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Rendiconti del Seminario Matematico della Università di Padova, tome 80 (1988), p. 127-137

<http://www.numdam.org/item?id=RSMUP_1988__80__127_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 80 (1988)

Postulation and Gonality for Projective Curves.

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We are interested in the interplay between intrinsic and projective properties of curves. In particular we are interested in the postulation of general k-gonal curves. A smooth curve $Y \subset \mathbf{P}^N$ is said to be canonical if $\mathcal{O}_Y(1) \cong K_Y$. A curve $Z \subset \mathbf{P}^N$ is said to have maximal rank if the restrictions maps $r_{Z,N}(k)$: $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(k)) \to H^0(Z, \mathcal{O}_Z(k))$ have maximal rank for all integers k. In [4] it was proved that for every $N > 3$, $g > N$, a general non-degenerate canonical curve with genus g in $\mathbf{P}^N$ has maximal rank.

Here we prove the following results (over $\mathbf{C}$).

THEOREM 1. For all integers $N > 3$, $g > N$, a general trigonal (resp. bielliptic) non-degenerate canonical curve of genus g in $\mathbf{P}^N$ has maximal rank.

THEOREM 2. For all integers N, d, g, with $g > N > 3$, $d > 2g$, a general embedding of degree d in $\mathbf{P}^N$ of a general hyperelliptic curve of genus g has maximal rank.

The proofs of theorem 1 and theorem 2 is modulo a smoothing result given in § 1, almost the same that the proofs in [4]; in particular in § 3 we omit the details which can be found in [4] or [3]. For § 1 we use the theory of admissible coverings ([7]) which is very useful to obtain results about general k-gonal curves by degeneration techniques. The inductive method of § 2, § 3, (the Horace's method) was introduced in [8].

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In § 4 we show that, by the theory of admissible covers, the results proved in [6] about the minimal free resolution of general canonical curves are true (with the same proof) for general k-gonal curves for suitable k.

0. Notations.

Let V be a variety (over $\mathbb{C}$) and S a closed subscheme of V; $\mathfrak{J}_{S,V}$ is the ideal sheaf of S in V and $N_{S,V}$ its normal sheaf (or normal bundle). Assume that we have fixed an embedding of V into $\mathbb{P}^k$, so that $\mathcal{O}_V(t)$ and $\mathcal{O}_S(t)$ are defined. Then $r_{S,V}(t)\colon H^0(V, \mathcal{O}_V(t)) \to H^0(S, \mathcal{O}_S(t))$ is the restriction map. If $V = \mathbb{P}^k$, we write often $\mathfrak{J}_{V,k}$, $r_{S,k}(t)$, $N_{S,k}$ instead of $\mathfrak{J}_{S,V}$, $r_{S,V}(t)$, $N_{S,V}$.

A curve $T \subset \mathbb{P}^N$ is called a bamboo of degree d if it is reduced, connected, with at most nodes as singularities, $\deg(T) = d$, its irreducible components are lines, and each line in T intersects at most two other irreducible components of T; equivalently, we may order the lines $L_1, \ldots, L_d$ of T so that $L_i \cap L_j \neq \emptyset$ iff and only if $|i-j| < 2$. A connected, reduced curve $X \subset \mathbb{P}^N$, $\deg(X) = 2d$, X with only ordinary nodes as singularities, is called a chain of d conics if its irreducible components $C_1, \ldots, C_d$ are conics, $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| < 2$, $\operatorname{card}(C_i \cap C_{i+1}) = 2$ if $0 < i < d$. Sometimes we will allow the reducibility of some of the conics C_i in a chain of conics; if C_i is reducible, we assume that every line of C_i intersects the adjacent conics. A line (resp. a conic) of a bamboo (resp. chain of conics) T is called final if it intersects at most another irreducible component of T.

