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λ-satisfiability, λ-consistency property, the downward Lowenheim Skolem theorem, and the failure of the interpolation theorem for $L_{k,k}$ with k a strong limit cardinal of cofinality λ

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λ-Satisfiability, λ-Consistency Property, the Downward Lowenheim Skolem Theorem, and the Failure of the Interpolation Theorem for $L_{k,k}$ with k a Strong Limit Cardinal of Cofinality λ.

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SUMMARY - In this paper, the infinitary languages $L_{k,k}$ where k is a strong limit cardinal of cofinality λ are considered. The notions of λ-satisfiability and of λ-seq-consistency property are introduced and the model existence theorem, its inverse and the downward Lowenheim Skolem theorem are proved. Also, the notions introduced are compared for different values of λ, and it is shown that the interpolation theorem fails for these languages when $\lambda \neq \omega$.

0. Introduction.

In the study of infinitary logics, some finitistic features should be kept in order to classify the subject as logic. In particular, the main finitistic feature in the infinitary languages $L_{\alpha,\beta}$ is the construction of the formulas.

Among these languages, the infinitary language $L_{k,k}$ with k a strong limit cardinal of cofinality ω are particularly well behaved [2], [4], [5], [6], [7]. Indeed for these languages it is possible to introduce the notion of ω-satisfiability, weaker than the notion of satisfiability, with respect to which it is possible to recover some important the-

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orems in general lost for infinitary logics, namely the model existence theorem, the downward Lowenheim Skolem theorem and the interpolation theorem.

But between one such cardinal, say k_1, and the next one, say k_2, there might be a large gap, and one could be interested in considering an infinitary sentence not expressible in L_{k_1,k_1} but for which L_{k_2,k_2} is far too large. Hence the interest in infinitary languages $L_{\alpha,\alpha}$ where α is a cardinal with different characteristics.

Here we consider the case in which the index of the infinitary language is k, a strong limit cardinal of cofinality λ.

For the case under consideration, we will introduce a convenient notion of λ-satisfiability, compare it with Karp's notion of ω-satisfiability, and obtain the model existence theorem for a λ-seq-consistency property, its inverse and the downward Lowenheim Skolem theorem. To stress the difference between this case and Karp's one, we will show that the interpolation theorem fails this time.

1. Preliminaries.

Let k be a strong limit cardinal of cofinality λ. Assume that $k = \bigcup\{k_n\colon n \in \lambda\}$ with $k_{n'} \geqslant 2^{k_n}$ whenever $n' > n$.

A λ-split structure is obtained from a structure $\langle A, \boldsymbol{R}, \boldsymbol{C}\rangle$ by splitting its universe A in λ subsets A_n, $n \in \lambda$, called the components of the universe, in such a way that $A = \bigcup\{A_n\colon n \in \lambda\}$ and $A_{n_1} \subset A_{n_2}$ whenever $n_1 < n_2$; $\boldsymbol{R}$ is an indexed set $\{R_i\colon i \in I\}$ of relations R_i on A; $\mu\colon I \to \omega$ is the function that assigns to each index of a relation its ariety; $\boldsymbol{C}$ is an indexed set $\{c_j\colon j \in J\}$ of designated elements and there is $n_0 \in \lambda$ such that $\boldsymbol{C} \subset A_{n_0}$.

We will denote a λ-split structure $\overline{\mathfrak{A}}$ as $\langle\{A_n\colon n \in \lambda\}, \boldsymbol{R}, \boldsymbol{C}\rangle$.

$\langle I, \mu, J\rangle$ is the similarity type both of the λ-split structure and of the related structure $\langle\bigcup\{A_n\colon n \in \lambda\}, \boldsymbol{R}, \boldsymbol{C}\rangle$.

Let $L_{k,k}$ be the infinitary language with identity adeguate for the similarity type $\langle I, \mu, J\rangle$.

A bounded assignment to a set of variables with respect to a λ-split structure is a map from the variables of the set in one of the components of the universe of the λ-split structure.

If t is a term, a valuation $\sigma = (\overline{\mathfrak{A}}, \boldsymbol{a})$ of t in a λ-split structure $\overline{\mathfrak{A}}$ under a bounded assignment $\boldsymbol{a}$ whose domain includes t if t is a variable, denoted by $\sigma(t)$, is the element of A defined as follows:

if t is the individual constant $\boldsymbol{c}_j$, then $\sigma(t)$ is the designated element c_j, while if t is a variable then $\sigma(t)$ is $\boldsymbol{a}(t)$.

We say that a λ-split structure $\overline{\mathfrak{A}}$ λ-satisfies an $L_{k,k}$ formula φ under a bounded assignment $\boldsymbol{a}$ to the free variables occurring in φ, $\overline{\mathfrak{A}}, \boldsymbol{a} \models^{\lambda} \varphi$, when the following holds:

a) if φ is the atomic formula $P_i(t_i, \ldots, t_{\mu(i)})$ then $\overline{\mathfrak{A}}, \boldsymbol{a} \models^{\lambda} \varphi$ if $\langle \sigma(t_1), \ldots, \sigma(t_{\mu(i)}) \rangle \in R_i$;

b) if φ is the atomic formula $t_1 = t_2$ then $\overline{\mathfrak{A}}, \boldsymbol{a} \models^{\lambda} \varphi$ if $\sigma(t_1)$ is $\sigma(t_2)$;

c) if φ is $-\psi$ then $\overline{\mathfrak{A}}, \boldsymbol{a} \models^{\lambda} \varphi$ if it is not the case that $\overline{\mathfrak{A}}, \boldsymbol{a} \models^{\lambda} \psi$;

d) if φ is $\& \Phi$ then $\overline{\mathfrak{A}}, \boldsymbol{a} \models^{\lambda} \varphi$ if $\overline{\mathfrak{A}}, \boldsymbol{a} \models_{\psi} \psi$ for all $\psi \in \Phi$;

e) if φ is $\exists \bar{v} \psi$ then $\overline{\mathfrak{A}}, \boldsymbol{a} \models^{\lambda} \varphi$ if there is a bounded assignment $\boldsymbol{b}$ to $\bar{v}$ such that $\overline{\mathfrak{A}}, \boldsymbol{a} \cup \boldsymbol{b} \models^{\lambda} \psi$.

St(L) will denote the set of sentences of the language L.

