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Rank-Two Torsion-Free Modules over Valuation Domains.

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1. Introduction.

Let R be a valuation domain and Q its field of quotients. We denote by $\mathcal{F}_0(R)$ the class of torsion-free R-modules of finite rank, and by $\mathcal{C}_0(R)$ the class of finitely generated torsion R-modules.

If R is a maximal valuation domain, there exists an evident symmetry between $\mathcal{F}_0(R)$ and $\mathcal{C}_0(R)$: in fact, every module in $\mathcal{F}_0(R)$ is a direct sum of rank-one modules (which are isomorphic to submodules of Q), and every module in $\mathcal{C}_0(R)$ is a direct sum of cyclic modules.

This symmetry is lost if R is an almost maximal valuation domain which fails to be maximal: in fact, there exist indecomposable modules in $\mathcal{F}_0(R)$ of rank larger than one (see [8], [13] and [6]), while every module in $\mathcal{C}_0(R)$ is still a direct sum of cyclic modules (see [7]). The symmetry is lost mainly because all torsion factors of R are complete, while R itself is not complete (in the topologies of non-zero ideals).

Recent investigations of the class $\mathcal{C}_0(R)$, for R a non almost maximal valuation domain, (see [9], [11], [12], [14], [15], [16]) have shown that many relevant facts of the theory developed for $\mathcal{F}_0(\mathbb{Z}_p)$, as the classification by Kurosh-Malcev-Derry, the Arnold's duality, the notion of quasi-isomorphism, have their analogue in $\mathcal{C}_0(R)$. This

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situation suggests that the evident symmetry existing between $\mathcal{F}_0(R)$ and $\mathcal{C}_0(R)$ in the case of R maximal should reappear in the case of R non almost maximal, at a more underhand level.

This is in fact the case. The symmetry appears soon in the investigation of the class $\mathcal{F}_2(R)$, consisting of indecomposable torsion-free R-modules of rank two, whose resemblance with the class $\mathcal{C}_2(R)$ of two-generated indecomposable torsion modules, investigated in [14] and [11], is transparent. This investigation is the goal of this paper, which is the first step in developing the theory of $\mathcal{F}_0(R)$ for general valuation domains symmetrically to the theory of $\mathcal{C}_0(R)$.

Torsion-free modules of rank two over an almost maximal valuation domain R were investigated by Viljoen [13], who gave a classification by means of a complete and independent set of invariants. His results were extended by Fuchs and Viljoen [6] to modules in $\mathcal{F}_0(R)$ with basic rank equal to one. It is noteworthy that their classification is a generalization, by means of vector spaces, of the classical classification by matrices given for modules in $\mathcal{F}_0(\mathbb{Z}_p)$ by Kurosch (see [3], [1]).

After the preliminary section 2, in which we collect notions and tools needed later, our approach starts in section 3 with the concrete construction of an indecomposable torsion-free R-module of rank two, where R is a non almost maximal valuation domain, using a unit of a maximal immediate extension S of R; this construction is similar to the classical one given in [8, Th. 19] for the category $\mathcal{F}_0(\mathbb{Z}_p)$.

Conversely, in section 4, given an indecomposable rank-two torsion-free R-module M, we associate to M a canonical basis and a unit u of S, which determine a triple of submodules of Q: (L, H, I), where L is isomorphic to a basic submodule B of M, H is isomorphic to M/B, and I is the breadth ideal of u.

We show how triples determined by different canonical bases and units of S are related. This allows us to obtain, in section 5, a complete classification of modules in $\mathcal{F}_2(R)$, which, in the particular case of R almost maximal, is equivalent to the Viljoen's classification.

In the last section 6 we compare the invariants used in [14] to classify modules in $\mathcal{C}_2(R)$ with the invariants obtained in section 5. We define, in a canonical way, a map $\Phi\colon [\mathcal{C}_2(R)] \to [\mathcal{F}_2(R)]$, where the brackets $[\mathcal{C}]$ denotes the set of isomorphy classes in the category $\mathcal{C}$, which turns out to be never injective, and whose image contains all the isomorphy classes of modules $M \in \mathcal{F}_2(R)$ such that M/B is not divisible, for B a basic submodule of M. There follows that Φ is surjective if and only if R is complete.

2. Preliminaries.

We denote by R an arbitrary valuation domain, i.e. a commutative integral domain with linearly ordered set of ideals. We denote by Q the field of quotients of R, by P the maximal ideal of R, by S a fixed maximal immediate extension of R. Recall that the ring structure of S is not unique up to isomorphism, unless R is almost maximal; the R-module structure of S is unique up to isomorphism, since $S \simeq PE(R)$, the pure-injective envelope of R (see [5]). S is still a valuation domain, whose field of quotients is QS. If T is a valuation domain, we denote by $U(T)$ its group of units.

We recall that every torsion-free R-module M contains a basic submodule B (see [4], [5]), which is a direct sum of uniserial submodules, pure-essential in M; B is unique up to isomorphism. If H is a subset of M, the pure submodule of M generated by H is the minimal pure submodule of M containing H.

For the definition and the properties of the height of an element $x \in M$ we refer to [5]; we recall here that, if $Q \geqslant L \geqslant R$, then Lx is the pure submodule generated by x in M if and only if the height of x in M is: $h_M(x) = L/R$.

If R is a submodule of Q, then $L^{\#} = \{r \in R: rL < L\}$ is a prime ideal of R, which coincides with the ideal $\bigcup \{rL: rL \leqslant R,\ r \in Q\}$. An ideal of R is said to be a v-ideal if it is the intersection of the ideals properly containing it.

Let $u \in U(S) \setminus R$; the breadth ideal of u is the ideal of R defined as follows (see [10]):

$$B(u) = \{a \in R: u \notin aR + S\}\,.$$

It is proved in [10] that an ideal I of R coincides with $B(u)$ for some $u \in U(S) \setminus R$ if and only if I is a v-ideal and R/I is not complete (in the R/I-topology). We shall use in the following the fact, whose proof is an easy exercise, that $B(u) = B(u^{-1})$.

We resume now the notion of compatible triple of ideals, which was fundamental in our investigation of two-generated modules in [11]; with respect to that paper, we generalize here this notion to our purposes. Recall that, given two submodules L and H of Q, then $L: H = \{q \in Q: qH \leqslant L\}$ and $L\because H = \{q \in Q: qH < L\}$ are still submodules of Q.

DEFINITION 2.1. Let L, H, I be submodules of Q. We say that (L, H, I) is a *compatible triple* if:

1) $L < H$;

2) $I = L{:}H$ and $I < rL$ for all $r^{-1} \in H \setminus L$;

3) R/I is not complete in the R/I-topology.

Note that $L < H$ implies that $I < R$, and that 2) shows that $I = \bigcap_{r^{-1} \in H \setminus L} rL$ is a v-ideal. Note also that, for $H = Q$, necessarily we have $I = 0$.

We collect in the next lemma the results on compatible triples that we shall use later.

LEMMA 2.2. *Let (L, H, I) be a compatible triple. Then*

1) *if* $a, b \in Q$, $a \notin H$ *and* $b \in H$, *there exists an* $r \in R$ *such that* $rb \in L$ *and* $ra \notin L$;

2) $H = L{::}I$;

3) $I^{\#} \leqslant L^{\#}$.