We will write $((a;b))$ for the binomial coefficient; thus $((a;b)) := (a!)/((a-b)!\,b!)$. A triple of integers $(d, g; N)$ with $d \geq g + N$, $N > 2$, $g \geq 0$ has critical value k if k is the first integer $t > 0$ such that $td + 1 - g \leq ((N+t; N))$. We define integers $r(k,g,N)$, $q(k,g,N)$ by the following relations:

$$kr(k,g,N) + 1 - g - q(k,g,N) = ((N+k; N)), \quad 0 \leq q(k,N) < k$$

A smooth curve T of degree d and genus g in $\mathbb{P}^N$ with $r(k-1,g,N) < d \leq r(k,g,N)$ and $h^1(T, \mathcal{O}_T(1)) < 2 < k$, has critical value k (i.e. $(d,g;N)$ has critical value k); T has maximal rank if and only if $r_{T,N}(k-1)$ is injective and $r_{T,N}(k)$ is surjective (Castelnuovo-Mumford's Lemma).

We define integers $c(k, N)$, $e(k, N)$ by the following relations:

$$(2k-1)c(k, N) + 2 + e(k, N) = ((N + k; N)), \quad 0 \leqslant e(k, N) \leqslant 2k - 2 .$$

A chain F of $c(k, N)$ conics in $\mathbb{P}^N$ has critical value k; it has maximal rank if and only if $r_{F,N}(k-1)$ is injective and $h^0(\mathbb{P}^N, \mathcal{I}_{F,N}(k)) = e(k, N)$. Define integers $y(k, N)$, $k > 0$, $N > 2$, in the following way. Set $y(1, N) := c(1, N)$ and assume defined $y(k-1, N)$. Set $y(k, N) = y(k-1, N) + (c(k, N) - c(k-1, N)) + e$ with $e = -1$ if $e(k, N) < e(k-1, N)$, $e = 0$ otherwise. Hence $c(k, N) \geqslant y(k, N) > c(k, N) - k$. Define integers $x(k, N)$, $j(k, N)$ by the following relations:

$$(2k-1)x(k, N) + j(k, N) = ((N + k; N)), \quad 0 \leqslant j(k, N) \leqslant 2k - 2 .$$

A canonical curve C of degree d in $\mathbb{P}^N$ has critical value $k > 1$ if and only if $2x(k-1, N) + 2 \leqslant d \leqslant 2x(k, N)$. Set $y'(k, N) := y(k, N) - [(N+5)/3] - 1$.

A finite subset $S \subset \mathbb{P}^k$ is said to be in Linear general position if every subset W of S spans a linear space of dimension min $(k$, card $(W) - 1)$.

A bielliptic curve is a smooth, connected, complete curve with a degree two morphism onto an elliptic curve.

Let $Y = A \cup B \subset \mathbb{P}^3$, A chain of conics, B bamboo, A intersecting B at a unique point, P, and quasi-transversally, P belonging to a final line of B and a final conic of A. An irreducible component of Y is called free if it intersects only another irreducible component of Y.

1. A smooth, connected curve $E \subset \mathbb{P}^n$ is called canonical if $\mathcal{O}_E(1) \simeq K_E$. Let $C(g, n)$ be the closure in Hilb $(\mathbb{P}^n)$ of the set of smooth canonical curves of genus g in $\mathbb{P}^n$, and $C(g, n, k)$ (resp. $C(g, n;$ biel)) the closure in Hilb $(\mathbb{P}^n)$ of the set of smooth canonical curves of genus g in $\mathbb{P}^n$, which are k-gonal (resp. bielliptic) as abstract curves.

PROPOSITION 1.1. Fix a smooth canonical trigonal curve $C \subset \mathbb{P}^n$, deg $(C) = 2g - 2$, 2 points A, B in the same fiber of a g_3^1 on C, and a chain D of r conics in $\mathbb{P}^n$, D intersecting C quasi-transversally and exactly at A and B, A and B belonging to a final conic of D. Then $C \cup D \in C(g + r, n, 3)$.

PROOF. By definition $C(g + r, n, 3)$ is closed in Hilb $(\mathbb{P}^n)$. Hence we may assume C, A, D general. We know that $C \cup D \in C(g, n)$ ([5], [4], § 2).