We assume, without loss of generality, that in our formulas, sentences, sets of sentences, or sequences of sets of sentences 1) no variable occurs both free and bound and 2) no variable occurs in more than one set of variables immediatly after a quantifier and 3) if a variable occurs in a set $\bar{v}$ of variables immediately after a quantifier then it occurs free in the immediate subformula following the set $\bar{v}$.

We assume also that $L_{k,k}$ has no functions. We will denote with C' the set of individual constants of $K_{k,k}$ and assume that $|C'| \leqslant k_0$. These assumptions will hold throughout this paper.

Let C_n, $n \in \lambda$ be a set of individual constants such that $|C_n| = k_{n+1}$ and for all m, n in λ if $m \neq n$ then $C_m \cap C_n = \emptyset$, and $C_m \cap C' = \emptyset$.

For all $n \in \lambda$ let L_n be the language obtained from $L_{k,k}$ by adding $\bigcup \{C_i \colon i \in n\}$ as individual constants.

Definition. A λ-sequence $S = \langle s_n \colon n \in \lambda \rangle$ of sets of sentences is called a good λ-sequence of sets of sentences if

a) $|\bigcup \{s_n \colon n \in \lambda\}| \leqslant k$, and

b) for all $n > 0$, n not a limit ordinal, all the sentences in s_n are of the form $-F(\bar{v}_F/f)$ where f is a $1-1$ function, $f \colon \bar{v}_F \to C_{n-1}$, and the sentence $-\forall \bar{v}_F F \in \bigcup \{s_j \colon j < n\}$, while $s_n = \emptyset$ for all limit ordinal $n < \lambda$, and

c) there is an ordinal number n_S such that for all $n > n_S$ we have that $|s_n| \leqslant |s_{n_S}| \leqslant k_{n_S}$ and $s_0 \subset \mathrm{ST}(L_{n_S})$, n_S is called the index of S, and

d) for all $n > 0$, $s_n \subset \mathrm{ST}(L_n)$.

REMARKS.

1) If all the sentences occurring is a good λ-sequence $S = \langle s_n: n \in \lambda \rangle$ are in $\mathrm{ST}(L_0)$ then $s_n = \emptyset$ for $n > 0$.

2) If φ is a sentence, for any pair $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ of bounded assignments $\overline{\mathfrak{A}}, \boldsymbol{a}_1 \models^{\lambda} \varphi$ iff $\overline{\mathfrak{A}}, \boldsymbol{a}_2 \models^{\lambda} \varphi$. Hence if φ is a sentence we will say that the λ-split structure $\overline{\mathfrak{A}}$ λ-satisfies φ, $\overline{\mathfrak{A}} \models^{\lambda} \varphi$, if there is a (for all) bounded assignment $\boldsymbol{a}$ such that $\overline{\mathfrak{A}}, \boldsymbol{a} \models^{\lambda} \varphi$.

3) If a sentence φ is satisfiable then it is also λ-satisfiable (a λ-split structure whose components are all equal to the universe satisfies the same sentences as the related structure).

4) If the cardinalities of the sets of variables following a quantifier in a sentence φ are smaller than λ, then if φ is λ-satisfiable then it is also satisfiable. Indeed any assignment to the variables following a quantifier is a bounded assignment.

DEFINITIONS. We say that a λ-split structure $\overline{\mathfrak{A}}$ λ-satisfies a good λ-sequence $S = \langle s_n: n \in \lambda \rangle$ of sets of sentence, $\overline{\mathfrak{A}} \models^{\lambda} S$, if for all $p \in \lambda$, $\overline{\mathfrak{A}} \models^{\lambda} \bigcup \{s_n: n < p\}$ (i.e. $\overline{\mathfrak{A}} \models^{\lambda} \varphi$ for all $\varphi \in \bigcup \{s_n: n < p\}$; of course this means that $\bigcup \{C_n: n < \max(p, n_S)\} \cup C'$ is evaluated within a component of the universe of $\overline{\mathfrak{A}}$ depending only on p).

A good λ-sequence S is said to be λ-satisfiable if there is a λ-split structure $\overline{\mathfrak{A}}$ such that $\overline{\mathfrak{A}} \models^{\lambda} S$.

If S_1 and S_2 are good λ-sequences, say $S_1 = \langle s_{1,n}: n \in \lambda \rangle$ and $S_2 = \langle s_{2,n}: n \in \lambda \rangle$, then by $S_1 \subset S_2$ we will mean that $s_{1,n} \subset s_{2,n}$ for all $n \in \lambda$.

If $S = \langle s_n: n \in \lambda \rangle$ is a good λ-sequence, then we let $\boldsymbol{S} = \bigcup \{s_n: n \in \lambda\}$ and $\boldsymbol{S}^m = \bigcup \{s_n: n \leqslant m\}$ for any $m \in \lambda$.

2. λ-satisfiability.

Up to now we consider the notion of λ-satisfiability for sentences in a language $L_{k,k}$ where the cofinality of k is λ, but we may as well consider the notion of λ'-satisfiability for $L_{k,k}$ where the cofinality of k is $\lambda \neq \lambda'$. Of course we will say that an $L_{k,k}$ formula φ $(\mathrm{cf}(k) = \lambda)$ is λ'-satisfiable if there is a λ'-split structure $\overline{\mathfrak{A}}'$ and an assignment $\boldsymbol{a}'$ bounded in $\overline{\mathfrak{A}}'$ to the free variables occurring in φ such that $\overline{\mathfrak{A}}', \boldsymbol{a}' \models^{\lambda'} \varphi$ with the obvious meaning of the notation.

We can compare the λ'-satisfiability and the λ''-satisfiability of an $L_{k,k}$ formula $(\mathrm{cf}(k) = \lambda)$. The following results will be usefull.

THEOREM 1. Assume that $\mathrm{cf}(\lambda'') = \lambda'$. Any $L_{k,k}$ sentence φ is λ'-satisfiable iff it is λ''-satisfiable.

PROOF. Let $\overline{\mathfrak{A}}''$ be a λ''-split structure that λ''-satisfies φ. Let μ_j, $j \in \lambda'$, be an increasing sequence of ordinals such that $\bigcup\{\mu_j : j \in \lambda'\} = \lambda''$, such a sequence exists since $\mathrm{cf}(\lambda'') = \lambda'$.

