigma, x, \xi), x, \xi)$ and

$$\hat{k}_\pm = \sum_0^\infty \hat{k}_\pm^{-\nu} \tag{2.5}$$

with $\hat{k}_\pm^{-\nu}(\varrho(\sigma, x, \xi), x, \xi)$ homogeneous (in ξ) of degree $-\nu$. We have, by means of a tedions calculation (we omit $\pm$)

$$\begin{aligned} P(\exp[i\varphi]k) = &-i\varrho^{-1/4}\exp[i\varphi]\cdot \\ &\cdot\Big\{2\,[(1-\sigma\sum a_{ij}b'_{x_i}b'_{x_j})(\theta'_\sigma + \sqrt{\varrho}\,\varrho'_\sigma)\,\varrho'_\sigma - \sigma\sum a_{ij}b'_{x_i}(\theta'_{x_j} + \sqrt{\varrho}\,\varrho'_{x_j})\,\varrho'_\sigma - \\ &-\sigma\sum a_{ij}b'_{x_i}\varrho'_{x_j}(\theta'_\sigma + \sqrt{\varrho}\,\varrho'_\sigma) - \sigma\sum a_{ij}(\theta'_{x_i} + \sqrt{\varrho}\,\varrho'_{x_i})\,\varrho'_{x_j}]\,\frac{\partial\hat{k}}{\partial\varrho} + \\ &+ L(\sqrt{\varrho}, x, D_x)\hat{k} - i(1-\sigma)\sum a_{ij}b'_{x_i}b'_{x_j}\cdot \\ &\cdot\left[\frac{5}{16}\varrho^{-2}\varrho'^2_\sigma\hat{k} - \frac{1}{2}\varrho^{-1}\varrho'^2_\sigma\frac{\partial\hat{k}}{\partial\varrho} - \frac{1}{4}\varrho^{-1}\varrho''_{\sigma\sigma}\hat{k} + \varrho''_{\sigma\sigma}\frac{\partial\hat{k}}{\partial\varrho} + \varrho'^2_\sigma\frac{\partial^2\hat{k}}{\partial\varrho^2}\right] + \dots\Big\} = 0 \end{aligned} \tag{2.6}$$

and $L(s, x, D_x)$ is a differential first order operator with C^∞ coefficients in (s, x). From (2.4) it follows that (for small $\sigma > 0$) we can make the change of variable $\sigma \to \varrho(\sigma, x, \xi)$. Then, as in the hyperbolic case, we insert (2.5) in (2.6), and we equal to zero the terms which have the same homogeneity. Therefore we obtain the following transport equations:

$$(2.7)_0 \quad G\hat{k}_0 = 2\sqrt{\varrho}\Big\{(1-\sigma\sum a_{ij}b'_{x_i}b'_{x_j})(\theta'_\sigma/\sqrt{\sigma}+\varrho'_\sigma)\varrho'_\sigma - -\frac{\sigma}{\sqrt{\varrho}}\sum a_{ij}b'_{x_i}(\,'_{x_j}\theta+\sqrt{\varrho}\,\varrho'_{x_j})\varrho'_\sigma - \frac{\sigma}{\sqrt{\varrho}}\sum a_{ij}b'_{x_i}\varrho'_{x_j}(\theta'_\sigma+\sqrt{\varrho}\,\varrho'_\sigma) - -\frac{\sigma}{\sqrt{\varrho}}\sum a_{ij}(\theta'_{x_i}+\sqrt{\varrho}\,\varrho'_{x_i})\varrho'_{x_j}\Big\}\frac{\partial\hat{k}_0}{\partial\varrho} + L(\sqrt{\varrho}, x, D_x)\hat{k}_0 = 0$$

and for $\nu\geqslant 1$:

$$(2.7)_\nu \quad G\hat{k}^{-\nu} = i(1-\sigma\sum a_{ij}b'_{x_i}b'_{x_j})\left[\frac{5}{16}\varrho^{-2}\varrho'^2_\sigma\hat{k}^{-\nu+1}+\frac{1}{2}\varrho^{-1}\varrho'^2_\sigma\frac{\partial\hat{k}^{-\nu+1}}{\partial\varrho}+ +\frac{1}{4}\varrho z^1\varrho''_{\sigma\sigma}\hat{k}^{-\nu+1}-\varrho''_{\sigma\sigma}\frac{\partial\hat{k}^{-\nu+1}}{\partial\varrho}-\varrho'^2_\sigma\frac{\partial^2\hat{k}^{-\nu+1}}{\partial\varrho^2}\right]+\dots .$$

