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ENRICO GREGORIO

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Dualities Over Compact Rings.

ENRICO GREGORIO (*)

0. Introduction.

Let R_τ be a topological ring: we can consider the categories $\mathfrak{L}\text{-}R_\tau$ and $R_\tau\text{-}\mathfrak{L}$ of locally compact right and left modules over R_τ . It is known that there is always a duality, namely Pontrjagin duality, between them (see section 1). It is also known that if R_τ is the ring \mathbb{Z} of integers, endowed with the discrete topology, then Pontrjagin duality is the unique duality between $\mathfrak{L}\text{-}\mathbb{Z}$ and $\mathbb{Z}\text{-}\mathfrak{L}$.

In [S] L. Stoyanov showed that also in the case when R_τ is *compact and commutative*, Pontrjagin duality is the unique duality between $\mathfrak{L}\text{-}R_\tau$ and $R_\tau\text{-}\mathfrak{L}$.

In this paper we extend Stoyanov's result to the non-commutative case. The most important tool we shall make use of is the characterization we gave in our earlier work [G] of the equivalences between categories of discrete modules over linearly topologized rings. The results of [G] can be used here because, as it is well known, every compact ring is linearly topologized (l.t.): indeed it has a basis of neighbourhoods of zero, or more briefly a local basis, consisting of (two-sided) ideals.

The proof of the main result of this paper relies on the classification (Theorem 3.17) of all c -progenerators (see section 3 for the definition)

(*) Indirizzo dell'A.: Dipartimento di Matematica pura ed applicata, via Belzoni 7, 35131 Padova (Italy).

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over a given compact ring R_τ and to the observation that every duality between $\mathfrak{L}\text{-}R_\tau$ and $R_\tau\text{-}\mathfrak{L}$ induces a duality between the categories $\text{Mod-}R_\tau$ and $R_\tau\text{-}CM$ of all discrete right modules and of all compact left modules over R_τ respectively.

In section 1 we introduce briefly Pontrjagin duality over an arbitrary topological ring, while in section 2 we study locally compact modules over compact rings. In this section we prove, using essentially the methods of [R] that every locally compact module over a compact ring is linearly topologized.

In section 3 we recall the main result from [G] and apply it to show (Theorem 3.6) that two compact rings are similar when regarded as right linearly topologized rings if and only if they are similar as left l.t. rings (two right l.t. rings R_τ and A_σ are said to be similar if there exists an equivalence between the categories $\text{Mod-}R_\tau$ and $\text{Mod-}A_\sigma$ of discrete right modules over R_τ and A_σ). Then the characterization of c -progenerators over the compact ring R_τ is given, which is based on a decomposition of projective modules in $CM\text{-}R_\tau$ as a topological product of projective covers of simple modules, analogous to the decomposition of injective modules over noetherian rings as a direct sum of indecomposable injectives. This characterization allows us to associate to a compact ring another compact ring, called its basic ring, which has properties similar to those of the basic ring of a semi-perfect ring.

Section 4 is devoted to the proof of the main theorem; in section 5 we show that our concept of basic ring coincides with other existing in the literature ([DO₂] and [MO]).

All rings and modules considered in this paper are, unless the contrary is stated, endowed with a Hausdorff topology and all morphisms are continuous. The use of the word *algebraic* means that the topology is not considered. All rings have an identity $1 \neq 0$ and all modules are unitary. If M and N are modules over the ring R_τ , then $\text{Chom}_R(M, N)$ denotes the group of all continuous morphisms of M into N .

All categories and functors considered are additive. If R_τ is a topological ring, we denote respectively by: (1) $TM\text{-}R_\tau$, (2) $LTC\text{-}R_\tau$, (3) $\mathfrak{L}\text{-}R_\tau$, (4) $CM\text{-}R_\tau$ and (5) $\text{Mod-}R_\tau$ the categories of (1) topological, (2) linearly topologized and complete, (3) locally compact, (4) compact and (5) discrete right modules over R_τ . We shall use analogous notations for left modules.

1. Pontrjagin duality.

1.1 Let $\mathbf{T} = \mathbb{R}/\mathbb{Z}$ be the topological group of real numbers modulo 1: \mathbf{T} is a compact group. If G is a topological abelian group, the *Pontrjagin dual* of G is

$$\Gamma(G) = \text{Chom}_{\mathbf{Z}}(G, \mathbf{T})$$

endowed with the topology of uniform convergence on compacta, which has as a local basis the family of all subsets of the form

$$\mathcal{W}(K; U) = \{\xi \in \text{Chom}_{\mathbf{Z}}(G, \mathbf{T}) : \xi(K) \subseteq U\},$$

as K runs through the compact subsets of G and U through the neighbourhoods of zero in \mathbf{T} . The elements of $\Gamma(G)$ are called the characters of G .

If $f: G \rightarrow H$ is a continuous morphism of topological abelian groups, we define

$$\Gamma(f) = f^*: \Gamma(H) \rightarrow \Gamma(G)$$

by $f^*(\xi) = \xi \circ f$ (for $\xi \in \Gamma(H)$).

Then f^* is a continuous morphism and so Γ is a (contravariant) functor from the category of topological abelian groups into itself.

1.2 If G is a topological abelian group, there exists a morphism (non continuous, in general)

$$\omega_G: G \rightarrow \Gamma^2(G)$$

defined, for $x \in G$ and $\xi \in \Gamma(G)$, by

$$\omega_G(x): \xi \rightarrow \xi(x).$$

1.3. **THEOREM.** *If G is a locally compact abelian group, then $\Gamma(G)$ is locally compact and ω_G is a topological isomorphism. Thus Γ induces a duality of the category \mathcal{L} of locally compact abelian groups with itself.*

For a proof of this result see [P].

1.4 THEOREM. *The functor Γ induces a duality between the categories $\text{Mod-}\mathbb{Z}$ of discrete abelian groups and $\text{CM-}\mathbb{Z}$ of compact abelian groups.*

PROOF. It is sufficient to prove that Γ carries discrete groups to compact ones and vice versa. If G is a discrete group and $f: \mathbb{Z}^{(X)} \rightarrow G$ is an epimorphism, then $f^*: \Gamma(G) \rightarrow \mathbb{T}^X$ is a topological immersion, so that $\Gamma(G)$ is compact. If G is compact and U is a neighbourhood of zero in \mathbb{T} which contains no non-zero subgroups, then $\mathcal{W}(G; U) = 0$ is a neighbourhood of zero in $\Gamma(G)$.

1.5 Let R be a ring endowed with the discrete topology and M a topological right R -module; we can define a left action of R on the topological group $\Gamma(M)$ by setting

$$(r\xi)(x) = \xi(xr) \quad (r \in R, x \in \Gamma(M), x \in M)$$

and in such a way $\Gamma(M)$ becomes a topological left R -module. In a similar way we can define a structure of topological right R -module over $\Gamma(N)$, where N is any topological left R -module

1.6 NOTATIONS. If S (resp. X) is a subset of a ring R (resp. of a right R -module), we shall denote by $X \cdot Y$ the set

$$X \cdot S = \{xs : x \in X, s \in S\},$$

while XS will denote the *submodule* spanned by $X \cdot S$. If one of the sets X or S consists of one element, we shall omit parentheses. Similar notations will hold for left modules.