PROOF. 1) If $b \in L$, take $r = 1$. Let $b \notin L$; since $b \in H \setminus L$, then $b^{-1}L > I$ Let $r \in b^{-1}L \setminus I$; then $rb \in L$ and, from $a \notin H$, we get: $a^{-1}L \leqslant \leqslant I < rR$; thus $raR > L$, i.e. $ra \notin L$;

2) (see [11, Lemma 4]) Since $L > r^{-1}I$ for all $r^{-1} \in H \setminus L$, there follows that $H \leqslant L{::}I$. Conversely, let $r^{-1} \in L{::}I$ $(r \neq 0)$. Then $rL > I = \bigcap_{s^{-1} \in H \setminus L} sL$. So $rL > sL$ for some $s^{-1} \in H \setminus L$, so that $H > s^{-1}R > > r^{-1}R$, i.e. $r^{-1} \in H$;

3) (see [11, Th. 6.2]) We must show that $aL = L$ $(a \in R)$ implies $aI = I$. This follows from:

$$I = \bigcap_{r^{-1} \in H \setminus L} rL = \bigcap_{r^{-1} \in H \setminus L} raL = aI . \quad \square$$

For all unexplained notation and for general facts about modules over valuation domains, we refer to [5].

3. Existence and first properties.

The classical construction of an indecomposable rank-two torsion-free $\mathbb{Z}_p$-module, which makes use of a p-adic unit not in $\mathbb{Z}_p$ (see [8, Th. 19]), gives us the start-off for our construction.

THEOREM 3.1. *Let $Q \geqslant H > L \geqslant R$ and let $I = L:H$. If I is a v-ideal such that R/I is not complete, then there exists an indecomposable torsion-free R-module M of rank two with basic submodule $B \simeq L$ and $M/B \simeq H$.*

PROOF. By [10] there exists an element $u \in U(S) \setminus R$ such that $I = B(u)$. From the equality $I = \bigcap_{r^{-1} \in H \setminus L} rL$ it follows that, for each $r^{-1} \in H \setminus L$, there exists an element $u_r \in R$, necessarily a unit, such that

$$u - u_r \in rLS\,.$$

Note that, if $H \geqslant s^{-1}R > r^{-1}R > L$, then $u_s - u_r \in rL$. Let V be a Q-vector space of dimension two and let $\{x, y\}$ be a basis of V. Let us consider the R-submodule of V defined by generators as follows:

$$M = \langle Lx, r^{-1}(x + u_r y) : r^{-1} \in H \setminus L \rangle\,.$$

We set $w_r = x + u_r y$; note that, if $H \geqslant s^{-1}R > r^{-1}R > L$, then

$$sr^{-1}(w_s s^{-1}) - w_r r^{-1} \in Ly \tag{1}$$

and analogously

$$sr^{-1}(w_s s^{-1} u_s^{-1}) - w_r r^{-1} u_r^{-1} \in Lx\,. \tag{1'}$$

We proceed now by steps.

Step 1. $h_M(x) = L/R$.

Since $Lx \leqslant M$, $h_M(x) \geqslant L/R$. To prove the converse inequality, let us assume, by way of contradiction, that $x \in rM$ for some $r^{-1} \in H \setminus L$. Then we get the relation:

$$x = r(ax + a_1 r_1^{-1} w_{r_1} + \ldots + a_n r_n^{-1} w_{r_n}) \tag{2}$$

where $a \in L$, $a_1, \ldots, a_n \in R$ and $Rr_1 > \ldots > Rr_n > L$. We can assume n to be minimal; then $n \geq 1$, otherwise $(1 - ra)x = 0$ implies $1 = ra$, which is absurd, because $r^{-1} \in H \setminus L$ and $a \in L$.

We claim that $r_n r_{n-1}^{-1}$ divides a_n in R. Developing the right side in (2), since the coefficient of y is zero, we obtain:

$$a_1 r_1^{-1} u_{r_1} + \ldots + a_{n-1} r_{n-1}^{-1} u_{r_{n-1}} + a_n r_n^{-1} u_{r_n} = 0$$

that implies:

$$u_{r_n} a_n r_n^{-1} r_{n-1} = -\left(r_{n-1} a_1 r_1^{-1} u_{r_1} + \ldots + a_{n-1} u_{r_{n-1}}\right).$$

Since $a_1, \ldots, a_{n-1} \in R$ and $Rr_{n-1} < Rr_i$ for $i = 1, \ldots, n-2$, there follows that the right side is in R, hence $a_n = r_n r_{n-1}^{-1} b$ for some $b \in R$, as we claimed.

By (1′) we get

$$a_n r_n^{-1} w_{r_n} = b r_n r_{n-1}^{-1} w_{r_n} r_n^{-1} = b w_{r_{n-1}} r_{n-1}^{-1} u_{r_n} u_{r_{n-1}}^{-1} + cx$$

for a suitable $c \in L$; substituting this expression of $a r_n^{-1} w_{r_n}$ in (2) we get a contradiction to the minimality of n.

From $h_M(x) = L/R$ there follows that Lx is pure in M. Similarily one can show that $h_M(y) = L/R$, hence also Ly is pure in M.

Step 2. M/Lx is isomorphic to H.

In order to define an isomorphism $\varphi \colon H \to M/Lx$, choose $r^{-1} \in H \setminus L$ and set:

$$\varphi(r^{-1}) = w_r u_r^{-1} r^{-1} + Lx .$$

Then φ extends to a well-defined homomorphism because, if $H \geq s^{-1}R > r^{-1}R$, then (1′) gives:

$$sr^{-1}(w_s u_s^{-1} s^{-1}) + Lx = w_r u_r^{-1} r^{-1} + Lx ,$$

or, equivalently, $sr^{-1}\varphi(s^{-1}) = \varphi(r^{-1})$. Clearly φ is monic, since every proper quotient of H is torsion and M/Lx is torsion-free.

Moreover, φ is surjective, because M/Lx is generated by the elements $w_r u_r^{-1} r^{-1} + Lx$ $(H \geq r^{-1}R > L)$.

Step 3. M is indecomposable.

Assume, by way of contradiction, that M is not indecomposable. Then M is a direct sum of two uniserial torsion-free non-zero submodules. By [5, Th. 5.6, p. 192], Lx is a summand of M, so $M = Lx \oplus Z$, with $Z \simeq H$. If $r^{-1} \in H \setminus L$, then for suitable a_r and a in L, and z_r, z in Z, we have:

$$w_r r^{-1} = a_r x + z_r\,, \qquad y = -\,ax + z\,.$$

Thus we obtain:

$$x = w_r - u_r y = (ra_r - au_r)x + (rz_r - zu_r)$$

therefore $1 = ra_r - au_r$, thus $1 - au_r \in rL$, so that $a \in U(R)$ and $a^{-1} - u_r \in rL$. There follows that $u - a^{-1} \in \bigcap_{r^{-1} \in H \setminus L} rLS = IS$, contradicting the equality $I = B(u)$. $\square$

REMARKS. 1) From the hypothesis of the preceding theorem there follows that H is not cyclic. This fact trivially follows from $M/B \simeq H$ and M indecomposable. It is possible to derive that H is not cyclic directly from the existence of $u \in U(S) \setminus R$ such that $B(u) = I = L{:}H$.

2) If R is almost maximal, then necessarily $H = Q$ (see [5, Lemma 1.4, p. 271]), thus $I = L{:}Q = 0$. This agrees with the fact that S is the completion of R, so every unit of S not in R has breadth ideal equal to zero.