From the λ''-split structure $\overline{\mathfrak{A}}''$ obtain a λ'-split structure $\overline{\mathfrak{A}}'$ in the following way: it has the same universe, relation and constants as $\overline{\mathfrak{A}}''$ while the components of the universe will be the following sets for $j \in \lambda'$:

$$A'_j = \bigcup\{A''_n : n \in \mu_j \text{ and } A''_n \text{ is a component of the universe of } \overline{\mathfrak{A}}''\}.$$

At this point it can be proved that φ is λ'-satisfied in $\overline{\mathfrak{A}}'$.

On the other hand let $\overline{\mathfrak{A}}'$ be a λ'-split structure that λ'-satisfies φ. Again let μ_j, $j \in \lambda'$ be an increasing sequence of ordinals such that $\bigcup\{\mu_j : j \in \lambda'\} = \lambda''$.

From the λ'-split structure $\overline{\mathfrak{A}}'$ obtain a λ''-split structure $\overline{\mathfrak{A}}''$ in the following way: it has the same universe, relations and constants as $\overline{\mathfrak{A}}'$ while the components of the universe will be the following sets: for all $n \in \lambda''$ let j_n be the least ordinal $j < \lambda'$ such that $\mu_j \geqslant n$ and take $A_n = A_{\mu_{j_n}}$.

At this point it can be proved that φ is λ''-satisfied in $\overline{\mathfrak{A}}''$. □

COROLLARY. If $\mathrm{cf}(\lambda') = \mathrm{cf}(\lambda'')$ and φ is an $L_{k,k}$ sentence then φ is λ'-satisfiable iff it is λ''-satisfiable.

THEOREM 2. If λ' is a regular cardinal and $\lambda' > k$ then an $L_{k,k}$ formula is λ'-satisfiable iff it is satisfiable.

PROOF. See Remark 4) in the previous section.

FACTS. Let $\lambda' < \lambda \leqslant k$, with λ' and λ'' regular cardinals. The $L_{k,k}$ sentence

$$[\forall\{v_0\ \exists v_1 P(v_0, v_1)\}]\ \&\ [\forall\{v_i : i < \lambda'\} - \&\{P(v_i, v_{i+1}) : i < \lambda'\}]$$

is λ'-satisfiable but not λ''-satisfiable. On the other hand, the $L_{k,k}$ sentence

$$[\forall\{v_i: i<\lambda''\}\ \forall\{v_j: j<\lambda''\} - \&\{v_i \neq v_j: i<j<\lambda''\}]\ \&$$
$$\&\left[\&\{\exists\{v_i: i<\delta\}\ \exists\{v_j: j<\delta\}\ \&\{v_i\neq v_j: i<j<\delta\}: \delta<\lambda''\}\right]$$

is λ''-satisfiable but not λ'-satisfiable. □

We can conclude that λ'-satisfiability and λ''-satisfiability are:

1) the same notion if cf $(\lambda') =$ cf (λ''),
2) two different notions neither one implying the other if cf $(\lambda') \neq$ cf (λ'') and at least one of the two cofinalities is smaller than k,
3) coincident with the notion of satisfiability if the cofinalities of λ' and of λ'' are greater than k.

This conclusion holds also if $\lambda' = \omega$ and $\lambda'' = \lambda$, and it permits to compare the notions of ω-satisfiability of Karp and the present notion of λ-satisfiability.

3. λ-seq-consistency property.

Now let us define the notion of λ-seq-consistency property for $L_{k,k}$.

Σ is a λ-seq-consistency property for $L_{k,k}$ with respect to $\{C_n: n\in\lambda\}$ if Σ is a set of good λ-sequences $S = \langle s_n: n\in\lambda\rangle$ of sets s_n of sentences such that all of the following conditions hold.

C0) If Z is an atomic sentence then either $Z \notin S$ or $-Z\notin S$, and if Z is of the form $-(c=c)$, c a constant, then $Z\notin S$.

C1) Suppose that $|I|<k$,

a) if $\{c_i = d_i: i\in I\}\subset s_0$ and $\langle s_n: n\in\lambda\rangle = S\in\Sigma$ with c_i and d_i constants, then the good λ-sequence $S' = \langle s'_n: n\in\lambda\rangle$ such that $s'_0 = s_0 \cup \{d_i = c_i: i\in I\}$ and $s'_n = s_n$ for $n>0$, $n\in\lambda$, belongs to Σ;

b) if $\{Z_i(c_i), c_i = d_i: i\in I\}\subset s_0$ and $\langle s_n: n\in\lambda\rangle = S\in\Sigma$ and the Z_i are atomic sentences and the c_i and the d_i are constants, then the good λ-sequence $S' = \langle s'_n: n\in\lambda\rangle$ such that $s'_0 =$

$= s_0 \cup \{Z_i(d_i): i \in I\}$ and $s'_n = s_n$ for $n > 0$, $n \in \lambda$, belongs to Σ. Here $Z_i(d_i)$ indicates the sentence obtained replacing d_i for some of the occurrences of c_i in $Z_i(c_i)$.

C2) If $\{- - F_i: i \in I\} \subset S^m$ for some $m \in \lambda$ and $\langle s_n: n \in \lambda\rangle = S \in \Sigma$ and $|I| < k$, then the good λ-sequence $S' = \langle s'_n: n \in \lambda\rangle$ such that $s'_0 = s_0 \bigcup \{F_i: i \in I\}$, $s'_n = s_n$ for $n > 0$, $n \in \lambda$, belongs to Σ.

C3) If $\{\& \boldsymbol{F}_i: i \in I\} \subset s_0$ and $|I| < k$ and there is $m' \in \lambda$ such that for all $i \in I$, $0 < |\boldsymbol{F}_i| < k_{m'}$ and $\langle s_n: n \in \lambda\rangle = S \in \Sigma$ then the good λ-sequence $S' = \langle s'_n: n \in \lambda\rangle$ such that $s'_0 = s_0 \cup \{F: F \in \boldsymbol{F}_i, i \in I\}$, $s'_n = s_n$ for $n > 0$, $n \in \lambda$, belongs to Σ.

C4) If $\{\forall \bar{v}_i F_i: i \in I\} \subset s_0$ and $|I| < k$ and there is $m' \in \lambda$ such that for all $i \in I$, $0 < |\bar{v}_i| < k_{m'}$ and $\langle s_n: n \in \lambda\rangle = S \in \Sigma$, and n_S is the index of S, then the good λ-sequence $S' = \langle s'_n: n \in \lambda\rangle$ such that

$$s'_0 = s_0 \cup \{F_i(\bar{v}_i/f): f \in {}^{\cup\{\bar{v}_i : i \in I\}}(\bigcup\{C_h: h < n_S\} \cup C'),\ i \in I\},$$

$s'_n = s_n$ for $n > 0$, $n \in \lambda$, belongs to Σ.