Given two subsets A and B of an abelian group, we set

$$A + B = \{a + b : a \in A, b \in B\}, \quad -A = \{-a : a \in A\}.$$

1.7 LEMMA. *Let R_τ be a topological ring, M be a locally compact right module over R_τ and U be a neighbourhood of zero in M . Then:*

(i) *for every compact subset K of M there exists a neighbourhood of zero V in R_τ such that $K \cdot V \subseteq U$ (equivalently, the set $\{r \in R : K \cdot r \subseteq U\}$ is a neighbourhood of zero in R_τ);*

(ii) *for every compact subset C of R_τ , there exists a neighbourhood of zero W in M such that $W \cdot C \subseteq U$ (equivalently, the set $\{x \in M : x \cdot C \subseteq U\}$ is a neighbourhood of zero in M).*

PROOF. (i) For every $x \in K$ there exist a neighbourhood of zero V_x in R_τ and a neighbourhood of zero W_x in M such that $(x + W_x) \cdot V_x \subseteq U$. Being K compact, there are x_1, \dots, x_n in K such that

$$\bigcup_{i=1}^n (x_i + W_{x_i}) \supseteq K$$

and, for $V = \bigcap V_{x_i}$, we have

$$K \cdot V \subseteq \bigcup_{i=1}^n (x_i + W_{x_i}) \cdot V \subseteq U.$$

The proof of (ii) is analogous.

This property of compact subsets of topological modules is known as *boundedness* (see [K]).

1.8 PROPOSITION. *Let R_τ be a topological ring and let M be a locally compact right module over R_τ . Then $\Gamma(M)$ is a topological left module over R_τ .*

PROOF. We must show that the multiplication $R_\tau \times \Gamma(M) \rightarrow \Gamma(M)$ defined in 1.5 is continuous. Fix $r \in R$, $\xi \in \Gamma(M)$ and a neighbourhood of zero in $\Gamma(M)$ of the form $\mathcal{W}(K; U)$, where K is a compact subset of M and U is a neighbourhood of zero in \mathbf{T} . We want to find:

(i) a neighbourhood of zero V in R_τ such that, for all $s \in V$ and all $x \in K$, $\xi(xs) \in U$, i.e.

$$V \cdot \xi \subseteq \mathcal{W}(K; U);$$

(ii) a compact subset K' of M and a neighbourhood U' of zero in \mathbf{T} such that

$$r \cdot \mathcal{W}(K'; U') \subseteq \mathcal{W}(K; U);$$

(iii) a neighbourhood of zero V' in R_τ , a compact subset K'' of M and a neighbourhood U'' of zero in \mathbf{T} such that, for all $s \in V'$ and all $\eta \in \mathcal{W}(K''; U'')$,

$$s\eta \in \mathcal{W}(K; U).$$

We now proceed to the proof.

(i) By 1.6 there exists a neighbourhood of zero V in R_τ such that $K \cdot V \subseteq \xi^{-1}(U)$; this means that $\xi(K \cdot V) \subseteq U$, so that $V \cdot \xi \subseteq \mathcal{W}(K; U)$.

(ii) Let $K' = K \cdot r$: then K' is a compact subset of M . Let $\eta \in \mathcal{W}(K'; U)$ and $x \in K$: then $(r\eta)(x) = \eta(rx) \in U$, so that $r \cdot \mathcal{W}(K'; U) \subseteq \mathcal{W}(K; U)$.

(iii) Let W be a compact neighbourhood of zero in M : by 1.6 there exists a neighbourhood of zero V' in R_τ such that $K \cdot V' \subseteq W$.

Now, for $s \in V'$ and $\eta \in \mathcal{W}(W; U)$,

$$(s\eta)(K) = \eta(K \cdot s) \subseteq \eta(K \cdot V') \subseteq \eta(W) \subseteq U.$$

Therefore $s\eta \in \mathcal{W}(K, U)$ and $V' \cdot \mathcal{W}(W, U) \subseteq \mathcal{W}(K, U)$: so we can take $K'' = W$.

Putting together statements (i), (ii) and (iii) and setting

$$V_0 = V \cap V', \quad K_0 = K' \cap K'', \quad U_0 = U' \cap U'',$$

we get

$$(r + V_0) \cdot (\xi + \mathcal{W}(K_0; U_0)) \subseteq r + \mathcal{W}(K; U) + \mathcal{W}(K; U), \quad \text{q.e.d.}$$

1.9. COROLLARY. *Let R_τ be a topological ring. The functor Γ induces a duality between the categories of left and right locally compact modules over R_τ .*

1.10 We shall denote by $\mathfrak{L}\text{-}R_\tau$ (resp. $R_\tau\text{-}\mathfrak{L}$) the category of right (resp. left) locally compact modules over R_τ and by

$$\Gamma_R: \mathfrak{L}\text{-}R_\tau \rightarrow R_\tau\text{-}\mathfrak{L}, \quad {}_R\Gamma: R_\tau\text{-}\mathfrak{L} \rightarrow \mathfrak{L}\text{-}R_\tau$$

the Pontrjagin functors. We shall denote moreover by $CM\text{-}R_\tau$ (resp. $\text{Mod-}R_\tau$) the category of compact (resp. discrete) right modules over R_τ . Analogous notation will be used for left modules.

1.11 COROLLARY. *Let R_τ be a topological ring. Then Γ induces a duality between $\text{Mod-}R_\tau$ and $R_\tau\text{-}CM$.*

2. Compact rings and their locally compact modules.

The followings results are well known: the proofs we give here are essentially those of [R], with some modifications of the exposition.

2.1 THEOREM. *Let R_τ be a compact topological ring. Then R_τ has a local basis consisting of open two-sided ideals.*

PROOF. Consider the family \mathfrak{C} of all τ -closed right ideals I of R such that R/I is finite: this is a local basis for a Hausdorff topology on R coarser than τ , hence equal to τ (the Hausdorff property follows from the duality theorem 1.8: if $x \in R$, $x \neq 0$, there exists a character $\xi \in \Gamma(R_\tau)$ such that $\xi(x) \neq 0$; then $I = \text{Ann}_R(x) \in \mathfrak{C}$, since, according to 1.11, $\Gamma(R_\tau)$ is a discrete module; clearly $x \notin I$).

If now I is a τ -open right ideal of R , consider $V = \{x \in R: \forall r \in R, rx \in I\}$: V is a two-sided ideal of R and $I \supseteq V$. According to 1.6, V is open.

The following lemma is well known (see e.g. [P]).