We observe that, for small σ, the coefficient fo $\partial\hat{k}_0/\partial\varrho$ in $(2.7)_0$ is of the form $\sqrt{\varrho}\,q(\sqrt{\varrho}, x, \xi)$ with $q(0, x, \xi)\neq 0$. Hence by the change of variable $\varrho = s^2$ and by setting $\hat{k}(s, x, \xi) = k(s^2, x, \xi)$, we can write the transport equations as follows (we add the initial condition on k^0)

$$(2.8)_0 \qquad \frac{\partial\hat{k}_0}{\partial s} + M(s, x, D_x)\hat{k}^0 = 0\,, \qquad \hat{k}(0, x, \xi) = 1\,,$$

$$(2.8)_\nu \qquad \left(\frac{\partial}{\partial s}+M\right)\hat{k}^{-\nu} = \text{right hand side depending on } \hat{k}_0, \dots, \hat{k}^{-\nu+1} \text{ which is singular for } s = 0\,.$$

From (2.3), (2.4) it follows that M is a first order differential operator with C^∞ coefficents.

Now we can apply the calculus developed in [5]. Precisely, we can take $k^{-\nu}$, solution of $(2.8)_0$, $(2.8)_\nu$ in such a way that for $s\geqslant 1$,

$$(2.9) \qquad \hat{k}_\pm^{-\nu}(s, x, \xi) = c^\pm_{\nu-}s^{-3\nu/2}[1+s\mu^\pm_{-\nu}(s, x, \xi)]$$

where $c^\pm_{-\nu}$ is a complex constant and $\mu^\pm_{-\nu}\in S^{1,-1/3}\cap S^{0,0}$. More precisely, if $\mathrm{Ai}(z)$ is the Airy function and $A_\pm(z) = 2\pi\exp[\pm\frac{2}{3}\pi i]\,\mathrm{Ai}\,(\exp[\pm\frac{2}{3}\pi i](-z))$, then $c^\pm_{-\nu}$ are the coefficients of the asymptotic expansion of $A_\pm$; i.e.

$$A_\pm(z) \approx z^{-1/4} \exp[2z^{3/2}/3] \sum_0^\infty c^\pm_{-\nu} z^{-3\nu/2}$$

(see for example Wasow [6]).

We recall that the class $S^{M,Q}$ is the set of all $C^\infty([1, \infty[\times \mathbb{R}^N \times \mathbb{R}^N)$ functions $a(s, x, \xi)$ satisfying the following estimates (locally in x)

$$|D_s^j D_\xi^\alpha D_x^\beta a(s, x, \xi)| \leqslant C s^{M-j} |\xi|^{Q-|\alpha|}, \qquad s \geqslant 1, \ |\xi| \geqslant 1,$$

We observe that if a $(s, x, \xi) \in S^{-\infty, Q}$, then

$$c(\sigma, x, \xi) = a\big(\sqrt{\varrho(\sigma, s, \xi), x}, \xi\big) \in S^Q_{1,2/3,0}$$

for $\varrho(\sigma, x, \xi) \geqslant 1$, i.e. $|D_\sigma^j D_\xi^\alpha D_x^\beta c(\sigma, x, \xi)| \leqslant c(1 + |\xi|)^{Q-|\alpha|+2j/3}$.

Then we can sum the $\hat{k}_\pm^{-\nu}$ in a standard way and we can define $\hat{k}_\pm(s, x, \xi) = \sum_0^\infty \hat{k}_\pm^{-\nu}(s, x, \xi)$ modulo an error $\in S^{-\infty,0}$. Finally take $k(\sigma, x, \xi) = \varrho^{-1/4} \hat{k}\,\big(\sqrt{\varrho(\sigma, x, \xi)}, x, \xi\big)$. If $\chi \in C^\infty(\mathbb{R}^+)$, $\chi(\tau) = 0$ when $0 \leqslant \tau \leqslant \delta$, $\chi(\tau) = 1$, $\tau \geqslant 2\delta$, $\delta > 1$, $\chi(\varrho)\ \chi_\pm$ is a C^∞ function and we have that

$$P\big(\exp[i\varphi_\pm](\chi k_\pm)\big) = \exp[i\theta]\, h_\pm$$

with $(s, x, \xi) \to h_\pm\,\big(\sigma(s^2, x, \xi), x, \xi\big) \in S^{-\infty, 4/3}$, i.e. $h_\pm$ is flat for $\varrho \to +\infty$ and moreover $h_\pm = 0$ for $\varrho \leqslant \delta$ (fig. 1)