C5) If $\{- \& \boldsymbol{F}_i: i \in I\} \subset S^m$ for some $m \in \lambda$ and $|I| < K$ and there is $m' \in \lambda$ such that for all $i \in I$, $0 < |\boldsymbol{F}_i| < k_{m'}$ and $\langle s_n: n \in \lambda\rangle = S \in \Sigma$, then there is $g \in \times\{\boldsymbol{F}_i: i \in I\}$ such that the good λ-sequence $S' = \langle s'_n: n \in \lambda\rangle$ where $s'_0 = s_0 \cup \{- g(i): i \in I\}$, $s'_n = s_n$ for $n > 0$, $n \in \lambda$, belongs to Σ.

C6) If $\{- \forall \bar{v}_i F_i: i \in I\} \subset S^m$ for some $m \in \lambda$ and there is m' the least natural number such that $|i| < k_{m'}$ and, for all $i \in I$, $0 < |\bar{v}_i| < k_{m'}$ and $m \leqslant m'$ and $S^m \subset \mathrm{ST}(L_{m'})$ and $\langle s_n: n \in \lambda) = S \in \Sigma$, then there is a λ-partition $P = \langle I_p: p \in \lambda\rangle$ of I and there is a set $\{f_p: p \in \lambda\}$ of $1-1$ functions $f_p \in {}^{\cup\{\bar{v}_i : i \in I_p\}}(C_{m'+p} - \{c: c$ is a constant occurring in $S\})$ such that the good λ-sequence $S' = \langle s'_n: n \in \lambda\rangle$ where for $n \leqslant m'$, $s'_n = s_n$ and, for all $p \in \lambda$, $s'_{m'+p+1} = s_{m'+p+1} \cup \{- F_i(\bar{v}_i/f_p): i \in I_p\}$, belongs to Σ.

C7) If m is a limit ordinal and $m \in \lambda$ and

$$\langle S_i: i \in m\rangle = \langle\langle s_{i,n}: n \in \lambda\rangle: i \in m\rangle$$

is a sequence of members of Σ such that, for each $i \in m$, S_{i+1} is obtained from S_i through a finite number of applications of clauses

C1)-C6) and $s_{i,n} = \bigcup\{s_{j,n}: j \in i\}$ for each limit ordinal $i \in m$ (and hence if $i' < i < m$ then $S_{i'} \subset S_i$), then $S_m = \langle s_{m,n}: n \in \lambda\rangle$, where $s_{m,n} = \bigcup\{s_{i,n}: i < m\}$, belongs to Σ.

4. Model existence theorem.

THEOREM. Model existence. If $S = \langle s_n: n \in \lambda\rangle$ is a good λ-sequence of sets of sentences of $L_{k,k}$ and $S \in \Sigma$, Σ a λ-seq-consistency property with respect to $\{C_i: i \in \lambda\}$, and $|S| = k_0 < k$, then S is λ-satisfiable in a λ-split structure $\overline{\mathfrak{A}}$. Moreover the n-th component of $\overline{\mathfrak{A}}$ has cardinality at most k_n.

The following proof is somehow similar to the proof of the analogous theorem for ω-consistency properties and seq-consistency properties given in [3] and in [5] and again it is an Hintikka type argument.

PROOF. By a good split of a set s of at most k sentences we shall mean a partition $\langle s_p: p \in \lambda\rangle$ of s such that $|s_p| \leqslant k_p$, every sentence of the form either $\& \boldsymbol{F}$ or $- \& \boldsymbol{F}$ in s_p has $|\boldsymbol{F}| \leqslant k_p$, every sentence of the form either $\forall \bar{v}_F F$ or $- \forall \bar{v}_F F$ in S_p has $|\bar{v}_F| \leqslant k_p$.

It is easy to see that any set of sentences has a good split.

Let us define by induction on $m \in \lambda$, good λ-sequences $S_m = \langle s_{m,n}: n \in \lambda\rangle \in \Sigma$ such that all the sentences occurring in $\boldsymbol{S}_m^m$ are in L_m, $|\boldsymbol{S}_m^m| \leqslant k_m$; good splits $\langle \boldsymbol{S}_{m,p}^m: p \in \lambda\rangle$ of each $\boldsymbol{S}_m^m$ such that $\boldsymbol{S}_{m+1,p}^{m+1} = \boldsymbol{S}_{m,p}^m$ for $p \leqslant m$ and $\boldsymbol{S}_{m,p}^m \subset \boldsymbol{S}_{m+1,p}^{m+1}$ for $p \geqslant m$ and, if m is a limit ordinal $\boldsymbol{S}_{m,p}^m = \bigcup\{\boldsymbol{S}_{m',p}^{m'}: m' < m\}$ for all p; and for all $q \in \lambda$, if m is a successor ordinal, sets S_m^q of existential sentences in $\boldsymbol{S}_{m-1,m-1}^{m-1}$ and $1-1$ functions $f_{m,q}$ from $\bigcup\{\bar{v}_F: - \forall \bar{v}_F F \in S_m^q\}$ into C_{m+q+1} such that for all i and j, $i \leqslant m$, $j \leqslant m$, if $i \neq j$ and $i + q = j + q'$ then range $(f_{i,q}) \cap$ range $(f_{j,q'}) = \emptyset$, while if m is a limit ordinal $S_m^q = \emptyset$ and $f_{m,q} = \emptyset$.

Let $S_0 = S$; let $\langle \boldsymbol{S}_{0,p}^0: p \in \lambda\rangle$ be any good split of $\boldsymbol{S}_0^0$; and for all $q \in \lambda$ let $S_0^q = \emptyset$ and $f_{0,q} = \emptyset$.

Suppose that S_h, $\langle S_{h,p}^h: p \in \lambda\rangle$, S_h^q, $f_{h,q}$ have been defined for all $q \in \lambda$ and for all $h \leqslant m < \lambda$ with the above mentioned properties.