First assume $r = 1$. Taking a projection, we may assume $n = g$, $C \cup D$ spanning $\mathbf{P}^n$, C spanning a hyperplane M, and $h^1(C, N_{C,M}) = 0$. We know that $h^1(C \cup D, N_{C\cup D,n}) = 0$ ([4], proof of 2.1), hence $C \cup D$ is a smooth point of Hilb $(\mathbf{P}^n)$. Since $C \cup D$ is semi-stable, we have a morphism h from a neighborhood of $C \cup D$ in Hilb $(\mathbf{P}^n)$ to the moduli scheme $\bar{M}_{g+1}$ of stable curves of genus $g + 1$ such that $h(C \cup D)$ is the curve C' obtained from C pinching together the points A, B. By the generality of C, A, we may assume Aut $(C') = \{1\}$, i.e. C' is a smooth point of $\bar{M}_{g+1}$. To obtain 1.1 for $r = 1$, it is sufficient to check that h is flat, hence open, at $C \cup D$. By the smoothness of Hilb $(\mathbf{P}^n)$ and $\bar{M}_{g+1}$ at the corresponding points, it is sufficient to check that the fiber $h^{-1}(C')$ has the right dimension $n^2 + 2n = \dim(\text{Aut}(\mathbf{P}^n))$ in a neiborhood of $C \cup D$. A priori near $C \cup Dh^{-1}(C')$ contains either curves abstractly isomorphic to C' (i.e. irreducible canonical stable curves) or curves isomorphic to $C \cup D$. The first type of curves has dimension $n^2 + 2n$. Since Pic $(C \cup D)$ has a 1-dimensional non-compact factor, we see easily that, up to projective transformations, there is exactly a one dimensional family of curves $C'' \cup D'' \cong C \cup D$. However for any 2 triples $\{E_i\}$, $\{F_i\}$, $i = 1, 2, 3$, of distinct points of D'', there is $m \in \text{Aut}(\mathbf{P}^n)$ with $m(E_i) = F_i$ for every i. Hence the stabilizer of any $C'' \cup D''$ in Aut $(\mathbf{P}^n)$ is one-dimensional, concluding the proof of the case $r = 1$. By induction on r, if $r > 1$ it is sufficient to prove the following claim stronger than the case $r = 1$ just proven.

Claim. Assume $r = 1$ and fix 2 general points E, F of D; then there is a flat family $X \to T$, T smooth irreducible affine curve, $X \subset T \times \mathbf{P}^n$, with $X_0 = C \cup D$, X_t smooth, canonical and trigonal for $t \in T$, $t \neq 0$, and a family m_t of 3-coverings, $m_t\colon X_t \to \mathbf{P}^1$ such that $m_0|C$ is the given g_3^1, $m_0|D$ sends A, B to one point of $\mathbf{P}^1$ and E, F to another point of $\mathbf{P}^1$.

By the theory of admissible coverings ([7]) there is a morphism $b\colon T \to \bar{M}_{g+1}$, with $b(0) = C'$, $b(t)$ a smooth 3-gonal curve for $t \neq 0$. By the first part of the proof we may assume $X_t = b(t)$. By [7], proof of th. 5(a), we may assume the existence of a family $m_t\colon X_t \to \mathbf{P}^1$, $t \in T \setminus \{0\}$, which, as t goes to 0, tends to an admissible covering m_0 with $m_0(A) = m_0(B)$, $m_0(E) = m_0(F)$. Taking a suitable fiber product, we obtain the claim. ■

PROPOSITION 1.2. Fix a canonical bielliptic curve $C \subset \mathbf{P}^n$, $\deg(C) = 2g - 2$, 2 points A, B on C with the same image under the 2 to 1 map of C to an elliptic curve, and a chain D of r conics in $\mathbf{P}^n$, D intersecting C quasi-transversally and exactly at A and B, A and B belonging to a final conic of D. Then $C \cup D \in C(g + r, n;$ biel).