Now we shall show that the situation described in Theorem 3.1 is quite general. From now on in this section *M will denote an indecomposable torsion-free R-module of rank two.* Let x, $y \in M$ be two independent elements; let Lx and $L'y$ be the pure submodules generated by x and y, respectively. Since all basic submodules of M are isomorphic, we have $Lx \simeq L'y$; therefore, possibly substituting a multiple of y to y, we can assume that $L = L'$, i.e. that $h_M(x) = L/R = h_M(y)$. It is noteworthy that L is necessarily a proper submodule of Q, since M is indecomposable. We set: $\bar{y} = y + Lx \in M/Lx$. Let $h_{M/Lx}(\bar{y}) = H/R$; then we get

(A) $\quad Q \geqslant H > L \geqslant R.$

The exact sequence: $0 \to Lx \to M \to M/Lx \to 0$ does not split, hence it is not balanced (see [5, Lemma 2.3, p. 274]). Thus $h_{M/Lx}(\bar{y}) = H/R > h_M(y) = L/R$.

Since $h_{M/Lx}(\bar{y}) = H/R$, for every $r^{-1} \in H \setminus L$ there exists an element $w_r \in M$ such that

$$rw_r = y + x_r \tag{3}$$

for a suitable $x_r \in Lx$. Note that, from $h_M(x) = L/R$, there follows that $h_M(x_r) = L/R$, thus $x_r = k_r x$ for a $k_r \in U(R_{L^\#})$ (see [5]; $R_{L^\#}$ is the localization of R at $L^\#$). If $H \geqslant Rs^{-1} > Rr^{-1} > L$, from (3) and its analogue

$$sw_s = y + x_s$$

where $x_s = k_s x$ $(k_s \in U(R_{L^\#}))$, we get:

$$r(w_r - sr^{-1}w_s) = (k_r - k_s)x\,.$$

Thus $(k_r - k_s)x \in rM$, therefore $(k_r - k_s)r^{-1} \in L$, or, equivalently:

$$k_r - k_s \in rL\,. \tag{4}$$

This implies that $Rk_r = Rk_s$, because k_r, $k_s \in U(R_{L^\#})$ and $rL \leqslant L^\#$; therefore there exists a $k \in U(R_{L^\#})$ such that $k_r = u_r^{-1}k$ for all $r^{-1} \in H \setminus L$, where $u_r^{-1} \in U(R)$ and, if $H \geqslant s^{-1}R > r^{-1}R > L$, then

$$u_r^{-1} - u_s^{-1} \in k^{-1}rL = rL \tag{5}$$

which is obviously equivalent to: $u_r - u_s \in rL$. Since

$$h_M(k^{-1}y) = kL/R = L/R \qquad \text{and} \qquad h_{M/Lx}(k^{-1}\bar{y}) = kH/R > kL/R = L/R,$$

substituting $k^{-1}y$ to y and kH to H, we can assume in (3), without loss of generality, that $x_r = u_r^{-1}x$ for suitable units u_r^{-1} of R satisfying (5).

Since S is a maximal immediate extension of R, there exists a unit $u \in S$ such that

$$u - u_r \in rLS \qquad \text{for all } r^{-1} \in H \setminus L\,. \tag{6}$$

(B) $u \in U(S) \setminus R$ and $B(u) = L{:}H < rL$ for all $r^{-1} \in H \setminus L$.

It is straightforward to verify that, if $u \in R$, then $M = Lx \oplus \oplus H(x + uy)$. Thus necessarily $u \notin R$. (6) shows that $B(u) \leqslant$

$\leqslant \bigcap_{r^{-1}\in H\setminus L} rL = L{:}H$. If $B(u) < L{:}H$, then there exists $t \notin B(u)$ such that $t \in L{:}H$. By the definition of $B(u)$, there exists $a \in R$ such that $u - a \in tS < L{:}H$; this implies that $u_r - a \in rL$ for all $r^{-1} \in H\setminus L$ and, as before, $M = Lx \oplus H(x + ay)$, a contradiction. Thus we have proved that $B(u) = L{:}H$. The last strict inclusion trivially follows from $u - u_r \in rLS$ for all $r^{-1} \in H\setminus L$.

(C) $(L, H, B(u))$ is a compatible triple.

It is an immediate consequence of (A) and (B).

In view of the preceding discussion we introduce the following definitions.

Two independent elements x and y of the indecomposable rank-two torsion-free R-module M are said to be a *canonical basis* of M if

$$M = \langle Lx, (x + u_r y)r^{-1}{:}r^{-1} \in H\setminus L\rangle$$

where: Lx is a basic submodule of M, $h_{M/Lx}(\bar{y}) = H/R$, and the u_r's are units of R; note that necessarily $h_M(x) = L/R = h_M(y)$.

A unit $u \in U(S)\setminus R$ such that $u - u_r \in rLS$ for all $r^{-1} \in H\setminus L$ is said to be an *associated unit* of M with respect to the canonical basis $\{x, y\}$.

Remarks. 1) Given the canonical basis $\{x, y\}$ with associated unit u, and given $0 \neq a \in L$, then $\{ax, ay\}$ is again a canonical basis of M with u as associated unit. Obviously $h_M(ax) = a^{-1}L/R$ and $h_{M/Lx}(\overline{ay}) = a^{-1}H/R$; note that $a^{-1}L{:}a^{-1}H = L{:}H$.

2) As already observed by Fuchs and Viljoen [6] in the special case of R almost maximal, our module M is canonically a module over $R_{L^\#}$, where L is isomorphic to a basic submodule of M.

4. Change of canonical bases.

Let M be the indecomposable rank-two torsion-free R-module of the preceding section, with canonical basis $\{x, y\}$ and associated unit u. We set $B(u) = I$, and take fixed other notation as in section 3.

Lemma 4.1. *Let a, $b \in Q$ be such that $ax + by \in M$. Then a, $b \in H$, and $a \in L$ if and only if $b \in L$.*

PROOF. Clearly $b \in L$ if and only if $by \in M$, if and only if $ax \in M$, if and only if $a \in L$. Let $0 \neq r \in R$ be such that ra, $rb \in L$, so that rax, $rby \in M$. Then $r(M/Lx)$ contains $r(\overline{ax + by}) = \overline{rax} + \overline{rby} = rb\bar{y}$, so that $b\bar{y} \in M/Lx$. Since $h_{M/Lx}(\bar{y}) = H/R$, this implies that $b \in H$. Assume now, by way of contradiction, that $a \notin H$. Then, by Lemma 2.2, there exists $r \in R$ such that $ra \notin L$ and $rb \in L$, so that $rax + rby \in M$, with $rax \notin M$ and $rby \in M$, which is absurd. □

LEMMA 4.2. *Let* $a, b \in Q$ *be such that* $x' = ax + by \in M$ *and* $h_M(x') = L/R$. *Then* a^{-1}, $b^{-1} \notin I$.

PROOF. Since $a \in H$, by Lemma 4.1, and H is not principal, by remark 1 after Theorem 3.1, we have the proper inclusion $H > Ra$. For every $r^{-1} \in H \setminus (aR + L)$ we have:

$$x' = a(x + u_r y) + (b - au_r)y \tag{7}$$

where $a(x + u_r y) = aw_r \in arM \leqslant M$. Assume, by way of contradiction, that $a^{-1} \in I$. Then $a^{-1}R \leqslant I < rL$ implies $r^{-1}a^{-1}R < L$. Since $h_M(x') = L/R$, there follows that $x' \in arM$, so that $(b - au_r)y \in arM$. Then $(ba^{-1} - u_r)y \in rM$, so that $r^{-1}(ba^{-1} - u_r) \in L$, or, equivalently, $ba^{-1} - u_r \in rL$. Since this happens for each $r^{-1} \in H \setminus (ar + L)$, we deduce that $ba^{-1} - u \in IS$, which contradicts the equality $I = B(u)$.