2.2 LEMMA. *Let G be a totally disconnected locally compact abelian group. Then G has a local basis consisting of compact open subgroups.*

2.3 LEMMA. *Let M_R be a locally compact module over the compact ring R_τ . Then M is totally disconnected.*

PROOF. Let $\xi: M \rightarrow \mathbf{T}$ be a character of M and U be a neighbourhood of zero in \mathbf{T} containing no non-zero subgroups. Then $\xi^{-1}(U)$ is a neighbourhood of zero in M ; put

$$V = \{x \in M: \forall r \in R, xr \in \xi^{-1}(U)\};$$

by 1.6, V is a neighbourhood of zero in M and moreover $V \cdot R \subseteq \xi^{-1}(U)$ and $V \subseteq V \cdot R$. If $x \in V$, then $xR \subseteq V$, so that $\xi(xR) \subseteq U$: thus $\xi(xR) = 0$, by the hypotheses we made on U . Hence $V \cdot R \subseteq \ker \xi$ and $\ker \xi$ is open. From 1.3 it follows that for any $x \in M$, $x \neq 0$, there exists $\xi \in \Gamma(M)$ with $\xi(x) \neq 0$.

2.4 THEOREM. *Let M_R be a locally compact module over the compact ring R_τ . Then M_R is linearly topologized over R_τ , i.e. M_R has a local basis consisting of open compact R -submodules.*

PROOF. From 2.3 and 2.2 we know that M has a local basis consisting of compact open subgroups. If H is an open subgroup of M then, by virtue of 1.6.

$$V = \{x \in M: \forall r \in R, xr \in H\}$$

is an open submodule of M_R contained in H .

3. Progenerators and c -progenerators.

3.1 In [G, Definition 2.4] we introduced a new notion of progenerator, which generalizes the classic one from Morita theory. Progenerators are the fundamental tool in the study of equivalences between categories of discrete modules over linearly topologized rings.

DEFINITION. Let R be a ring (endowed with the discrete topology) and let P_R be a linearly topologized module over R ; denote by $\mathcal{F}(P_R)$ the family of all open R -submodules of P . Let τ be a right linear topology on R .

We say that P_R is:

(1) *topologically finitely generated* if, for all $V \in \mathcal{F}(P_R)$, the quotient P/V is a finitely generated R -module;

(2) *topologically quasiprojective* if, for all $V \in \mathcal{F}(P_R)$ and all continuous R -morphisms $f: P \rightarrow P/V$ (where P/V is given the discrete topology), there exists a continuous R -endomorphism $g: P \rightarrow P$ such that $f = g\pi$ (π is the canonical projection $P \rightarrow P/V$);

(3) a *selfgenerator* if, for all $V \in \mathcal{F}(P_R)$, the closure in P of

$$t_P(V) = \sum \{\text{Im } f: f \in \text{Chom}_R(P, V)\}$$

coincides with V ;

(4) a τ -*generator* if it is a topological module over R_τ and, for each non-zero morphism $g: M \rightarrow N$ in $\text{Mod-}R_\tau$, there exists a continuous morphism $g: P \rightarrow M$ such that $fg \neq 0$.

We say that P_R is:

(a) a *quasiprogenerator* if it satisfies (1), (2), (3) and is complete;

(b) a τ -progenerator (or simply a *progenerator* if it is clear which topology we are considering) if it is a quasiprogenerator and a τ -generator.

(c) a c - τ -progenerator if it is a τ -progenerator and the ring $\text{Chom}_R^u(P, P)$ (i.e. the ring of all continuous endomorphism of P_R endowed with the topology of uniform convergence) is compact.

We recall here the main theorem of [G], which characterizes the equivalences between categories of discrete modules over linearly topologized rings in terms of progenerators.

3.2 DEFINITION. Let R_τ and A_σ be right linearly topologized rings. We say that R_τ and A_σ are *similar* if there exists an equivalence (called a *similarity*)

$$(F: \text{Mod-}A_\sigma \rightarrow \text{Mod-}R_\tau, G: \text{Mod-}R_\tau \rightarrow \text{Mod-}A_\sigma)$$

(with F and G additive functors).

3.3 THEOREM [G, Theorem 4.9]. *Let R_τ and A_σ be similar complete right linearly topologized rings and let $(F: \text{Mod-}A_\sigma \rightarrow \text{Mod-}R_\tau, G: \text{Mod-}R_\tau \rightarrow \text{Mod-}A_\sigma)$ be a similarity between them. Then there exists a τ -progenerator P_R such that A_σ is canonically isomorphic to the ring $\text{Chom}_R^u(P, P)$ (endowed with the topology of uniform convergence) and the functor G is naturally isomorphic to the functor $\text{Chom}_R(P, -)$. If, moreover, $Q_A = \text{Chom}_R^u(P, R_\tau)$, then Q_A is a σ -progenerator, R_τ is canonically isomorphic to $\text{Chom}_A^u(Q, Q)$ and F is naturally isomorphic to $\text{Chom}_A(Q, -)$.*

3.4 REMARKS. (a) Every progenerator over a compact ring is compact. A progenerator is a c -progenerator if and only if, for all finite modules $M \in \text{Mod-}R_\tau$, the group $\text{Chom}_R(P, M)$ is finite.

(b) Given a linearly topologized module P_R over a right linearly topologized ring R_τ , if $A_\sigma = \text{Chom}_R^u(P, P)$, we can define a functor $\text{Chom}_R^u(P, -)$ from the category $LTC\text{-}R_\tau$ to $LTC\text{-}A_\sigma$, which sends the complete linearly topologized module M_R over R_τ to $\text{Chom}_R^u(P, M)$, the A -module of continuous morphism from P to M , endowed with the topology of uniform convergence (see [G, 3.1]).

3.5 THEOREM. *Let R_τ be a compact ring and let P_R be a c - τ -progen-*

erator. Set $A_\sigma = \text{Chom}_R^u(P, P)$. Then the functor

$$\text{Chom}_R^u(P, -): LTC\text{-}R_\tau \rightarrow LTC\text{-}A_\sigma$$

is an equivalence which sends compact modules to compact modules and locally compact modules to locally compact modules.

PROOF. It is not difficult to prove that $\text{Chom}_R^u(P, -)$ preserves limits and is full and faithful.

If M_R is a compact module over R_τ , then it is the limit (in $LTC\text{-}R_\tau$) of a family of finite modules. Since P_R is a c -progenerator, $\text{Chom}_R(P, -)$ takes finite modules to finite modules, so that $\text{Chom}_R^u(P, M)$ is compact, being the limit of a family of finite A -modules. Let now M_R be a locally compact module over R_τ : if V is an open compact submodule of M_R then $\text{Chom}_R^u(P, V)$ is topologically isomorphic to $\mathfrak{J}(V) = \{f \in \text{Chom}_R^u(P, M) : f(P) \subseteq V\}$, which is a typical neighbourhood of zero in $\text{Chom}_R^u(P, M)$. Hence $\text{Chom}_R^u(P, M)$ has a local basis consisting of compact open submodules (2.4).

In Definition 3.2 we have defined similarity between right linearly topologized rings by using, as it is obvious, right modules. But, if we are given compact rings, Theorem 2.1 enables us to speak about right or left similarity, since any compact ring is both right and left linearly topologized. The following theorem shows that no distinction is needed.