Let $S_m'' = \bigcup\{\boldsymbol{S}_{m,i}^m: i \in \lambda\}$; and let

$$S_m' = S_{m,m}^m \bigcup \{\forall \bar{v}_F F: \forall \bar{v}_F F \in S_m''\} \cup \{c = d: c = d \in S_m''\} \cup \cup \{Z: Z \text{ is an atomic sentence in } S_m''\}.$$

Clearly $S_m' \subset \bigcup\{s_{m,i}: i \leqslant m\}$, $|S_m'| \leqslant k_m$, all conjunction and quantification sets in S_m' have cardinality at most k_m.

Let $S_m^{(1)} = \langle s_{m,n}^{(1)} \colon n \in \lambda \rangle$ where $s_{m,n}^{(1)} = s_{m,n}$ if $n > 0$ and

$$s_{m,0}^{(1)} = s_{m,0} \bigcup \{d = c \colon c = d \in S'_m\} \cup$$
$$\cup \{Z(d) \colon Z(c) \text{ is an atomic sentence in } S'_m \text{ and } c = d \in S'_m\} .$$

Let $S_m^{(2)} = \langle s_{m,n}^{(2)} \colon n \in \lambda \rangle$ where $s_{m,n}^{(2)} = s_{m,n}^{(1)}$ if $n > 0$ and

$$s_{m,0}^{(2)} = s_{m,0}^{(1)} \cup \{F \colon - - F \in S'_m\} .$$

Let $S_m^{(3)} = \langle s_{m.n}^{(3)} \colon n \in \lambda \rangle$ where $s_{m,n}^{(3)} = s_{m,n}^{(2)}$ if $n > 0$ and

$$s_{m,0}^{(3)} = s_{m,0}^{(2)} \cup \{F \colon F \in \boldsymbol{F}, \ \& \boldsymbol{F} \in S'_m\} .$$

Let $S_m^{(4)} = \langle s_{m,n}^{(4)} \colon n \in \lambda \rangle$ where $s_{m,n}^{(4)} = s_{m,n}^{(3)}$ if $n > 0$ and

$$s_{m,0}^{(4)} = s_{m,0}^{(3)} \cup \{- g(\boldsymbol{F}) \colon - \& \boldsymbol{F} \in S'_m\}$$

where g is a function, $g \in \times \{\boldsymbol{F} \colon - \& \boldsymbol{F} \in S'_m\}$, such that if $S_m^{(3)} \in \Sigma$ then also $S_m^{(4)} \in \Sigma$ (such a function exists due to C5)).

Let $\langle S_{m+1}^q \colon q \in \lambda \rangle$ be a λ-partition of $\{- \forall \bar{v}_F F \colon - \forall \bar{v}_F F \in S'_m\}$ and let $\{f_{m+1,q}^* \colon q \in \lambda\}$ be a set of $1-1$ functions $f_{m+1,q}^*$ from

$$\bigcup \{\bar{v} \colon - \forall \bar{v}_F F \in S_{m+1}^q\} \text{ to } (C_{m+q} - \{c \colon c \text{ is a constant occurring in } S_m\})$$

such that if $S_m^{(4)} \in \Sigma$ then the good λ-sequence $S_m^* = \langle s_{m,n}^* \colon n \in \lambda \rangle \in \Sigma$ where $s_{m,n}^* = s_{m,n}^{(4)}$ for all $n \leqslant m$, while for all q we have that $s_{m,m+q+1}^*$ is equal to $s_{m,m+q+1}^{(4)} \cup \{- F(\bar{v}_F / f_{m+1,q}^*) \colon - \forall \bar{v}_F F \in s_{m+1}^q\}$ (such a partition and functions exist due to C6)).

Let S_{m+1}^q be the set just mentioned and $f_{m+1,q}$ a choice of functions as above.

Let $S_m^{(5)} = \langle s_{m,n}^{(5)} \colon n \in \lambda \rangle$ where $s_{m,n}^{(5)} = s_{m,n}^{(4)}$ for $n \leqslant m$ while for all $q \in \lambda$

$$s_{m,n+q+1}^{(5)} = s_{m,m+q+1}^{(4)} \bigcup \{- F(\bar{v}_F / f_{m+1,q}) \colon - \forall \bar{v}_F F \in s_{m+1}^q\} .$$

Let $S_m^{(6)} = \langle s_{m,n}^{(6)} \colon n \in \lambda \rangle$ where $s_{m,n}^{(6)} = s_{m,n}^{(5)}$ if $n > 0$ while $s_{m,0}^{(6)}$ is equal to

$$s_{m,0}^{(5)} \cup \{F(\bar{v}_F / f) \colon f \in {}^{\cup \{\bar{v}_F \colon \forall \bar{v}_F F \in S'_m\}} \bigcup \{C_j \colon j \leqslant m\}, \ \forall \bar{v}_F F \in S'_m\} .$$

Define $S_{m+1} = S_m^{(6)}$, $\langle S^{m+1}_{m+1,n}\colon n \in \lambda\rangle$ any good split of S^{m+1}_{m+1} such that $S^{m+1}_{m+1,p} = S^m_{m,p}$ if $p \leqslant m$ while $S^m_{m,p} \subset S^{m+1}_{m+1,p}$ if $p > m$.

To procede with the inductive definition in the limit case, now suppose that m is a limit ordinal, $m < \lambda$, and suppose that S_h, $\langle S^h_{h,p}\colon p \in \lambda\rangle$, S^q_h, $f_{h,q}$ have been defined for all $q \in \lambda$ and for all $h < m$.

Define $S_m = \langle s_{m,n}\colon n \in \lambda\rangle$ where $s_{m,n} = \bigcup\{s_{h,n}\colon h < m\}$; $S^m_{m,p} = \bigcup\{S^h_{h,p}\colon h < m\}$; $S^q_m = \emptyset$; $f_{m,q} = \emptyset$.

Remark that $|S^m_{m,p}| \leqslant k_m$.

To complete the definition by induction we have only to remark that all the conditions on S_m, $\langle S^m_{m,p}\colon p \in \lambda\rangle$, S^q_m, $f_{m,q}$ are preserved.

Indeed $S_{m+1} \in \Sigma$ if S_m does too thanks to conditions C1)-C6), while if m is a limit ordinal $S_m \in \Sigma$ thanks to condition C7) since the hypothesis of this condition are met. Also the other conditions are met due to the type of construction that we used.

Remark that $S_m \subset S_{m+1}$ for all $m \in \lambda$ and hence $\mathbf{S}_m \subset \mathbf{S}_{m+1}$.