With a similar argument, starting from the equality:

$$u_r x' = (u_r a - b)x + b(u_r y + x)$$

one can show that, if $b^{-1} \in I$, then $ab^{-1} - u^{-1} \in IS$, which is also a contradiction, because $B(u^{-1}) = B(u) = I$. Thus also $b^{-1} \notin I$. □

Our main goal in this section is to compare the canonical basis $\{x, y\}$ with another canonical basis $\{x', y'\}$. By remark 1 at the end of section 3, we can assume, without loss of generality, that $h_M(x') = L/R = h_M(x)$.

Let $v \in U(S) \setminus R$ be an associated unit of M with respect to $\{x', y'\}$.

Since $M/Lx \simeq M/Lx'$, we have:

$$h_{M/Lx'}(\bar{y}') = \beta H/R \quad \text{for some } \beta \in Q$$

where $\bar{y}' = y' + Lx'$ and $\beta H > L$ (i.e. $\beta \notin L{:}H = I$). Obviously we get:

$$B(v) = L{:}\beta H = \beta^{-1}B(u) = \beta^{-1}I\,,$$

therefore, for every $r^{-1} \in \beta H \setminus L$, there exists $v_r \in U(R)$ such that:

$$v - v_r \in rLS .$$

Consider the invertible matrix of $M_2(Q)$:

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

such that:

$$\begin{aligned} x' &= ax + by , & x &= \Delta^{-1}(dx' - by') , \\ y' &= cx + dy , & y &= \Delta^{-1}(-cx' + ay') , \end{aligned} \tag{8}$$

where $\Delta = \det T$.

We shall see later that $\beta^{-1}H = \Delta H$, so that β can be assumed to be equal to Δ^{-1}. In the above notation we have the following

LEMMA 4.3. 1) *There exists* $t^{-1} \in H \setminus L$ *such that, for all* $s^{-1} \in H \setminus t^{-1}R$, $b - au_s$ *and* $d - cu_s$ *belong* to $U(R_{L\#})$.

2) *There exists* $t^{-1} \in \beta H \setminus L$ *such that, for all* $r^{-1} \in \beta H \setminus t^{-1}R$, $\Delta^{-1}(a + cv_r)R$ *and* $\Delta^{-1}(b + dv_r)$ *belong to* $U(R_{L\#})$.

PROOF. 1) By Lemma 4.2, $a^{-1} \notin I = \bigcap_{r^{-1} \in H \setminus L} rL$. So there exists $t^{-1} \in H \setminus L$ such that $a^{-1} \notin tL$, or, equivalently, $a^{-1}t^{-1}R > L$. If $s^{-1} \in H \setminus t^{-1}R$, then $a^{-1}s^{-1}R > a^{-1}t^{-1}R > L$. Thus we have:

$$x' = a(x + u_s y) + (b - au_s)y , \tag{9}$$

where $a(x + u_s y) \in asM \leqslant M$; since $h_M(a(x + u_s y)) \geqslant a^{-1}s^{-1}R/L > R/L$, from (9) and $h_M(x') = L/R$ we deduce that $h_M((b - au_s)y) = L/R$. Since $h_M(y) = L/R$, there follows that $b - au_s \in U(R_{L\#})$.

Analogously one can show that $d - cu_s \in U(R_{L\#})$.

2) Can be obtained by 1), reversing the roles of $\{x, y\}$ and $\{x', y'\}$. □

An easy consequence of Lemma 4.3 is the following

LEMMA 4.4. $b - au$, $d - cu$, $\Delta^{-1}(a + cv)$, $\Delta^{-1}(b + dv) \in U(S_{L\#S})$.

PROOF. If $s^{-1} \in H \setminus t^{-1}R$, where t^{-1} is as in Lemma 4.3, we have:

$$b - au = (b - au_s) - a(u - u_s)$$

where $a(u - u_s) \in asLS < S$; since $asL < L^\#$ and $b - au_s \in U(R_{L^\#})$ by Lemma 4.3, we get $b - au \in U(S_{L^\# S})$. Analogously one can prove the other inclusions. □

We can now prove the main result of this section

PROPOSITION 4.5. *In the above notation we have*

$$v \equiv -(b - au)(d - cu)^{-1} \quad (mod\, B(v)S) .$$

PROOF. We know that the canonical basis $\{x', y'\}$ satisfies: $x' + v_r y' \in rM$ for all $r^{-1} \in \beta H \setminus L$. By (8) we have:

$$(a + v_r c)\, x + (b + v_r d)\, y \in rM \quad \text{for all } r^{-1} \in \beta H \setminus L ,$$

so, setting $s = r\beta$, we get:

$$\text{(10)} \quad \beta(a + v_r c)\, x + \beta(b + v_r d)\, y \in sM \quad \text{for all } s^{-1} \in H \setminus (L + \beta^{-1}L) .$$

Moreover we have:

$$\text{(11)} \quad x + u_s y \in sM \quad \text{for all } s^{-1} \in H \setminus L .$$

Since $H > L + \beta^{-1}L$, by Lemma 4.3 there exists a $t^{-1} \in \beta H \setminus (\beta L + L)$ such that:

$$b - au_s, \quad d - cu_s \in U(R_{L^\#}) \quad \text{for all } s^{-1} \in H \setminus t^{-1}\beta^{-1}R ,$$

$$\Delta^{-1}(a + cv_r) , \quad \Delta^{-1}(b + dv_r) \in U(R_{L^\#}) \quad \text{for all } r^{-1} \in \beta H \setminus t^{-1}R .$$

Note that, ranging r^{-1} in $\beta H \setminus t^{-1}R$, the elements $a + v_r c$ have all the same value.

We distinguish now two cases.

1st Case: $\beta(a + v_r c) \in R$ $(r^{-1} \in \beta H \setminus t^{-1}R)$.

Subtracting from (10) the relation (11) multiplied by $\beta(a + v_r c)$ we obtain

$$\beta\big(b + v_r d - u_s(a + v_r c)\big)\, y \in sM$$

so, recalling that $h_M(y) = L/R$ and $s = r\beta$, we get:

$$b + v_r d - u_s(a + v_r c) \in rL$$

or equivalently:

$$v_r(d - cu_s) + (b - au_s) \in rL\,. \tag{12}$$

Then, since $(v - v_r)(d - cu_s) \in (d - cu_s)\, rLS = rLS$, from (12) we deduce:

$$b + vd - u_s(a + vc) \in rLS\,;$$

note that $\beta(a + vc)$ has the same value as $\beta(a + v_r c)(r^{-1} \in \beta H \setminus t^{-1}R)$, i.e. $\beta(a + vc) \in S$. Since $(u - u_s)(a + vc) \in (a + vc)\, r\beta LS \leqslant rLS$, we get:

$$b + vd - u(a + vc) \in rLS\,. \tag{13}$$

This holds for all $r^{-1} \in \beta H \setminus t^{-1}R$; recalling that $d - cu \in U(S_{L\#S})$ and $B(v) = \bigcap_{r^{-1}\in\beta H\setminus t^{-1}R} rL$, we obtain from (13):

$$v \equiv -(b - au)(d - cu)^{-1} \quad (\operatorname{mod} B(v)S)\,. \tag{14}$$

2nd Case $\beta(a + v_r c) \in Q \setminus R \quad (r^{-1} \in \beta H \setminus t^{-1}R)$