3.6 THEOREM. *The compact rings R_τ and A_σ are right similar if and only if they are left similar. If P_R is a c - τ -progenerator with $A_\sigma = \text{Chom}_R^u(P, P)$, then ${}_A P$ is a c - σ -progenerator and R_τ is canonically isomorphic to $\text{Chom}_A^u(P, P)$.*

PROOF. Assume that R_τ and A_σ are right similar and let P_R be a c - τ -progenerator with $A_\sigma = \text{Chom}_R^u(P, P)$. Put $Q_A = \text{Chom}_R^u(P, R_\tau)$: then Q_A is a c - σ -progenerator with $R_\tau = \text{Chom}_A^u(Q, Q)$ [G, 2.3 and 4.1].

By 3.5 we have the equivalence $C_R = \text{Chom}_R^u(P, -): CM\text{-}R_\tau \rightarrow CM\text{-}A_\sigma$, whose inverse must be $C_A = \text{Chom}_A^u(Q, -): CM\text{-}A_\sigma \rightarrow CM\text{-}R_\tau$. Then we have the equivalence

$$(F = \Gamma_A \circ C_R \circ_R \Gamma: \text{Mod}\text{-}A_\sigma \rightarrow \text{Mod}\text{-}R_\tau,$$

$$G = \Gamma_R \circ C_A \circ_A \Gamma: \text{Mod}\text{-}R_\tau \rightarrow \text{Mod}\text{-}A_\sigma)$$

which makes R_τ and A_σ left similar.

Denote by ${}_{\tau}\mathcal{F}$ the filter of all open left ideal of R_{τ} : the σ -progenerator which gives the (left) similarity between R_{τ} and A_{σ} is (see [G, 4.6, 1.2 and 1.3])

$${}_A X = \lim_{J \in {}_{\tau}\mathcal{F}} F(R/J).$$

From the definition of F we get

$$F(R/J) = \Gamma_A \left(\text{Chom}_R^u(P, {}_R \Gamma(R/J)) \right)$$

and the algebraic isomorphisms

$$F(R/J) \cong \text{Hom}_{\mathbf{Z}} \left(\text{Hom}_R(R/J, \Gamma_R(P)), \mathbf{T} \right) \cong \text{Hom}_{\mathbf{Z}}(\text{Ann}(J), \mathbf{T})$$

(where $\text{Ann}_{\Gamma_R(P)}(J) = \{\xi \in \Gamma_R(P) : r\xi = 0\}$). Then, taking limits and colimits for $J \in {}_{\tau}\mathcal{F}$,

$$\begin{aligned} \lim F(R/J) &\cong \lim \text{Hom}_{\mathbf{Z}}(\text{Ann}(J), \mathbf{T}) \cong \\ &\cong \text{Hom}_{\mathbf{Z}}(\text{colim } \text{Ann}_{\Gamma_R(P)}(J), \mathbf{T}) \cong \text{Hom}_{\mathbf{Z}}(\Gamma_R(P), \mathbf{T}). \end{aligned}$$

Endowing $\text{Hom}_{\mathbf{Z}}(\Gamma_R(P), \mathbf{T})$ with the topology of pointwise convergence (which, in this case, coincides with the topology of uniform convergence on compacta), we have a topological isomorphism

$$X = \lim F(R/J) \cong \Gamma^2(P) \cong P$$

which is easily seen to be a bimodule isomorphism.

We devote the rest of this Section to the study of c -progenerators over a given compact ring R_{τ} .

3.7 THEOREM. *Let R_{τ} be a compact ring and $P_R \in \text{CM-}R_{\tau}$. The following conditions are equivalent:*

(a) P_R is a c - τ -progenerator;

(b) P_R is a projective generator of $\text{CM-}R_{\tau}$ and, for all open right ideals J of R_{τ} , the closure of PJ is an open submodule of P_R .

PROOF. (a) \Rightarrow (b): P_R is a projective generator of $\text{CM-}R_{\tau}$ because, if $A_{\sigma} = \text{Chom}_R^u(P, P)$, the functor $\text{Chom}_R^u(P, -)$ induces an equivalence

between $CM\text{-}R_\tau$ and $CM\text{-}A_\sigma$ (3.5). It follows from 3.6 that ${}_A P$ is a σ -progenerator and $R_\tau \cong \text{Chom}_A^u(P, P)$ canonically. Thus, if I is an open left ideal of R_τ , \overline{PI} is open in P (recall definition 3.1). Let J be an open right ideal of R_τ : then J contains an open two-sided ideal I and so, from $\overline{PJ} \supseteq \overline{PI}$ we get that \overline{PJ} is open in P .

(b) \Rightarrow (a): it is clear that P_R is a topologically finitely generated quasiprojective selfgenerator. Let $f: M \rightarrow N$ be a non zero morphism in $\text{Mod-}R_\tau$ and choose $x \in M$ such that $f(x) \neq 0$: then xR is finite, hence compact with the discrete topology and so there exists a continuous morphism $g: P \rightarrow xR$ whose image is not contained in $\ker f \cap xR$; if $i: xR \rightarrow M$ is the inclusion, then $fig \neq 0$ and P_R is a τ -generator. Let us verify that P_R is τ -projective. If ${}_R K = \Gamma_R(P_R)$, then ${}_R K$ is injective in $R_\tau\text{-Mod}$; let $M \in \text{Mod-}R_\tau$, $L \subseteq M$ and $f: P \rightarrow M/L$ be a continuous morphism. Applying the functor Γ_R , we obtain the diagram in $R_\tau\text{-}\mathcal{L}$, with exact row (in $R_\tau\text{-}CM$)

$$\begin{array}{ccc}
 & & {}_R K \\
 & & \uparrow \\
 & & \Gamma_R(f) \\
 & & \uparrow \\
 0 & \rightarrow & \Gamma_R(M/L) \rightarrow \Gamma_R(M)
 \end{array}$$

and there exists a continuous morphism that completes it. Dualizing again we have the result. It remains to see that the functor $\text{Chom}_R^u(P, -)$ takes finite modules to finite modules, but it is sufficient to prove that $\text{Chom}_R(P, R/J)$ is finite, for all open right ideals J of R_τ . Consider $f \in \text{Chom}_R(P, R/J)$, $p \in P$ and $r \in J$: then $f(pr) = f(p)r = 0$, so that $PJ \subseteq \ker f$; thus $\overline{PJ} \subseteq \ker f$ and so, since \overline{PJ} is open, $\text{Chom}_R(P, R/J) \cong \text{Hom}_R(P/\overline{PJ}, R/J)$ is finite.

3.8 REMARK. In proving implication (b) \Rightarrow (a) in 3.7, we have seen that, if R_τ is a compact ring, then every projective generator of $CM\text{-}R_\tau$ is a τ -progenerator; the following example shows that not every projective generator of $CM\text{-}R_\tau$ is a $c\text{-}\tau$ -progenerator.