Now let $s_\lambda = \bigcup\{\mathbf{S}_m\colon m \in \lambda\}$.

This set s_λ has the same properties as the analogous s_ω in [3], hence it is an Hintikka set, and it can be used to build a λ-split structure $\overline{\mathfrak{A}}$ whose components have the prescribed cardinalities and which λ-satisfies the good λ-sequences in the λ-seq-consistency property that we have used, in particular S, i.e. $\overline{\mathfrak{A}} \models^\lambda S$. □

5. The inverse of the model existence theorem.

THEOREM. Let $S = \langle s_n\colon n \in \lambda\rangle$ be a good λ-sequence of sets of sentences of $L_{k,k}$ that is λ-satisfiable. Let $|S| \leqslant k$. Let C_j, $j \in \lambda$, be sets such that $|C_j| = k_j$ and for all i and j belonging to λ if $i \neq j$ then $C_i \cup C_j = \emptyset$. Under these assumptions there is a k-seq-consistency property Σ with respect to $\{C_i\colon i \in \lambda\}$ such that $S \in \Sigma$.

PROOF. Partition each C_i in exactly $i+1$ parts, $\langle C_{i,j}\colon j \leqslant i\rangle$, such that $|C_{i,j}| = k_j$. Let L'_m be the language obtained from $L_{k,k}$ by adding $\bigcup\{C_{i,m'}\colon m' < m, i \in \lambda\}$ as a set of constants.

Remark that $L'_0 = L_{k,k}$; $L'_m = L_m \cup (\bigcup\{C_{i,j}\colon i \geqslant m \text{ and } j < m\})$; $L'_{m+1} = L'_m \cup (\bigcup\{C_{i,m}\colon i \geqslant m\})$; and $L'_m = \bigcup\{L'_{m'}\colon m' < m\}$ if m is a limit ordinal smaller than λ.

For $n \in \lambda$ let $\langle s^p_n\colon p \in \lambda\rangle$ be a λ-partition of s_n such that $|s^p_n| \leqslant k_p$ and, for $n > 0$, if $-F(\bar{v}_F/f) \in s^p_n$ then $-\forall\bar{v}_F F \in \bigcup\{s^p_{n'}\colon n' < n\}$.

Let $\overline{\mathfrak{A}}$ be a λ-split structure that λ-satisfies S.

Let us define by induction on $m \in \lambda$:

a) good λ-sequences $S_m = \langle s_{m,n}\colon n \in \lambda\rangle$ in L'_m such that if $m' < m$ then $S_{m'} \subset S_m$ and $|\boldsymbol{S}_m - \boldsymbol{S}_0| \leqslant k_m$;

b) λ-partitions $\langle s^p_{m,n}\colon p \in \lambda\rangle$ of $s_{m,n}$ such that $|s^p_{m,n}| \leqslant k_p$ and, for all $m' < m < \lambda$, $s^p_{m',n} \subset s^p_{m,n}$ and if $-F(\bar{v}_F/f) \in s^p_{m,n}$ then $-\forall \bar{v}_F F \in \bigcup \{s^{p'}_{m,n'}\colon n' < n,\ p' < p\}$;

c) $1-1$ functions $f_{m,q}$ with $q \in \lambda$;

d) λ-split structures $\overline{\mathfrak{A}}_m$ which are expansions $\overline{\mathfrak{A}}$of $_{m'}$, for each $m' < m$, to the language L'_m such that $\overline{\mathfrak{A}}_m \models^\lambda S_m$;

in the following way.

$$S_0 = S\,, \qquad \overline{\mathfrak{A}}_0 = \overline{\mathfrak{A}}\,, \qquad f_{0,q} = \emptyset\,, \qquad s^p_{0,n} = s^p_n\,.$$

Suppose that S_h, $\overline{\mathfrak{A}}_h$, $f_{h,q}$, $s^p_{h,n}$ have already been defined for all $h < m < \lambda$.

If m is a successor ordinal, $m = m'' + 1$, do as follows.

Let

$$s''_{m''} = \bigcup \{s^p_{m'',0}\colon p \leqslant m''\}\,, \qquad S''_{m''} = \bigcup \{s^p_{m'',n}\colon n \leqslant m'',\ p \leqslant m''\}\,.$$

Let

$$\begin{aligned} s'_{m''} = \{c = d\colon c = d \in s''_{m''}\} \cup \{Z\colon Z \text{ is an atomic sentence and } Z \in s''_{m''}\} \cup \\ \bigcup\{--F\colon --F \in (S''_{m''} - \bigcup\{S''_{m'}\colon m' < m''\})\} \cup \\ \cup \{\& \boldsymbol{F}\colon \text{ either } \& \boldsymbol{F} \in (s''_{m''} - \bigcup\{s''_{m'}\colon m' < m''\}) \text{ and } |\boldsymbol{F}| \leqslant k_{m''} \\ \text{or } \& \boldsymbol{F} \in \bigcup\{s''_{m'}\colon m' < m''\} \text{ and } |\boldsymbol{F}| = k_{m''}\} \cup \\ \cup \{\forall \bar{v}_F F\colon \forall \bar{v}_F F \in s''_m \text{ and } |\bar{v}_F| \leqslant k_m\} \cup \\ \cup \{-\& \boldsymbol{F}\colon \text{ either } -\& \boldsymbol{F} \in (S''_{m''} - \bigcup\{S''_{m'}\colon m' < m''\}) \\ \text{and } |\boldsymbol{F}| \leqslant k_{m''} \text{ or } -\& \boldsymbol{F} \in \bigcup\{S''_{m'}\colon m' < m''\} \text{ and } |\boldsymbol{F}| = k_{m''}\}\,. \end{aligned}$$

Let

$s'_{m'',q} = \{-\forall \bar{v}_F F\colon q$ is the least ordinal smaller than λ such that there is a bounded assignment $\boldsymbol{a}_F$ within the $(m''+q)$-th component of $\overline{\mathfrak{A}}_{m''}$ such that $\overline{\mathfrak{A}}_{m''}, \boldsymbol{a}_F \models^{\lambda} F$ and either $-\forall \bar{v}_F F \in (S''_{m''} - \bigcup\{S''_{m'}: m' < m''\})$ and $|\bar{v}_F| \leqslant k_{m''}$ or $-\forall \bar{v}_F F \in \bigcup\{S''_{m'}: m' < m''\}$ and $|\bar{v}_F| = k_{m''}\}$.