Subtracting (11) from the eq. (10) multiplied by $\beta^{-1}(a + v_r c)^{-1} \in R$, we obtain:

$$\big(u_s - (b + v_r d)(a + v_r c)^{-1}\big)\, y \in sM$$

from which we get:

$$u_s - (b + v_r d)(a + v_r c)^{-1} \in sL\,.$$

Since $u - u_s \in sLS$ we have

$$u - (b + v_r d)(a + v_r c)^{-1} \in sLS$$

so that

$$u(a + v_r c) - (b + v_r d) \in r\beta(a + v_r c)LS \geqslant rLS\,. \tag{15}$$

Recalling that $d - cu \in U(S_{L^\# S})$, from (15) we get:

$$v_r + (b - au)(d - cu)^{-1} \in r\beta(a + v_r c)LS$$

and, since $v - v_r \in rLS$, we deduce:

$$v(d - cu) + (b - au) \equiv 0 \pmod{r\beta(a + v_r c)LS}\,. \tag{16}$$

Recalling now that $\Delta^{-1}(a + v_r c) \in U(R_{L^\#})$ and that $\bigcap_{r^{-1}\in\beta H\setminus L} r\beta L = \beta B(v) = B(u)$, multiplying (16) by Δ^{-1} we get:

$$\Delta^{-1}v(d - cu) + \Delta^{-1}(b - au) \equiv 0 \pmod{B(u)S}$$

or equivalently:

$$v \equiv -(b - au)(d - cu)^{-1} \pmod{\Delta B(u)S}\,. \tag{17}$$

Comparing the two relations (14) and (17), obtained in the first case and in the second case, respectively, we see that, in order to conclude the proof, it is enough to show that in the second case $\Delta B(u) \leqslant B(v)$.

By way of contradiction, assume that $\Delta B(u) > B(v)$. Choose an $r^{-1} \in \beta H \setminus L$ such that:

$$v - v_r \in rLS < \Delta B(u)S\,, \qquad \Delta^{-1}(a + cv_r) \in U(R_{L^\#})\,.$$

Then from (17) we get:

$$v_r \equiv -(b - au)(d - cu)^{-1} \pmod{\Delta B(u)S}\,.$$

But $d - cu \in U(S_{L^\# S})$ implies that

$$u(a + cv_r) - (b + dv_r) \in \Delta B(u)S$$

so, multiplying by Δ^{-1}:

$$u\Delta^{-1}(a + cv_r) - \Delta^{-1}(b + dv_r) \in B(u)S\,;$$

since $\Delta^{-1}(a + cv_r) \in U(R_{L^\#})$, we get the contradiction from the relation:

$$u - (b + dv_r)(a + cv_r)^{-1} \in B(u)S .$$

It is noteworthy that, from the preceding proof, there follows that $\Delta B(u) \leqslant B(v)$. Actually, the equality holds: $\Delta B(u) = B(v)$, as we shall see soon in the next section. $\square$

5. The classification theorem.

Before giving the classfication theorem of rank-two indecomposable torsion-free R-modules, we need two technical results on the units of S. Let $u, v \in U(S)\setminus R$ be such that $B(u) = L:H$ and $B(v) = L:H'$ for some submodules of $Q: Q \geqslant H,\ H' \geqslant L \geqslant R$. Let u_s (resp. v_r) be units of R such that $u - u_s \in sLS$ for all $s^{-1} \in H\setminus L$ (resp. $v - v_r \in rLS$ for all $r^{-1} \in H'\setminus L$). In this notation we have

LEMMA 5.1. *Let $0 \neq q_1, q_2 \in Q$ be such that $q_1 + q_2 u \in U(S_{L^\# S})$. Then there exists $t^{-1} \in H\setminus L$ such that $q_1 + q_2 u_s \in U(R_{L^\#})$ for all $s^{-1} \in H\setminus t^{-1}R$.*

PROOF. By contradiction, assume that for each $t^{-1} \in H\setminus L$ there exists $s^{-1} \in H\setminus t^{-1}R$ such that $q_1 + q_2 u_s \notin U(R_{L^\#})$. Then from the equality

$$q_1 + q_2 u - (q_1 + q_2 u_s) = q_2(u - u_s) \in q_2 sLS$$

we deduce that $q_1 + q_2 u$ has value at least equal to the value of $q_2(u - u_s)$, therefore $q_1 + q_2 u \in \bigcap_{t^{-1}\in H\setminus L} q_2 tLS$, or, equivalently, $q_1 q_2^{-1} + u \in \bigcap_{t^{-1}\in H\setminus L} tLS = B(u)S$, which is absurd. $\square$

LEMMA 5.2. *Let*

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in M_2(Q) , \qquad \textit{with } \Delta = \det T \neq 0 ,$$

be such that $d - uc,\ \Delta^{-1}(a + vc) \in U(S_{L^\# S})$ and

$$v \equiv - (b - ua)(d - uc)^{-1} \pmod{B(v)S} .$$

Then $\Delta B(u) = B(v)$.

PROOF. From the hypothesis, recalling that $B(v)^{\#} \leqslant L^{\#}$, we get: $u(a+vc)-(b+vd) \in B(v)S$, or, equivalently,

$$u\Delta^{-1}(a+vc)-\Delta^{-1}(b+vd) \in \Delta^{-1}B(v)S\,. \tag{18}$$

By contradiction, assume that $\Delta B(u) \neq B(v)$; we distinguish the two possible inclusions.

If $\Delta^{-1}B(v) > B(u)$, let $t^{-1} \in H \setminus L$ be such that $u-u_t \in tLS \leqslant \leqslant \Delta^{-1}B(v)S$ and $d-u_t c \in U(R_{L^{\#}})$; such a t does exist, in view of LEMMA 5.1. Since $\Delta^{-1}(a+vc) \in U(S_{L^{\#}S})$ and $B(v) \leqslant L^{\#}$, there follows that $\Delta^{-1}(a+vc)B(v) = B(v)$; so (18) implies that

$$u_t\Delta^{-1}(a+vc)-\Delta^{-1}(b+vd) \in \Delta^{-1}B(v)S\,,$$

and, multiplying by Δ, we obtain

$$(u_t a - b) + v(u_t c - d) \in B(v)S$$

which is absurd because $d-u_t c \in U(R_{L^{\#}})$ by the choice of t.

If $\Delta^{-1}\ B(v) < B(u)$, let $t^{-1} \in H' \setminus L$ be such that

$$\Delta^{-1}(v-v_t) \in \Delta^{-1}tLS \leqslant B(u) \qquad \text{and} \qquad \Delta^{-1}(a+v_t c) \in U(R_{L^{\#}})\,;$$

such a t does exist, again by Lemma 5.1. From (18) now we get:

$$v\Delta^{-1}(d-uc)+\Delta^{-1}(b-ua) \in \Delta^{-1}B(v)S < B(u)S\,,$$

so, since $(v-v_t)\Delta^{-1}(d-uc) \in \Delta^{-1}(d-uc)tLS = \Delta^{-1}tLS \leqslant B(u)S$, we obtain: $v_t\Delta^{-1}(d-uc)+\Delta^{-1}(b-ua) \in B(u)S$, from which we easily deduce

$$u \equiv (b+v_t d)(a+v_t c)^{-1} \qquad (mod\ B(u)S)$$

a contradiction. □

It is noteworthy that, in the hypothesis of Proposition 5.3, from the preceding lemma we have that $\Delta B(u) = B(v)$. Since $B(v) = = L{:}\beta H = \beta^{-1}B(u)$ for a suitable β, we see that *we can assume* β *equal to* Δ^{-1}.