3.9 EXAMPLE. Let R_τ be a compact ring and X be a set: it is shown in [G, 6.2] that the compact right R -module R^X , endowed with the product topology, is a τ -progenerator (it is in fact a projective generator of $CM\text{-}R_\tau$), but it is clear that R^X is a $c\text{-}\tau$ -progenerator if and only if X is finite.

We can now proceed to the classification of all c - τ -progenerators over a given compact ring R_τ .

3.10 LEMMA. *Every injective module in $R_\tau\text{-Mod}$ can be decomposed into a direct sum of indecomposable injective modules.*

PROOF. The proof can be obtained in the same way as for noetherian rings (see, e.g. [SV]).

If $M \in R_\tau\text{-Mod}$, we denote by $E_\tau(M)$ its injective envelope in $R_\tau\text{-Mod}$: it is easily seen that, denoting by $E(M)$ the injective envelope of M in $R\text{-Mod}$, one has

$$E_\tau(M) = \{x \in E(M) : \text{Ann}_R(x) \text{ is } \tau\text{-open}\}.$$

3.11 LEMMA. *Let $E \in R_\tau\text{-Mod}$ be an injective module. Then E is indecomposable if and only if there exists a simple module $S \in R_\tau\text{-Mod}$ such that $E \cong E_\tau(S)$.*

PROOF. It is plain that, for all simple modules $S \in R_\tau\text{-Mod}$, $E_\tau(S)$ is indecomposable. Conversely, if $E \in R_\tau\text{-Mod}$ is indecomposable and $x \in E$, $x \neq 0$, we have $E = E_\tau(xR)$. Now xR is a finite module and so it has a simple submodule.

3.12 LEMMA. *Let $S \in R_\tau\text{-Mod}$ be a simple module. Then S is the only simple submodule of $E_\tau(S)$.*

PROOF. Trivial.

3.13 Let $(S_\delta)_{\delta \in \Delta}$ be a complete set of representatives of the simple modules in $R_\tau\text{-Mod}$. Put $T_\delta = {}_R\Gamma(S_\delta)$: then T_δ is a simple module in $\text{Mod-}R_\tau$ and $(T_\delta)_{\delta \in \Delta}$ is a complete set of representatives of the simple modules in $\text{Mod-}R_\tau$. Let $E \in R_\tau\text{-Mod}$ be an injective module: there exists (3.10 and 3.11) a decomposition

$$E = \bigoplus_{\delta \in \Delta} E_\tau(S_\delta)^{(\nu_\delta)}$$

and if $E = \bigoplus_{\delta} E_\tau(S_\delta)^{(\mu_\delta)}$ is another decomposition, then, since isomorphic modules have isomorphic socles and socles commute with direct sums, we get by 3.12 that

$$\bigoplus_{\delta \in \Delta} S_\delta^{(\nu_\delta)} \cong \bigoplus_{\delta \in \Delta} S_\delta^{(\mu_\delta)}$$

and so $\nu_\delta = \mu_\delta$ for all $\delta \in \Delta$.

3.14 DEFINITION. Let ${}_R E = \bigoplus_{\delta \in \Delta} E_{\tau}(S_{\delta})^{(\nu_{\delta})}$ be an injective module in $R_{\tau}\text{-Mod}$. The family $(\nu_{\delta})_{\delta \in \Delta}$ of cardinal numbers is called the *grade* of E and E is said to have *finite grade* if ν_{δ} is finite for all $\delta \in \Delta$. It is obvious that ${}_R E$ is a cogenerator of $R_{\tau}\text{-Mod}$ if and only if $\nu_{\delta} \geq 1$ for all $\delta \in \Delta$.

If R_{τ} is a compact ring, then every compact module $M \in CM\text{-}R_{\tau}$ has a *projective cover*, which we denote by $P_{\tau}(M)$; it is, obviously,

$$P_{\tau}(M) = {}_R \Gamma(E_{\tau}(\Gamma_R(M))).$$

3.15 DEFINITION. Let P_R be a projective module in $CM\text{-}R_{\tau}$: then $\Gamma_R(P_R)$ is an injective module in $R_{\tau}\text{-Mod}$. The *grade* of P_R is the grade of its Pontrjagin dual $\Gamma_R(P)$. We say that P_R has *finite grade* if $\Gamma_R(P)$ has finite grade.

If the decomposition of $\Gamma_R(P)$ is $\Gamma_R(P) = \bigoplus_{\delta} E_{\tau}(S_{\delta})^{(\nu_{\delta})}$, then we can write

$$P_R \cong \prod_{\delta \in \Delta} P_{\tau}(T_{\delta})^{\nu_{\delta}}.$$

A projective module P_R in $CM\text{-}R_{\tau}$ is a generator if and only if $\nu_{\delta} \geq 1$ for all $\delta \in \Delta$.

3.16 LEMMA. Let $T \in CM\text{-}R_{\tau}$ be a simple module and let $P_R = P_{\tau}(T)$. Then, for each open two-sided ideal I of R_{τ} , \overline{PI} is open on P . Moreover $\overline{PI} \neq P$ if and only if I is contained in the annihilator of T in R .

PROOF. It is clear (see 3.12) that P_R has a unique maximal open submodule M and $P/M \cong T$. Let $\mathfrak{m} = \text{Ann}_R(T) = \{r \in R: Pr \subseteq M\}$: \mathfrak{m} is an open two-sided ideal of R_{τ} . Let I be another open two-sided ideal of R_{τ} : if I is not contained in \mathfrak{m} , then PI is not contained in M , so that $\overline{PI} = P$; assume now $I \subseteq \mathfrak{m}$ and consider the finite ring $S = R/I$ with the discrete topology and the module $Q_s = P/\overline{PI} \in CM\text{-}S$. Then T is in a natural way a simple S -module and there is a continuous epimorphism of S -modules $f: Q \rightarrow T$, since $I \subseteq \mathfrak{m}$ implies $\overline{PI} \subseteq M$. We want to verify that Q is a projective cover (in $CM\text{-}S$) of $T_s: Q_s$ is clearly projective; next, given an epimorphism $Q' \rightarrow T$, with Q' projective, let P' be the projective cover of Q' in $\text{Mod-}R_{\tau}$: then there exists an epimorphism $P' \rightarrow P \rightarrow 0$ and, taking the annihilators of I , we are done.

Thus Q is finite and so \overline{PI} is open in P_R :

3.17 THEOREM. *Let R_τ be a compact ring and let $P_R \in CM\text{-}R_\tau$. The following conditions are equivalent:*

- (a) P_R is a $c\text{-}\tau$ -progenerator;
- (b) P_R is a projective generator of finite grade of $CM\text{-}R_\tau$.

PROOF. (a) \Rightarrow (b): P_R is a projective generator of $CM\text{-}R_\tau$ by 3.7. Moreover we have, using the same notations as in 3.15,

$$P_R \cong \prod_{\delta \in \Delta} P_\tau(T_\delta)^{v_\delta}.$$

If, for some $\delta \in \Delta$, v_δ were infinite, one could easily find a module $X \in \text{Mod}\text{-}R_\tau$ such that $\text{Chom}_R(P, X)$ is infinite.