Define $f_{m''+1,q}$ as a $1-1$ function from $\bigcup\{\bar{v}_F: -\forall \bar{v}_F F \in s'_{m'',q}\}$ in $C_{m''+q,m''}$. Such functions exist since $|\mathrm{dom}\,(f_{m''+1,q})| \leqslant k_{m''}$.

Let $\boldsymbol{a}_{m'',q} = \bigcup\{\boldsymbol{a}_F: -\forall \bar{v}_F F \in s'_{m'',q}\}$.

Define $\overline{\mathfrak{A}}_{m''+1}$ as the expansion of $\overline{\mathfrak{A}}_{m''}$ to the language $L'_{m''+1}$ obtained by interpreting each constant c belonging to $\bigcup\{f_{m''+1,q}(\bar{v}_F): -\forall_F F \in s'_{m'',q}\}$ in $\boldsymbol{a}_{m'',q}(f^{-1}_{m''+1,q}(c))$, for all $q \in \lambda$, and, again for all $q \in \lambda$, each constant of $C_{m''+q,m''} - \bigcup\{f_{m''+1,q}(\bar{v}_F): -\forall \bar{v}_F F \in s'_{m'',q}\}$, in a fixed element of the first component of $\overline{\mathfrak{A}}$.

Let $g_{m''} \in \times\{\boldsymbol{F}: -\&\,\boldsymbol{F} \in s'_{m''}\}$ such that $-g(\boldsymbol{F})$ is λ-satisfied in $\overline{\mathfrak{A}}_{m''}$.

Define $S_{m''+1} = \langle s_{m''+1,n}: n \in \lambda\rangle$ as the good λ-sequence defined as follows.

If $0 < n \leqslant m''$ then $s_{m''+1,n} = s_{m'',n}$, otherwise

$$s_{m''+1,m''+1+q} = s_{m'',m''+1+q} \cup \left(\bigcup\{-F(\bar{v}_F/f_{m''+1,q}): -\forall \bar{v}_F \in s'_{m''}\}\right)$$

and

$$s_{m''+1,0} = s_{m'',0} \cup \{d = c: c = d \in s'_{m''}\} \cup \{Z(d): Z(c) \in s'_{m''} \text{ and } c = d \in s_{m''} \text{ and } Z \text{ is an atomic sentence}\} \cup \{F: --F \in s'_{m''}\} \cup \cup \{F: F \in \boldsymbol{F} \text{ and } \&\,\boldsymbol{F} \in s'_{m''}\} \cup \{F(\bar{v}_F/f_F): \forall \bar{v}_F F \in s'_{m''} \text{ and } f_F \text{ is a function from } \bar{v}_F \text{ to } (\bigcup\{C_i: i \leqslant m''\} \cup C')\} \cup \{-g_{m''}(\boldsymbol{F}): -\&\,\boldsymbol{F} \in s'_{m''}\}.$$

If $p \leqslant m''$ then define $s_{m''+1,n} = s^p_{m'',n}$.

Let $\langle I_{m'',n,r+1}: r \in \lambda\rangle$ be a λ-partition of $s_{m''+1,n} - s_{m'',n}$ such that $|I_{m'',n,r+1}| \leqslant k_{m''+r+1}$.

If $p > m''$ then define $s^p_{m''+1,n} = s^p_{m'',n} \cup I_{m'',n,p-m''}$.

If m is a limit ordinal, do as follows.

Define $S_m = \langle s_{m,n}: n \in \lambda\rangle$ as $\langle \bigcup\{s_{h,n}: h < m\}: n \in \lambda\rangle$.

Define $f_{m,q} = \emptyset$.

Define $\overline{\mathfrak{A}}_m$ as the expansion to the language L'_m of all the λ-split structures $\overline{\mathfrak{A}}_h$ with $h < m$.

Define $s^p_{m,n} = s^p_{m',n}$ for all $p < m$ and for an m' such that $p < m' < m$.

Define $s^p_{m,n} = \bigcup\{s^p_{m',n}: m' < m\}$ for all $p \geqslant m$.

So we have completed the definition by induction on m once we have remarked that the defined entities have the properties that they should have.

Indeed for $f_{m,\alpha}$ it is obvious.

For the good λ-sequences remark that: 1) they are in L'_m, 2) $S_{m'} \subset S_m$ for all $m' < m$ and 3) $|S_m - S_0| \leqslant k_m$ because of property b).

For the λ-partitions $s^{v}_{m,n}$ of $s_{m,n}$ remark that the cardinality and inclusion conditions are satisfied and that the condition on existential sentences is true due to the construction.

For the $\overline{\mathfrak{A}}_m$ remark that the unique not obvious point is the last one which claims that $\bar{\mathfrak{a}}_m \models^{\lambda} S_m$: when m is a successor ordinal it is true by construction, while, when m is a limit ordinal, it is still true because then the $S'_i s$, with $i < m$, are all true in $\overline{\mathfrak{A}}_m$, and

$$S_m = \langle \bigcup \{s_{i,n}: i < m\}: n \in \lambda \rangle .$$

Here it should be kept in mind that to λ-satisfy a λ-sequence as S_m it means that, for all $n' \in \lambda$, $S^{n'}_m$ is λ-satisfied in a λ-split structure in which all the constants are interpreted within a fixed component. Indeed if n_s is the index of S_m and $n'' = \max(n_s, n')$ we see that the constants in $S^{n'}_m$ are in $L_{n''}$ and these are interpreted in $\overline{\mathfrak{A}}_m$ within the n''-th component due to the construction performed.

Let us call Γ the set of the S_m, $m \in \lambda$, that have been constructed.

Now let

$$\Sigma = \{S': S' \text{ is a good } \lambda\text{-sequence and } S' \subset S_m \text{ for some } S_m \subset \Gamma\} .$$

We shall now prove that this Σ is a λ-seq-consistency property with respect to $\{C_i: i \in \lambda\}$ to which S belongs.

First, since $S_0 = S$ it is obvious that $S \in \Sigma$.

To proceed, let us check the less obvious clause of a λ-seq-consistency property, and leave the others to the reader.