We can now prove our isomorphy theorem

THEOREM 5.3. *Let M and M' be rank-two indecomposable torsion-free R-modules. Let $\{x, y\}$ and $\{x', y'\}$ be canonical bases of M and M', respectively, with associated units u and v, respectively. Let $h_M(x) = L/R$, $h_{M'}(x') = L'/R$. Then the following facts are equivalent*:

1) *M is isomorphic to M'*

2) *L is isomorphic to L' and there exists*

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in M_2(Q) \qquad \textit{with } \Delta = \det T \neq 0$$

such that $b - ua$, $\Delta^{-1}(a + vc) \in U(S_{L\# S})$ *and*

$$v \equiv -(b - ua)(d - uc)^{-1} \quad (\textit{mod } B(v)S)\ .$$

PROOF. 1) → 2) Let $\varphi: M \to M'$ be an isomorphism. Then $\{\varphi x, \varphi y\}$ is a canonical basis of M', such that $h_{M'}(\varphi x) = L/R$, therefore $L \simeq L'$. Obviously u is an associated unit of M' with respect to $\{\varphi x, \varphi y\}$. As in section 4, we can assume that $L = L'$. Let:

$$x' = a\varphi x + b\varphi y\ , \quad y' = c\varphi x + d\varphi y\ .$$

Then the matrix

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

is invertible; let $\Delta = \det T$. Then the claim follows from Lemma 4.4 and from Proposition 4.5.

2) → 1). Since $L \simeq L'$, we can assume, without loss of generality, that $h_{M'}(x') = L/R = h_{M'}(y')$. Since $\Delta \neq 0$, there exist elements x_1, $y_1 \in Qx' \oplus Qy' = V'$ such that

$$x' = ax_1 + by_1\ , \quad y' = cx_1 + dy_1\ .$$

Let us consider the R-submodule M'' of V' defined by generators as follows:

$$M'' = \langle Lx_1,\ s^{-1}(x_1 + u_s y_1) : s^{-1} \in H \setminus L \rangle$$

where $H/R = h_{M'Lx}(\bar{y})$ and the u_s's have the usual meaning. Clearly $M'' \simeq M$, and our goal now is to show that $M'' = M'$.

Let $H'/R = h_{M'/Lx}(\bar{y}')$; Lemma 5.2 shows that $\Delta B(u) = B(v)$; then $H = L::B(u)$ and $H' = L::B(v)$ imply that $\Delta^{-1}H = H'$. First we show that

$$M' = \langle Lx',\ r^{-1}(x' + v_r y') : r^{-1} \in \Delta^{-1}H \setminus L \rangle \leqslant M'' .$$

$Lx' < M''$: note that $a^{-1} \notin B(u)$; in fact, $d - uc \in U(S_{L\#S})$ implies that $b - ua \in U(S_{L\#S})$; moreover $a^{-1}(b - ua) = a^{-1}b - u \notin B(u)S$ implies $a^{-1} \notin B(u)$. Using Lemma 5.1, we can choose $s^{-1} \in H \setminus L$ such that:

$$b - au_s \in U(R_{L\#}) \qquad \text{and} \qquad a^{-1}R > sL > B(u) .$$

For such an s, we decompose x' as follows:

$$x' = a(x_1 + u_s y_1) + (b - au_s) y_1 .$$

Since $h_{M''}((b - au_s) y_1) = (b - au_s)^{-1} L/R = L/R$, it is enough to show that $a(x_1 + u_s y_1) \in M''$ and $h_{M''}(a(x_1 - u_s y_1)) \geqslant L/R$. In fact:

$$a(x_1 + u_s y_1) = as(s^{-1}(x_1 + u_s y_1)) \in asM''$$

where $as \in R$ (because $R > asL$), thus

$$h_{M''}(a(x_1 + u_s y)) \geqslant a^{-1}s^{-1}R/R > L/R$$

as we want. In the same way one can prove that $Ly' \leqslant M''$.

$r^{-1}(x' + v_r y') \in M''$ for all $r^{-1} \in \Delta^{-1}H \setminus L$: let us notice that

$$\bigcap_{r^{-1} \in \Delta^{-1}H \setminus L} rL = B(v) = \Delta B(u) = \bigcap_{s^{-1} \in H \setminus L} \Delta sL ;$$

using Lemma 5.1, we can choose $t^{-1} \in \Delta^{-1}H \setminus L$ such that, for each $r^{-1} \in \Delta^{-1}H \setminus t^{-1}R$:

$$\Delta^{-1}(a + v_r c) \in U(R_{L\#}) \qquad \text{and} \qquad rL = \Delta sL \ (\exists s^{-1} \in H \setminus L) .$$

Clearly it is enough to prove the claim for these $r^{-1} \in \Delta^{-1}H \setminus t^{-1}R$;

in fact, we have

$$r^{-1}(x' + v_r y') = r^{-1}\big((a + v_r c)\, x_1 + (b + v_r d)\, y_1\big)\,. \tag{19}$$

By hypothesis we also have:

$$ua - b + v(uc - d) \in B(u)\, S$$

from which, since $(v - v_r)(uc - d) \in rLS$, we get:

$$u(a + v_r c) - (b + v_r d) \in rLS\,. \tag{20}$$

Now we observe that, for $rL = \Delta sL$ $(s^{-1} \in H \setminus L)$ we have: $u - u_s \in \in sLS = r\Delta^{-1}LS$, so that

$$(u - u_s)(a + v_r c) \in r(a + v_r c)\Delta^{-1}LS = rLS\,.$$

Thus from (20) we deduce:

$$u_s(a + v_r c) - (b + v_r d) \in rL$$

or, equivalently, there exists $k \in L$ such that

$$b + v_r d = u_s(a + v_r c) + rk\,.$$

Moreover, since $rL = s\Delta L$, $r = s\Delta\eta$ for some $\eta \in U(R_{L^\#})$. Substituting in (19) we get:

$$\begin{aligned} r^{-1}(x' + v_r y') &= r^{-1}\big((a + v_r c)\, x_1 + u_s(a + v_r c) y_1 + rky_1\big) = \\ &= s^{-1}\eta^{-1}\Delta^{-1}(a + v_r c)(x_1 + u_s y_1) + ky_1\,. \end{aligned} \tag{21}$$

The last member in (21) is an element of M'', because $ky_1 \in M''$, since $k \in L$ and $h_{M''}(y_1) = L/R$, $s^{-1}(x_1 + u_s y_1) \in M''$ and $\eta^{-1}\Delta^{-1}(a + v_r c) \in \varepsilon\, U(R_{L^\#})$ (we use here the fact that M'' is an $R_{L^\#}$-module).