(a) \Rightarrow (b): let J be an open two-sided ideal of R_τ ; then

$$\overline{PJ} = \prod_{\delta \in \Delta} (\overline{P_\tau(T_\delta)J})^{v_\delta}$$

is open in P , since $\overline{P_\tau(T_\delta)J} = P_\tau(T_\delta)$ for all but a finite number of $\delta \in \Delta$, since R/J is a finite ring, so that the set of ideals of R containing J is finite. It is sufficient now to apply 3.16 to end the proof.

3.18 COROLLARY. *Let R_τ be a compact ring. There exists in $CM\text{-}R_\tau$ a projective generator Π_R which is minimal, in the sense that, if P_R is any projective generator in $CM\text{-}R_\tau$, then there is a continuous epimorphism $P_R \rightarrow \Pi_R \rightarrow 0$. Such a minimal projective generator is unique up to topological isomorphisms and is a $c\text{-}\tau$ -progenerator.*

We can now prove a result of Golman and Sah [GS].

3.19 COROLLARY. *Every compact ring is a topological product of compact local modules.*

PROOF. Let R_τ be a compact ring. If $T \in CM\text{-}R_\tau$ is simple, then $P_\tau(T)$ is local. Since R_τ is a $c\text{-}\tau$ -progenerator, the result follows from the decomposition given in 3.15.

We can easily apply 3.17 to primary rings.

3.20 DEFINITION. Let R_τ be a right linearly topologized ring. R_τ is:

(a) *primary* if there exists in $\text{Mod-}R_\tau$ a unique (up to isomorphism) simple module.

(b) *local* if it has a unique maximal open right ideal.

It is clear that the unique open maximal right ideal of a local ring is two-sided. Moreover, every local ring is primary and in the commutative case the two notions coincide. If R_τ is compact, then it is primary as right l.t. ring if and only if it is primary as left l.t. ring, since there is a bijection between the isomorphism classes of simple modules in $\text{Mod-}R_\tau$ and in $R_\tau\text{-Mod}$.

3.21 THEOREM. *Let R_τ be a compact primary ring. Then R_τ is the ring of $n \times n$ (for some positive integer n) matrices with entries in a compact local ring.*

PROOF. If T_R is the only simple module in $\text{Mod-}R_\tau$, the minimal projective generator in $\text{CM-}R_\tau$ is $\Pi_R = P_\tau(T)$. It is obvious that $B_\beta = \text{Chom}_R^u(P, P)$ is local. The c - τ -progenerator which gives the similarity between R_τ and B_β is $P_\beta(S)^n$, where S is the unique simple module in $\text{Mod-}B_\beta$. We can conclude by putting

$$A_\sigma = \text{Chom}_R^u(P_\beta(S), P_\beta(S)).$$

This theorem was proved by Kaplansky [K] using the lifting of idempotents. The method used here is similar to that of Theorem 5.1 in [DO₁].

4. The main theorem.

4.1. L. Stoyanov [S] proved the following theorem:

If R_τ is a compact commutative ring and $(\Delta_1: \mathfrak{L-}R_\tau \rightarrow R_\tau\text{-}\mathfrak{L}, \Delta_2: R_\tau\text{-}\mathfrak{L} \rightarrow \mathfrak{L-}R_\tau)$ is a duality, then Δ_1 and Δ_2 are naturally isomorphic to the Pontrjagin duality functors defined in 1.10.

In this section we shall extend this result to the non commutative case.

We shall fix a compact ring R_τ and denote by Π_R the minimal projective generator of $\text{CM-}R_\tau$ (3.18) and by ${}_R U$ the minimal injective cogenerator of $R_\tau\text{-Mod}$. It is clear from the definitions that ${}_R U =$

$= \Gamma_R(\Pi_R)$ and that $\text{Chom}_R(\Pi, \Pi)$ is algebraically isomorphic to $\text{Hom}_R(U, U)$. Finally we shall set $A_\sigma = \text{Chom}_R^u(\Pi, \Pi)$: A_σ is a compact ring (3.18).

4.2. PROPOSITION. *A_σ is topologically isomorphic to the ring $\text{Hom}_R(U, U)$ endowed with the topology of pointwise convergence.*

PROOF. The algebraic isomorphism $\varphi: A \rightarrow \text{Hom}_R(U, U)$ is given by $\varphi(\alpha) = \Gamma_R(\alpha)$. If V is an open submodule of Π_R , then $\mathfrak{J}(V) = \{\alpha \in A: \alpha P \subseteq V\}$ is open in A_σ and

$$\varphi(\mathfrak{J}(V)) = \{f \in \text{Hom}_R(U, U): \forall \xi \in \Pi, (\xi)f \in V\}.$$

Since we can think that $\Pi_R = {}_R\Gamma(U)$, we can assume that V is of the form

$$\mathfrak{W}(F) = \{\xi \in {}_R\Gamma(U): \forall x \in F, \xi(x) = 0\},$$

where F is a finite subset of U . The conclusion is now easy.

4.3 DEFINITION. Let R_τ be a compact ring and denote by Π_R the minimal projective generator of $CM\text{-}R_\tau$ (3.18): the compact ring

$$\mathbf{B}_r(R_\tau) = \text{Chom}_R^u(\Pi, \Pi)$$

is called the *right basic ring of R_τ* : every compact ring is similar to its right basic ring.

We can analogously define the notion of *left basic ring of R_τ* , $\mathbf{B}_l(R_\tau)$.

4.4 PROPOSITION. *Let R_τ be a compact ring and $A_\sigma = \mathbf{B}_r(R_\tau)$ be its right basic ring. Then A_σ , as a right module over itself, is the minimal projective generator of $CM\text{-}A_\sigma$.*

PROOF. If Π_R is the minimal projective generator of $CM\text{-}R_\tau$, then the functor $\text{Chom}_R^u(\Pi, -)$ is an equivalence between $CM\text{-}R_\tau$ and $CM\text{-}A_\sigma$ (3.18 and 3.5) and so it takes the minimal projective generator to the minimal projective generator.

4.5 DEFINITION. The compact ring R_τ is said to be *right basic* if it is, as right module over itself, the minimal projective generator of $CM\text{-}R_\tau$.

4.6 PROPOSITION. *Let R_τ be a right basic compact ring and let*

$$(F: \text{Mod-}R_\tau \rightarrow \text{Mod-}R_\tau, G: \text{Mod-}R_\tau \rightarrow \text{Mod-}R_\tau)$$

be a similarity. Then the functors F and G are naturally isomorphic to the identity.