C7) suppose that m' is a limit ordinal, $m' < \lambda$, and that $S_i = \langle s_{i,n}: n \in \lambda \rangle$, $i < m'$, is a sequence of members of Σ such that, for each $i < m'$, S_{i+1} is obtained from S_i through a finite number of applications of clauses C1)-C6) and $s_{i,n} = \bigcup \{s_{j,n}: j < i\}$ for each limit ordinal $i < m'$. Suppose further that each S_i, $i < m'$, is contained in some $S_{m_i} \in \Gamma$. Let $m_{m'} = \bigcup \{m_i: i < m'\}$. Since $m' < \lambda$ and $m_i < \lambda$ for each $i < m'$, and λ is a regular cardinal, it follows that $m_{m'} < \lambda$.

$\{m_i\colon i < m'\}$ may have a maximum or not. In any case

$\langle \bigcup \{s_{m_i,n}\colon i < m'\}\colon n \in \lambda \rangle = S_{m_{m'}} \in \Gamma$ and $\langle \bigcup \{s_{i,n}\colon i < m'\}\colon n \in \lambda \rangle \subset S_{m_{m'}}$,

and hence it belongs to Σ. □

6. The downward Lowenheim Skolem theorem.

In what we have done so far, it is implicit an easy proof of the following version of the

DOWNWARD LOWENHEIM SKOLEM THEOREM FOR $L_{k,k}$. If s is a set of sentences of $L_{k,k}$, $|s| < k$, and s is λ-satisfiable, then there is a λ-split structure, whose n-th component has cardinality at most k_n, that λ-satisfies s.

PROOF. Let $S = \langle s_n\colon n \in \lambda \rangle$ be such that $s_0 = s$ and $s_n = \emptyset$ for $0 < n < \lambda$. Clearly S is λ-satisfiable. Hence there is Σ, a λ-seq-consistency property with respect to $\{C_i\colon i \in \lambda\}$, to which S belongs due to the inverse of the model existence theorem. Now, taking advantage of the model existence theorem, since $S \in \Sigma$, we can claim that S is λ-satisfied in a λ-split structure whose n-th component has cardinality at most k_n. □

7. The failure of the interpolation theorem.

The proof given by Malitz of the failure of Beth's definability theorem for the usual notion of satisfiability in languages $L_{\alpha,\beta}$, with $\alpha > \aleph_0$ and $\beta > \aleph_0$, works also for λ-satisfiability for $\lambda > \omega$.

Let L' be the language whose only extralogical symbols are the unary predicates P_A and P_B and the binary predicates $\langle_A$, $\langle_B$ and R. There is a sentence σ in $L'_{\omega_1,\omega}$, and hence in $L'_{k,k}$, such that it is true exactly in the class K of $L'_{k,k}$ structures $\mathfrak{C}$ such that $\mathfrak{C} = \langle \mathfrak{A} \oplus \mathfrak{B}, R^\tau \rangle$ where $\oplus$ denotes the full direct sum and

1) the only extralogical symbol in the languages of both $\mathfrak{A}$ and $\mathfrak{B}$ is the binary predicate $\langle$,
2) the interpretation of $\langle$ in $\mathfrak{A}$ is a well ordering of $\mathfrak{A}$ and the interpretation of $\langle$ in $\mathfrak{B}$ is a well ordering of $\mathfrak{B}$,

3) $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, and if f is the (unique) isomorphism mapping $\mathfrak{A}$ into $\mathfrak{B}$ then $R^{\mathfrak{C}} = \{\langle x, y\rangle : f(x) = y$ of $f(y) = x\}$.

Since the sets of variables following a quantifier in σ are denumerable, and hence smaller than λ, σ is satisfiable iff it is λ-satisfiable, and hence it defines implicitly R also with respect to the notion of λ-satisfiability.

On the other hand R is not explicitly definable also with respect to the notion of λ-satisfiability, as it can be seen with the following argument.

Let L be the reduction of the language L' obtained by delating R. Suppose that the formula φ in $L_{k,k}$ with two free variables defines explicitly R with respect to σ. Replace R by φ in σ and obtain a new sentence ϱ in $L_{k,k}$.

Clearly, there is a λ-split structure $\mathfrak{C}^*$ adequate for $L_{k,k}$ such that $\mathfrak{C}^* \models^{\lambda} \varrho$ iff there is an expansion of $\mathfrak{C}^*$ to $\mathfrak{C}$ adequate for $L_{k,k}$ such that $\mathfrak{C} \models^{\lambda} \sigma$.

If we choose $\mathfrak{B} = \langle 2^{2^{\mu}}, \varepsilon\rangle$, $\mathfrak{B}'$ a proper μ, μ-elementary substructure of $\mathfrak{B}$ (it exists [8]), $\mathfrak{A}$ isomorph to $\mathfrak{B}$ (all this notions with respect to satisfiability), we can take μ so large that $\langle \mathfrak{A} \oplus \mathfrak{B}, R^{\mathfrak{C}}\rangle = \mathfrak{C}$ and $\mathfrak{C} \models \sigma$, $\mathfrak{C}^* = \langle \mathfrak{A} \oplus \mathfrak{B}\rangle$ and $\mathfrak{C}^* \models \varrho$ and $\mathfrak{C}^* \equiv_{k,k} \mathfrak{A} \oplus \mathfrak{B}'$ (see [8]).

These structures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{C}$ and $\mathfrak{C}^*$ can be thought of as λ-split structures where each element of the λ-split of the universe is the universe itself. For such structures the notions of satisfiability and of λ-satisfiability coincide. Hence $\mathfrak{A} \oplus \mathfrak{B}' \models^{\lambda} \varrho$ and there should be an expansion $\mathfrak{C}' = \langle \mathfrak{A} \oplus \mathfrak{B}', R^{\mathfrak{C}'}\rangle$ of $\mathfrak{A} \oplus \mathfrak{B}'$ such that $\mathfrak{C}' \models^{\lambda} \sigma$ (and $\mathfrak{C}' \models \sigma$, which is the same). This implies the existence of an isomorphism f of $\mathfrak{A}$ onto $\mathfrak{B}'$ such that $R^{\mathfrak{C}'} = \{\langle x, y\rangle : f(x) = y$ or $f(y) = x\}$. But this is impossible since $\mathfrak{B}'$ is a proper substructure of $\mathfrak{B}$.

This contradiction proves that R cannot be explicitly defined.

Hence Beth's definability theorem fails for the languages under consideration, and therefore also Craig's interpolation theorem fails for the same languages.

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