We have seen that $M' \leqslant M''$. In order to show that $M'' \leqslant M'$, consider the relations:

$$x_1 = \Delta^{-1} dx' - \Delta^{-1} by'\,, \qquad y_1 = -\,\Delta^{-1} cx' + \Delta^{-1} ay'$$

(obtained using the matrix T^{-1}) and the relation

$$u \equiv \Delta^{-1}(b + vd)(\Delta^{-1}(a + vc))^{-1} \qquad (\mathrm{mod}\, \Delta^{-1}B(v)S = B(u)S)$$

obtained by the congruence in the hypothesis. Note that, if

$$T^{-1} = \begin{bmatrix} a' & c' \\ b' & d' \end{bmatrix}$$

then $b' - a'v = -\Delta^{-1}(b + vd), \Delta(a' + uc') = -(d - uc) \in U(S_{L^\# S})$. So we can argue as before, exchanging the roles of x', y', v, T by x_1, y_1, u, T^{-1}, obtaining the desired inclusion $M'' \leqslant M'$. $\square$

REMARK. It is worthwhile to note that the proof of Theorem 5.3 shows, as a consequence of Lemma 5.2, that, if $h_{M/Lx}(\bar{y}) = H/R$ and $h_{M'/Lx'}(\bar{y}') = H'/R$, then H is isomorphic to H'; in fact it is showed that $H' = \Delta^{-1}H$, where $\Delta = \det T$.

Theorem 5.3 shoves us to the following definition.

Let us consider all the triples of the form (L, H, u), where $Q \geqslant H > > L > 0$, $u \in U(S) \setminus R$ and $(L, H, B(u))$ is a compatible triple. We say that two such triples (L, H, u) and (L', H', v) are *$\mathcal{F}$-equivalent* if the three following conditions are satisfied:

a) $L \simeq L'$;

b) there exists a matrix

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in M_2(Q)\,, \qquad \text{with } 0 \neq \Delta = \det T\,,$$

such that $v \equiv -(b - ua)(d - uc)^{-1}$, $(\mathrm{mod}\, B(v)S)$;

c) $b - ua$, $\Delta^{-1}(a + vc) \in U(S_{L^\# S})$.

Using Lemma 5.2 and the equality $L^\# = (L')^\#$ it is cumbersome but straightforward to verify that $\mathcal{F}$-equivalence is in fact an equivalence; we denote by $[L, H, u]_{\mathcal{F}}$ the equivalence class of the triple (L, H, u) with respect to $\mathcal{F}$-equivalence.

THEOREM 5.4. *The equivalence classes $[L, H, u]_{\mathcal{F}}$ form a complete and independent set of invariants for rank-two torsion-free indecomposable R-modules.*

PROOF. Let M and M' be rank-two torsion-free indecomposable R-modules, with basic submodules $B \simeq L$ and $B' \simeq L'$ respectively ($Q > L$, $L' \geqslant R$). Assume that $M/B \simeq H$, $M'/B' \simeq H'$ and that u and v are associated units of M and M', respectively, such that $B(u) = L:H$ and $B(v) = L':H'$. Since $(L, H, B(u))$ and $(L', H', B(v))$ are compatible triples (see (C) in section 3), we can associate to M and M' the equivalence classes $[L, H, u]_{\mathcal{F}}$ and $[L', H', v]_{\mathcal{F}}$ respectively. Theorem 5.3 shows that M is isomorphic to M' if and only if $[L, H, u]_{\mathcal{F}} = [L', H', v]_{\mathcal{F}}$. Moreover, given the equivalence class $[L, H, u]_{\mathcal{F}}$, where $Q \geqslant H > L > 0$, we can assume $L \geqslant R$. Then we associate the module M constructed in Theorem 3.1, where $I = B(u)$. □

If R is a valuation domain which is almost maximal but not maximal, then the unique ideal I such that R/I is not complete is $I = 0$. Therefore the only compatible triples are of the form $(L, Q, 0)$, and our Theorem 5.3 says that the two R-modules M and M' are isomorphic if and only if they have isomorphic basic submodules and

$$v = -(b - ua)(d - uc)^{-1} \tag{22}$$

where

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

is an invertible matrix of $M_2(Q)$ such that $b - ua$, $\Delta^{-1}(a + vc) \in U(S_{L\#S})$. Since $B(v) = 0$, the last condition can be removed. Then, if we consider the two Q-subspaces of QS:

$$V_M = Q \oplus Qu\,, \qquad V_{M'} = Q \oplus Qv$$

it is easily seen that (22) is equivalent to the equality $\varrho V_{M'} = V_M$ for a suitable $\varrho \in V_M$. This is actually the classification given by Viljoen in [13] and extended to modules with uniserial basic submodules by Fuchs and Viljoen in [6]. Therefore our classification generalizes that by Viljoen.

If R is a non-complete rank-one discrete valuation domain and $M \in \mathcal{F}_2(R)$, then a basic submodule B of M is necessarily isomorphic to R, so our Theorem 5.4 says that M is isomorphic to $M' \in \mathcal{F}_2(R)$ if and only if their associated units u and v satisfy (22). It is easy to see that this classification is equivalent to the classical Kurosch classification by matrix invariants (see [11] for a modernized version of the Kurosch theory).

6. Comparison with two-generated torsion modules.

In this section R will always denote a valuation domain which is *not almost maximal.* We shall compare the classification of rank-two torsion-free indecomposable R-modules obtained in section 5 with the classification of two-generated indecomposable torsion R-modules obtained in [14] and [12]. For this purpose, we readapt some concepts already introduced in that papers.

Recall that, given a two-generated indecomposable torsion module $X = Rx + Ry$, with Rx pure in X, we associate to it a compatible triple (A, J, I) of *ideals of* R (see Definition 2.1), where $A = \operatorname{Ann} x = \operatorname{Ann} X < J = \operatorname{Ann}(y + Rx)$, and $I = B(u)$ for a certain unit $u \in U(S) \setminus R$ depending on x and y. Changing the generators x and y, the compatible triple remains unchanged, while the unit u ranges over the class determined by the equivalence relation defined as follows:

given $u, v \in U(S) \setminus R$, we say that u and v are *$\mathcal{C}$-equivalent* if there exists an invertible matrix

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

in $M_2(R)$ such that

$$v \equiv -(b - ua)(d - uc)^{-1} \pmod{B(v)S}\ .$$

From this definition it is easy to deduce the following facts.

(i) $d - uc,\ b - ua,\ \Delta^{-1}(a + vc),\ \Delta^{-1}(b + vd) \in U(S)$.

In fact, $d - uc$ and $b - ua$ have the same value and

$$\Delta = \det T = \det \begin{bmatrix} a & c \\ b - ua & d - uc \end{bmatrix} \in U(R)\ .$$

The proof for the two other elements is similar.

(ii) $B(u) = B(v)$.

This equality follows now from Lemma 5.2, since $\Delta \in U(R)$.

The definition of $\mathcal{F}$-equivalence given in section 5, in the torsion-free case, has its counterpart in the following definition.

Consider the triples of the form (A, J, u), where $R \geqslant J > A > 0$ $u \in U(S) \setminus R$ and $(A, J, B(u))$ is a compatible triple. We say that two such triples (A, J, u) and (A', J', v) are *$\mathcal{C}$-equivalent* if the two following conditions are satisfied:

a) $A = A'$;

b) u and v are $\mathcal{C}$-equivalent.

As remarked above, *b*) implies $B(u) = B(v)$, so that *a*) implies $J = A :: B(u) = A' :: B(v) = J'$. We denote by $[A, J, u]_{\mathcal{C}}$ the equivalence class of the triple (A, J, u) with respect to $\mathcal{C}$-equivalence.

In the above notation, the classification theorem in [14] for two-generated modules reads now as follows.