PROOF. Applying 3.3 we find a c - τ -progenerator P_R with $F \cong \cong \text{Chom}_R(P, -)$ and $R_\tau \cong \text{Chom}_R^u(P, P)$ canonically. If we put $Q_R = \text{Chom}_R^u(P, R_\tau)$, then $G \cong \text{Chom}_R(Q, -)$ and $R_\tau \cong \text{Chom}_R^u(Q, Q)$. Since P_R is a c - τ -progenerator, it is, by 3.7, a projective generator of CM - R_τ and so, being R_τ the minimal projective generator of CM - R_τ , there exists, by 3.18, a continuous epimorphism

$$P_R \rightarrow R_\tau \rightarrow 0.$$

The functor $\text{Chom}_R^u(P, -)$ is an equivalence of CM - R_τ with itself (3.5): hence we get an epimorphism

$$R_\tau \cong \text{Chom}_R^u(P, P) \rightarrow Q_R = \text{Chom}_R^u(P, R_\tau) \rightarrow 0.$$

But Q_R is a c - τ -progenerator too, and so the existence of this epimorphism implies that $R_\tau \cong Q_R$. Since $P_R \cong \text{Chom}_R^u(Q, R_\tau)$, we have $P_R \cong R_\tau$.

4.7 THEOREM. *Let R_τ be a compact ring and let*

$$(F: \text{Mod-}R_\tau \rightarrow \text{Mod-}R_\tau, G: \text{Mod-}R_\tau \rightarrow \text{Mod-}R_\tau)$$

be a similarity. Then the functors F and G are naturally isomorphic to the identity.

PROOF. Let II_R be the minimal projective generator of CM - R_τ and put $A_\sigma = \mathbf{B}_r(R_\tau) = \text{Chom}_R^u(II, II)$: A_σ is right basic (4.4). The c - τ -progenerator II_R defines a similarity

$$(C: \text{Mod-}R_\tau \rightarrow \text{Mod-}A_\sigma, T: \text{Mod-}A_\sigma \rightarrow \text{Mod-}R_\tau)$$

(see [G, Theorem 4.9]), so that the pair of functors (CFT, CGT) is an equivalence of $\text{Mod-}A_\sigma$ with itself. Then, by 5.6,

$$CFT \cong \mathbf{1}_{\text{Mod-}A_\sigma}$$

and so

$$F \cong (TC)F(TC) \cong T(CFT)C \cong TC \cong \mathbf{1}_{\text{Mod-}R_\tau}.$$

We can show analogously that

$$G \cong \mathbf{1}_{\text{Mod-}R_\tau}.$$

4.8 COROLLARY. *Let R_τ be a compact ring and let*

$$(\Delta_1: \text{Mod-}R_\tau \rightarrow R_\tau\text{-CM}, \Delta_2: R_\tau\text{-CM} \rightarrow \text{Mod-}R_\tau)$$

be a duality. Then there exist two natural isomorphisms

$$\Delta_1 \cong \Gamma_R, \quad \Delta_2 \cong {}_R\Gamma.$$

PROOF. The functors $({}_R\Gamma \circ \Delta_1, \Delta_2 \circ \Gamma_R)$ give an equivalence of $\text{Mod-}R_\tau$ with itself.

4.9 COROLLARY. *Let R be a finite ring. Every autoequivalence of $\text{Mod-}R$ is naturally isomorphic to the identity.*

The next results will be useful in the proof of the main theorem.

4.10 LEMMA. *Let R_τ be a topological ring and let $(M_\lambda)_{\lambda \in \Lambda}$ be a family of discrete right modules over R_τ . Then the coproduct $\coprod_{\lambda} M_\lambda$ in $\mathcal{TM-}R_\tau$ is the direct sum endowed with the discrete topology.*

PROOF. It is easy to verify that $\bigoplus_{\lambda} M_\lambda$, endowed with the discrete topology, has the universal property of the coproduct.

4.11 LEMMA. *Let R_τ be a compact ring and let $M \in \mathcal{L-}R_\tau$:*

- (i) *M is compact if and only if it is a limit of finite modules in $\mathcal{L-}R_\tau$;*
- (ii) *M is discrete if and only if it is a colimit of finite modules in $\mathcal{L-}R_\tau$;*

PROOF. Obvious by 2.4 and 4.10.

We are now ready to state and prove our main theorem.

4.12 THEOREM. *Let R_τ be a compact ring and*

$$(\Delta_1: \mathfrak{L}\text{-}R_\tau \rightarrow R_\tau\text{-}\mathfrak{L}, \Delta_2: R_\tau\text{-}\mathfrak{L} \rightarrow \mathfrak{L}\text{-}R_\tau)$$

be a duality. Then the functors Δ_1 and Δ_2 take discrete modules to compact modules and vice versa, and there exists two natural isomorphisms

$$\Delta_1 \cong \Gamma_R, \quad \Delta_2 \cong {}_R\Gamma$$

(where Γ denotes Pontrjagin duality).

PROOF. Let $M \in \mathfrak{L}\text{-}R_\tau$ be a finite module. Then $\Delta_1(M) \in R_\tau\text{-}\mathfrak{L}$ has only a finite number of closed submodules and so it is discrete: recall that $\Delta_1(M)$ is linearly topologized and Hausdorff, so that the 0 submodule, which is closed, is the intersection of a finite number of open submodules and hence it is open.

If X is an infinite set, it follows from 4.9 that

$$\Delta_1(M^{(X)}) \cong \Delta_1(M)^X$$

is locally compact and so $\Delta_1(M)$ must be compact. Now $\Delta_1(M)$ is discrete and compact, hence finite.

Thus it follows easily from 4.11 that Δ_1 takes discrete modules to compact ones and vice versa. We have thus shown that (Δ_1, Δ_2) induces a duality between $\text{Mod-}R_\tau$ and $R_\tau\text{-CM}$, which, by 4.9, is equivalent to Pontrjagin duality.

If $M \in \mathfrak{L}\text{-}R_\tau$, it is obvious that M is the colimit in $\mathfrak{L}\text{-}R_\tau$ of the family $\mathbf{K}(M)$ of its compact submodules (2.4) and so

$$\Delta_1(M) = \Delta_1(\text{colim}_{L \in \mathbf{K}(M)} L) \cong \lim_{L \in \mathbf{K}(M)} \Delta_1(L) \cong \lim_{L \in \mathbf{K}(M)} \Gamma_R(L) \cong \Gamma_R(M).$$

5. The basic ring of a compact ring.

5.1 Dikranjan and Orsatti introduced in [DO₂] the notions of *basic* and *cobasic* ring of a linearly compact ring (recall that a topological ring R_τ is said to be *right linearly compact* if it is right l.t. and, given a family $(V_\lambda)_{\lambda \in \Lambda}$ of closed right ideals of R_τ and a family $(x_\lambda)_{\lambda \in \Lambda}$ of elements of R such that the family $(V_\lambda + x_\lambda)_{\lambda \in \Lambda}$ has the finite intersection property, then $\bigcap_\lambda (V_\lambda + x_\lambda) \neq \emptyset$).

Let R_τ be a right linearly compact ring: if W_R is the minimal injective cogenerator of $\text{Mod-}R_\tau$, the *cobasic ring* of R_τ is the left linearly compact ring

$$A_\sigma = \text{Hom}_R(W, W)$$

endowed with the topology of pointwise convergence.

The *basic ring* of R_τ is the cobasic ring of the cobasic ring of R_τ .