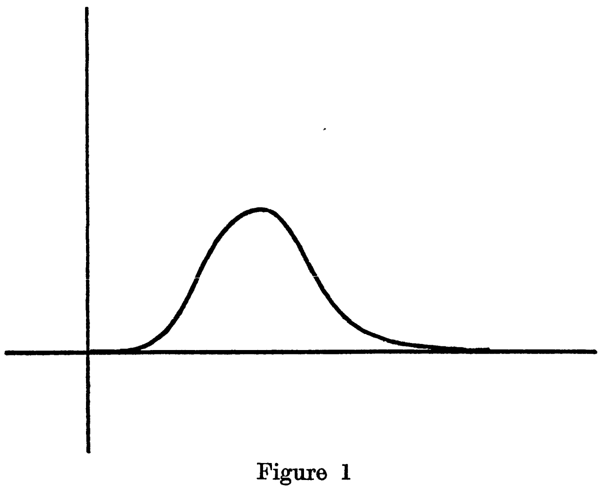

Figure 1

Unfortunately, the operators

$$f(x) \to \int \exp[i\theta(\sigma, x, \xi)]\, h_\pm(\sigma, x, \xi) \hat{f}(\xi)\, d\xi$$

are not regularizing. However, there exist $m_\pm(\sigma, x, \xi) \in S^0_{1,2/3,0}$ which compensate the error $h_\pm$, that is $P(\exp[i\theta] m_\pm) - \exp[i\theta] h_\pm$ is regularizing and the matrix

$$\begin{matrix} m_+ & m_- \\ n'_+ & m'_- \end{matrix}$$

is elliptic for $\sigma = 0$. The proof of this remarkable fact is given in [5] in a very similar case. Therefore, there exist two operators $E_\pm(\sigma)$ with

$$\begin{aligned} PE_\pm &= 1\,, & \sigma &> 0\,, \\ E_+(0) &\equiv 1\,, & E_-(0) &\equiv 0\,, \\ E'_+(0) &\equiv 0\,, & E'_-(0)&\,, \end{aligned}$$

and $E_\pm$ are of the form

$$E_\pm(\sigma) f(x) = \int \exp[i\theta(\sigma, x, \xi)]\, m_\pm(\sigma, x, \xi) \hat{f}(\xi)\, d\xi + \\ + \int \exp[i\varphi(\sigma, x, \xi)]\, m_\pm(\sigma, x, \xi) \hat{f}(\xi)\, d\xi$$

with $m_\pm$, $n_\pm \in S^0_{1,2/3,0}$ and $n_\pm(0, x, \xi) = 0$.

By the Duhamel's principle we can write a parametrix for the non-homogeneous Cauchy-problem with initial data at $\sigma = 0$.

Now we have to study the Dirichlet problem on $\sigma \leqslant 0$. We begin by study the case

$$Pu = 0\,, \qquad u|_{\sigma=0} = g\,, \quad \sigma < 0 \tag{2.10}$$

and we seek a parametrix of (2.10) on the form

$$Jg(\sigma)(x) = \int \exp[i\langle x, \xi\rangle]\, a(\sigma, x, \xi) \hat{g}(\xi)\, d\xi$$

with $a \in S^0_{1,2/3,0}$, i.e. $|D^j_\sigma D^\alpha_\xi D^\beta_x a| \leqslant C(1 + |\xi|)^{-|\alpha| + 2j/3}$. More precisely, we seek a as an asymptotic expansion $\sum_0^\infty a_{-\nu/2}$ with $a_{-\nu/2}$ pseudo-homogeneous of degree $-\nu/2$, i.e. $a_{-\nu/2}(\lambda^{-1}\sigma, x, \lambda^{3/2}\xi) = \lambda^{-\nu/2} a_{-\nu/2}(\sigma, x, \xi)$ $\forall \lambda > 0$. We can give a sense to the sum $\sum_0^\infty a_{-\nu/2}$, since $a_{-\nu/2} \in S^{-\nu/3}_{1,2/3,0}$ (see [4] for the details). From here, we can follow closely the calculation of [4], [5] to obtain a parametrix for $Pu = f$ for $\sigma \in]-\delta, \delta[$.