THEOREM 6.1 [14]. *The equivalence classes $[A, J, u]_{\mathcal{C}}$ form a complete and independent set of invariants for the class of two-generated indecomposable R-modules.* □

Let us denote by $[\mathcal{C}_2]$ and $[\mathcal{F}_2]$ the sets of isomorphy classes of two-generated indecomposable torsion R-modules and, respectively, of rank-two indecomposable torsion-free R-modules.

In view of Theorems 5.4 and 6.1, we identify $[\mathcal{C}_2]$ and $[\mathcal{F}_2]$ with the sets of $\mathcal{C}$-equivalence classes $[A, J, u]_{\mathcal{C}}$ and, respectively, of equivalence classes $[L, H, v]_{\mathcal{F}}$.

THEOREM 6.2. *There exists a canonical map $\Phi: [\mathcal{C}_2] \to [\mathcal{F}_2]$ defined by $\Phi([A, J, u]_{\mathcal{C}}) = [A, J, u]_{\mathcal{F}}$. Φ is never injective, and it is surjective if and only if R is complete.*

PROOF. Φ is well defined, because $[A, J, u]_{\mathcal{C}}$ is a subset of $[A, J, u]_{\mathcal{F}}$. To show that Φ is not injective, let $u \in U(S) \setminus R$ be such that $B(u) = I \neq 0$. Let $0 \neq q \in I$, so that $qI < I$. Setting $v = 1 + qu$, we have $B(v) = qB(u) = qI$ (see [10]). Consider now the invertible matrix of $M_2(Q)$:

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \equiv \begin{bmatrix} q & 0 \\ -1 & 1 \end{bmatrix}.$$

Then $b - ua = -v \in U(S)$, $(a + vc)\Delta^{-1} = 1 \in U(S)$. There follows that $[A, J, u]_{\mathcal{F}} = [qA, J, v]_{\mathcal{F}}$, but clearly $B(u) \neq B(v)$, so that $[A, J, u]_{\mathcal{C}} \neq \neq [A, J, v]_{\mathcal{C}}$. Finally, R is not complete if and only if there exists a unit $u \in U(S) \setminus R$ such that $B(u) = 0$. In this case the invariant $[R, Q, u]_{\mathcal{F}}$ does exist and obviously is not in the image of Φ.

REMARK. The Kurosch-Malcev-Derry theory, that classifies torsion-free abelian groups of finite rank by means of invariants, consisting of equivalence classes of matrix sequences, was considered of little utility (see [3, p. 158]). The reason is that the question of deciding wether two matrix sequences are in the same class is in general as difficult as to decide wether two groups are isomorphic.

Actually, in the local case, i.e. for $\mathbb{Z}_p$-modules, these invariants are more easy to handle (one deals with one matrix only and not with a sequence), and their benefit becomes more relevant, as is shown in the application to Arnold's duality [1]. The preceding Theorem 6.2, and also the concrete application in the next Example 6.4, give another evidence of the utility of the classification by matrices, in the Kurosch's fashion.

We conclude this section with two examples. In the first one we show that, for a suitable valuation domain R, we can have triples (A, J, u) and (A, J, v) such that $[A, J, u]_{\mathcal{C}} \neq [A, J, v]_{\mathcal{C}}$ and $[A, J, u]_{\mathcal{F}} = [A, J, v]_{\mathcal{F}}$, even if $B(u) = B(v)$. Compare this example with the proof of Theorem 6.2. In the second example we exhibit a valuation domain which admits only one rank-two torsion-free indecomposable module up to isomorphism, while the classes in $[\mathcal{C}_2]$ are infinitely many.

EXAMPLE 6.3. Let R be a valuation domain with a non-zero prime ideal I such that $R/I \simeq \mathbb{Z}_p$. There exists $u \in U(S) \setminus R$ such that $B(u) = I$ and $u + IS$ is transcendent over R/I. Choose $q \in P \setminus I$ and let $v = 1 + qu$. Then $B(v) = qB(u) = qI = I$. One can see that, if $0 \neq a \in I$ and $J = aR : : I$, then (aR, J, I) is a compatible triple and, as in the proof of Theorem 6.2, that $[aR, J, u]_{\mathcal{F}} = [aR, J, v]_{\mathcal{F}}$. Let us assume, by way of contradiction, that there exists an invertible matrix of $M_2(R)$

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

such that

$$v \equiv -(b - ua)(d - uc)^{-1} \qquad (mod\, IS) .$$

From $v = 1 + qu$ we get:

$$(1 - qu)(d - uc) + (b - ua) \equiv 0 \pmod{IS},$$

or, equivalently,

$$qcu^2 + (-qd + c + a)u - (d + b) \equiv 0 \pmod{IS}.$$

Since $u + IS$ is transcendent over R/I, we deduce that $qc, -qd + c + a$, $d + b$ are in I. Therefore $c + a \in P$, since $qd \in P$; hence

$$\det T = \det \begin{bmatrix} a + c & c \\ b + d & d \end{bmatrix} \in P,$$

a contradiction.

EXAMPLE 6.4. By the results in [2], there exists a complete valuation domain R which is discrete of rank (= Krull dimension) two, with chain of prime ideals $P = Rp > I > 0$, and such that S/IS has rank two as R/I-module. In the terminology of [10], R has completion defect at I equal to two. It is well known that every non-zero ideal K of R is isomorphic either to R or to I and that R/K is not complete (in the R/K-topology) if and only if $K \simeq I$.

We will show that two triples (L, H, u) and (L', H', v) are always $\mathcal{F}$-equivalent; so there exists only one rank-two torsion-free indecomposable R-module, up to isomorphism. Necessarily $H \simeq I \simeq H'$, since H and H' are not principal. Note that $L \simeq I$ implies that $B(u) = L:H$ is principal, which is impossible; therefore $L \simeq R$ and similarly, $L' \simeq R$, so that $L \simeq L'$.

There exist suitable $a, b \in R$ such that $B(u) = aI$ and $B(v) = bI$. Consider the unit $w = 1 + a^{-1}bu$, whose breadth ideal is:

$$B(w) = a^{-1}bB(u) = bI = B(v).$$

The completion defects at I and at bI are equal (see [10, Prop. 2.2]); this means that

$$c_0 + c_1 v + c_2 w \in bIS$$

for suitable $c_0, c_1, c_2 \in R$, where at most one $c_i \in P$. If $c_1 \in U(R)$, then

$$v \equiv -c_1^{-1}(c_0 + c_2 w) \pmod{B(v)S}$$

and substituting we get:

$$v \equiv -(c_0 + c_2 + a^{-1}bc_2u)c_1^{-1} \qquad (\mathrm{mod}\ B(v)S)\,.$$

It is easy to see that the matrix

$$T = \begin{bmatrix} -a^{-1}bc_2 & 0 \\ c_0 + c_2 & c_1 \end{bmatrix}$$

has the desired properties (trivially $c_2 \neq 0$, so $\Delta = \det T \neq 0$). If $c_2 \in U(R)$, then

$$c_2(1 + a^{-1}bu) + (c_0 + c_1v) \equiv 0 \qquad (\mathrm{mod}\, B(v)S)$$

from which we deduce

$$u \equiv -(c_0 + c_2 + c_1v)(c_2a^{-1}b)^{-1} \qquad (\mathrm{mod}\ B(u)S)\,.$$

As before, it is easy to see that the matrix

$$T = \begin{bmatrix} -c_1 & 0 \\ c_0 + c_2 & c_2a^{-1}b \end{bmatrix}$$

has the desired properties. Obviously there exist infinitely many non-isomorphic two-generated indecomposable torsion modules.

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