If the ring R_τ is *right strictly linearly compact* (shortened in *s.l.c.*), i.e. R_τ has a local basis consisting of right ideals I such that R/I is an artinian module, then both its basic and cobasic ring are s.l.c.

5.2 Menini and Orsatti [MO] extended the notion of basic ring to the locally artinian Grothendieck categories. We recall that a Grothendieck category is said to be *locally artinian* if it has a set of generators consisting of artinian objects. For example, if R_τ is a l.t. ring, then the Grothendieck category $\text{Mod-}R_\tau$ is locally artinian if and only if R_τ is strictly linearly compact.

Let \mathcal{A} be a locally artinian Grothendieck category: it can be proved that \mathcal{A} has a minimal injective cogenerator W ; put $A = \text{End}_{\mathcal{A}}(X)$. Consider next the class \mathcal{N} consisting of all subobjects of all finitely generated objects in \mathcal{A} ; if L is a subobject of $M \in \mathcal{A}$, we denote by L^\perp the canonical image of $\text{Hom}_{\mathcal{A}}(M/L, W)$ in $\text{Hom}_{\mathcal{A}}(M, W)$. The family of left ideals of A of the form L^\perp , as L runs through all the subobjects of W which belong to \mathcal{N} , is a local basis for a linearly compact topology σ on A .

The left l.t. ring A_σ is called the *cobasic ring* of \mathcal{A} .

Since A_σ is linearly compact, it has a cobasic ring in the sense of 5.1: this is called the *basic ring* of the category \mathcal{A} , and is denoted by $B(\mathcal{A})$. $B(\mathcal{A})$ is a strictly linearly compact ring [MO, Theorem 3.11].

We recall some results from [MO] which will allow us to show that the basic ring of the category $\text{Mod-}R_\tau$, where R_τ is a compact ring, coincides with the right basic ring of R_τ defined in 4.3.

5.3 THEOREM [MO, Theorem 3.12]. *Let \mathcal{A} be a locally artinian Grothendieck category and let $B_\beta = B(\mathcal{A})$ be its basic ring. Then there exists an equivalence between \mathcal{A} and $\text{Mod-}B_\beta$.*

5.4 THEOREM [MO, Theorem 3.13]. *Let \mathcal{A} and \mathcal{B} be locally artinian Grothendieck categories. Then \mathcal{A} and \mathcal{B} are equivalent if and only if their basic rings $B(\mathcal{A})$ and $B(\mathcal{B})$ are topologically isomorphic.*

5.5 PROPOSITION [MO, Theorem 3.11(d)]. *Let \mathcal{A} be a locally artinian Grothendieck category and denote by A_σ and B_β respectively the cobasic and the basic ring of \mathcal{A} . Then A_σ is (topologically isomorphic to) the cobasic ring of B_β .*

5.6 THEOREM. *Let R_τ be a compact ring. Then the rings $\mathbf{B}_r(R_\tau)$ and $\mathbf{B}(\text{Mod-}R_\tau)$ are topologically isomorphic.*

PROOF. It follows from 4.2 that $\mathbf{B}_r(R_\tau)$ is the cobasic ring of $R_\tau\text{-Mod}$. Hence $\mathbf{B}_r(R_\tau)$ coincides with its basic ring (5.5) and, since $\text{Mod-}R_\tau$ and $\text{Mod-}\mathbf{B}_r(R_\tau)$ are equivalent categories, we are done (5.4).

It is also obvious from the definition and 3.17 the following

5.7 PROPOSITION. *Let R_τ be a compact ring. Then there exists an idempotent e in R_τ such that $\mathbf{B}_r(R_\tau) \cong eRe$ with the induced topology.*

We end with some results about commutative rings; R_τ will be a fixed commutative compact ring.

5.8 COROLLARY. *Let R_τ be a compact commutative ring. Then R_τ is (left and right) basic.*

PROOF. If R_τ is commutative, then also $\mathbf{B}_r(R_\tau)$ is (5.7) and so, by [G, Corollary 7.2] and 5.3, $\mathbf{B}_r(R_\tau)$ is topologically isomorphic to R_τ .

5.9 PROPOSITION. *Let R_τ be a compact commutative ring and let \mathfrak{m} be a maximal open ideal of R_τ . If $x \neq 0$ is an element of the injective envelope $E_\tau(R/\mathfrak{m})$ of R/\mathfrak{m} in $\text{Mod-}R_\tau$, then $\text{Ann}_R(x)$ is an \mathfrak{m} -primary ideal.*

PROOF. It is exactly the same as in the case of noetherian rings (see, e.g. [SV]).

5.10 COROLLARY. *If S_1 and S_2 are simple modules in $\text{Mod-}R_\tau$ and there is a non zero morphism $f: E_\tau(S_1) \rightarrow E_\tau(S_2)$, then $S_1 \cong S_2$.*

PROOF. Let $\mathfrak{m}_1 = \text{Ann}_R(S_1)$ and $\mathfrak{m}_2 = \text{Ann}_R(S_2)$. If $x \in E_\tau(S_1)$ is such that $f(x) \neq 0$, then $\text{Ann}_R(x) \subseteq \text{Ann}_R(f(x)) \subseteq \mathfrak{m}_2$: Taking the radicals, we have

$$\mathfrak{m}_1 = \sqrt{\text{Ann}_R(x)} \subseteq \sqrt{\mathfrak{m}_2} = \mathfrak{m}_2.$$

5.11 The result above can be expressed in $CM-R_\tau$: if T_1 and T_2 are simple modules in $CM-R_\tau$ and there is a non zero continuous morphism $f: P_\tau(T_1) \rightarrow P_\tau(T_2)$, then $T_1 \cong T_2$.

In fact we have $\Gamma_R(P_\tau(T_i)) \cong E_\tau(\Gamma_R(T_i))$ ($i = 1, 2$).

We can use the preceding results to prove the following well-known

5.12 THEOREM. *Every commutative compact ring is topologically isomorphic to the topological product of (compact) local rings.*

PROOF. Let R_τ be a commutative compact ring and let $(T_\delta)_{\delta \in \Delta}$ be a complete system of representatives of the simple modules in $CM-R_\tau$. Then

$$\Pi_R = \prod_{\delta \in \Delta} P_\tau(T_\delta)$$

is the minimal projective generator of $CM-R_\tau$ and $\text{Chom}_R^u(\Pi, \Pi)$ is topologically isomorphic to R_τ (5.8). It follows from (5.11) that

$$\text{Chom}_R^u(\Pi, \Pi) \cong \prod_{\delta \in \Delta} \text{Chom}_R^u(P_\tau(T_\delta), P_\tau(T_\delta))$$

topologically. It is also obvious that the rings

$$\text{Chom}_R^u(P_\tau(T_\delta), P_\tau(T_\delta))$$

are compact and local.

This theorem was proved first by Kaplansky [K] without the assumption of identity and then by Zelinsky [Z] in the case of linearly compact rings. The proof given here is similar to that given in [DO₁].

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