3. Cauchy problem with initial at $t = 0$.

In the section 2. we have given a local parametrix for the Cauchy problem for P on $\sigma \geqslant 0$, that is on $t \geqslant -b(x)$. Here we shall prove that it is possible to construct a local parametrix for the Cauchy problem for P on $t \geqslant 0$ and x near to x_0 (with $b(x_0) = 0$) by starting from a parametrix for the Cauchy problem on $t \geqslant -b(x)$. Let $E_\pm$ be the operators constructed in the section 2., i.e.

$$E_\pm(\sigma)f(x) = \int \exp[i\theta(\sigma, x, \xi)]\, m_\pm(\sigma, x, \xi)\hat{f}(\xi)\, d\xi + \\ + \int \exp\left[i\left[\theta(\sigma, x, \xi) \pm \tfrac{2}{3}\varrho(\sigma, x, \xi)^{3/2}\right]\right] n_\pm(\sigma, x, \xi)\hat{f}(\xi)\, d\xi .$$

In [5] one proves that

$$m_\pm(\sigma, x, \xi) = \big(1 - \chi(\varrho(\sigma, x, \xi))\, A_\pm(\varrho(\sigma, x, \xi)) + b_\pm(\sigma, x, \xi)\big)$$

with $b_\pm \in S^{-1/3}_{1,2/3,0}$ and

$$\exp[\pm i\tfrac{2}{3}\varrho^{3/2}]\, n_\pm(\sigma, x, \xi) = \chi(\varrho(\sigma, x, \xi))\,(1 + q_\pm(\sigma, x, \xi))\, A_\pm(\varrho(\sigma, x, \xi))$$

with

$$q_\pm(\sigma(\varrho, x, \xi), x, \xi) \in S^{1/2,-1/3} .$$

Therefore we can rewrite the operators $E_\pm$ in the following way

$$E_\pm(\sigma)f(x) = \int \exp[i\theta(\sigma, x, \xi)]\,(1 + (\chi q_\pm)(\sigma, x, \xi))\, A_\pm(\varrho(\sigma, x, \xi))\hat{f}(\xi)\, d\xi + \\ + \int \exp[i\theta(\sigma, x, \xi)]\, b_\pm(\sigma, x, \xi)\hat{f}(\xi)\, d\xi .$$

Therefore the operators $Q_\pm$ defined by (here we write for simplicity b in the place of $-b$)

$$(3.1)\qquad Q_\pm(\sigma)f(x) = \iint \exp\left[i[\theta(\sigma, x, \xi) - \theta(b(y), y, \xi)]\right]\cdot \cdot\left\{(1 + (\chi q_\pm)(\sigma, x, \xi))\frac{A_\pm(\varrho(\sigma, x, \xi))}{A_\pm\big(\varrho(b(y), y, \xi)\big)} + b_\pm(\sigma, x, \xi)\right\} f(y)\, dy\, d\xi$$

are (locally) a solution of $Pu = 0$ on $\sigma \geqslant 0$.

We write $A_\pm = (1-\chi)A_\pm + \chi F_\pm \exp[\pm i\frac{2}{3}\varrho^{3/2}$, with $F_\pm \in C^\infty$ for $\varrho \geqslant 0$. Take $\zeta \in C^\infty(\overline{\mathbf{R}}_+)$, $\zeta \equiv 1$ on a neighborhood of 0. Then it follows that

$$(3.2)\quad A_\pm(\varrho) = \exp[\pm i\tfrac{2}{3}\varrho^{3/2}\zeta(\varrho)]\,[(1-\chi(\varrho))A_\pm(\varrho)\exp[\mp i\tfrac{2}{3}\varrho^{3/2}\zeta(\varrho) + + \chi(\varrho)F_\pm(\varrho)] = \exp[\pm i\varepsilon(\varrho)]\,C_\pm(\varrho)$$

with $C_\pm \in S^{0,0}$. By means of a simple calculation, we have that

$$(3.3)\qquad A_+ A'_- - A_- A'_+ = C_+ C'_- - C_- C'_+ - 2i\varepsilon' C_+ C_- = = \text{complex constant } k \neq 0.$$

We set

$$\psi_\pm(x, y, \xi) = \theta(b(x), x, \xi) - \theta(b(y), y, \xi) \pm \varepsilon\big(\varrho(b(x), x, \xi)\big) + + \varepsilon\big(\varrho(b(y), y, \xi)\big)$$

and

$$S_\pm = \text{principal contribution to } Q_\pm|_{\sigma = b(x)}$$

$$T_\pm = \text{principal contribution to } Q'_\pm|_{\sigma = b(x)} \circ |D_x|^{2/3}.$$

Hence, from (3.1), (3.2) we obtain

$$S_\pm f(x) = \iint \exp[i\psi_\pm(x, y, \xi)]\big(1 + (\chi q_\pm)(b(x), x, \xi)\big)\cdot \cdot\frac{C_\pm\big(\varrho(b(x), x, \xi)\big)}{C_\pm\big(\varrho(b(y), y, \xi)\big)} f(y)\, dy\, d\xi\,,$$

$$T_{\pm} f(x) = \iint \exp\left[i\psi_{\pm}(x, y, \xi)\right] \cdot$$
$$\cdot |\xi|^{-2/3} \left\{ i\left[\theta'_{\sigma}(b(x), x, \xi) \pm \varepsilon'_{\varrho}\big(\varrho(b(x), x, \xi)\big)\varrho'_{\sigma}(b(x), x, \xi)\right] \cdot \right.$$
$$\cdot \big(1 + (\chi q_{\pm})(b(x), x, \xi)\big) \frac{C_{\pm}\big(\varrho(b(x), x, \xi)\big)}{C_{\pm}\big(\varrho(b(y), y, \xi)\big)} +$$
$$+ (\chi q_{\pm})'_{\sigma}(b(x), x, \xi) \frac{C_{\pm}\big(\varrho(b(x), x, \xi)\big)}{C_{\pm}\big(\varrho(b(y), y, \xi)\big)} +$$
$$\left. + \big(1 + (\chi q_{\pm})(b(x), x, \xi)\big) \frac{C_{\pm}\big(\varrho(b(x), x, \xi)\big)}{C_{\pm}\big(\varrho(b(y), y, \xi)\big)} \varrho'_{\sigma}(b(x), x, \xi) \right\} f(y)\, dy\, d\xi\,.$$

From (2.3) and (2.4) we can write $\psi_{\pm}(x, y, \xi) = \langle x - y, \xi + \gamma_{\pm}(x, y, \xi)\rangle$ with

$$|\gamma_{\pm}(x, y, \xi)| \leqslant C\big(|x - x_0| + |y - y_0|\big)$$

and

$$|\nabla\xi\ \psi_{\pm}(x, y, \xi)| \leqslant C\big(|x - x_0| + |y - x_0|\big)$$

for (x, y) near to (x_0, x_0). Then we can consider the change of variable $\xi \to \eta = \xi + k_{\pm}(x, y, \xi)$. Since

$$|\chi q_{\pm}(\sigma, x, \xi)| \leqslant C\sqrt{\sigma} \quad \text{and} \quad |(\chi q_{\pm})'_{\sigma}(\sigma, x, \xi)| \leqslant C\sqrt{\sigma}|\xi|^{2/3}$$

for $\sigma \in [0, \delta]$ and x near to x_0, the matrix

$$\begin{bmatrix} S_{+} & S_{-} \\ T_{+} & T_{-} \end{bmatrix}$$

is locally invertible in a neighborhood of $x = 0$ if is invertible the matrix

$$J = \begin{bmatrix} S^{\#}_{+} & S^{\#}_{-} \\ T^{\#}_{+} & T^{\#}_{-} \end{bmatrix}$$

where

$$S^{\#}_{\pm} f(x) = \iint \exp\left[i\langle x - y, \eta\rangle\right] \cdot \frac{C_{\pm}\big(\varrho(b(x), x, \eta)\big)}{C_{\pm}\big(\varrho(b(y), y, \eta)\big)} f(y)\, dy\, d\eta$$

and

$$T_{\pm}^{\#}f(x) = \int\int \exp\left[i\langle x - y, \eta\rangle\right]|\eta|^{-2/3} \cdot$$

$$\cdot\left\{\left[i\theta_\sigma'(b(x), x, \eta) \pm \varepsilon_\varrho'\big(\varrho(b(x), x, \eta)\big)\varrho_\sigma'(b(x), x, \eta)\right]\cdot\right.$$

$$\left.\cdot \frac{C_\pm\big(\varrho(b(x), x, \eta)\big)}{C_\pm\big(\varrho(b(y), y, \eta)\big)} + \frac{C_\pm\big(\varrho(b(x), x, \eta)\big)}{C_\pm\big(\varrho(b(y), y, \eta)\big)}\varrho_\sigma'(b(x), x, \eta)\right\} f(y)\, dy\, d\eta\,.$$

The matrix J is invertible since, from (3.3) we have

$$|\eta|^{-2/3}\left[C_+(i\theta_\sigma' C_- - i\varepsilon_\varrho' C_- \varrho_\sigma' + C_-' \varrho_\sigma') - \right.$$
$$\left. - C_-(i\theta_\sigma' C_+ + i\varepsilon_\varrho' C_+ \varrho_\sigma' + C_+' \varrho_\sigma')\right] = |\eta|^{-2/3}\varrho_\sigma' k\,.$$

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