PROOF. The proof of 1.1 works with two minor twists. We use the notations introduced in the proof of 1.1. Since Aut (C) and Aut (C') is not trivial, $\bar{M}_{g+1}$ is not smooth at C'. Instead of $\bar{M}_{g+1}$, we may however use (over $\mathbb{C}$) the Kuranishi local deformation space ov C' (or a suitable rigidification of $\bar{M}_{g+1}$). Instead of admissible coverings of $\mathbf{P}^1$, we have to use admissible 2-coverings of curves of arithmetic genus 1. Since these coverings are cyclic, there is no need here of a general theory. Take a 2-covering $c: C \to Z$, Z elliptic curve, with $c(A) = c(B)$ (hence c not ramified at A, B) and a 2-covering $d: D \to \mathbf{P}^1$ with $d(A) = d(B)$ (hence unramified at A and B). Take as $Z \cup \mathbf{P}^1$ the glueing of Z and $\mathbf{P}'$ along $c(A)$ and $d(E)$. Then c, d induce a 2-covering $u: C \cup D \to Z \cup \mathbf{P}^1$. Let $a_1, \ldots, a_{2g}$ be the ramification points of u, with $a_i \in Z$ if and only if $i \leqslant 2g - 2$. Take any flat family $s: W \to T$, with $W_0 = Z \cup \mathbf{P}^1$, W_t smooth elliptic for $t \neq 0$, and, in an etale neighborhood of 0, any $2g$ sections $s_1, \ldots, s_2g$ of s with $s_i(0) = a_i$: The divisor $s_1(t) + \ldots + s_{2g}(t)$ on W_t induces a cyclic 2-covering which tends to u when t goes to 0. ∎

Let $Z(d, g, n, k)$ (resp. $Z(d, g, n;$ biel)) be the closure in Hilb $(\mathbf{P}^n)$ of the set of smooth, connected, k-gonal (resp. bielliptic) curves of degree d, genus g, and with non special hyperplane section. $Z(d, g, n, k)$ and $Z(d, g, n;$ biel) are irreducible (and not empty if $d \geqslant g + n$). We have the following result.

PROPOSITION 1.3. Fix $C \subset \mathbf{P}^n$, $C \in Z(d, g, n, k)$ (resp. $Z(d, g, n;$ biel)), 2 smooth points A, B on C with the same image under an admissible covering of degree k (resp. a 2-to-1 covering of a curve of genus 1) of C, and a chain D of r conics in $\mathbf{P}^n$, D intersecting C quasi-transversally, and exactly at A, B, A and B belonging to the same final conic of D. Then $C \cup D \in Z(d + 2r, g + r, n, k)$ (resp. $Z(d + 2r, g + r,$ conic of D. Then $C \cup D \in Z(d + 2r, g + r, n, k)$ (resp. $Z(d + 2r,$ $g + r, n;$ biel)).

PROOF. It is much easier than the proof of 1.1, 1.2. Take any flat family of admissible covering, $(W \to T, U \to T, W \to U)$ with $W_0 = C \cup D$, $U_0 = Z \cup \mathbf{P}^1$, $p_a(Z) = 0$ (resp. 1) and $d + 2r$ sections w_i of T with $w_1(0) + \ldots + w_{d+2r}(0)$ an hyperplane section of W_0. Embedd W_t using $|w_1(t) + \ldots + w_{d+2g}(t)|$, and, if necessary, project the result in $\mathbf{P}^n$. ∎

We shall use often the following fact ([9], [2], 3.5). Fix a smooth curve $C \subset \mathbb{P}^n$, $\deg(C) = d$, and a smooth, connected curve D, $\deg(D) = r$, D intersecting quasi-transversally C and exactly at a point. Assume D rational and that the hyperplane section of C is non-special. Then $C \cup D$ is a limit of embeddings of degree $d + r$ of C into $\mathbb{P}^n$.

2. In this section we construct curves $Y = Z \cup T \subset \mathbb{P}^3$, with $Z \cap T = \emptyset$, Z chain of conics, T bamboo, and with good postulation. This construction will be used in the next section to prove theorems 1, 2 in $\mathbb{P}^4$.

LEMMA 2.1. *Fix non-negative integers a, b, r, s, and a smooth quadric Q in $\mathbb{P}^3$. Assume either (i) $a = b > 2$, or (ii) $a > b > 1$. Then there is (Y, Z, T) with $Y = Z \cup T \subset \mathbb{P}^3$, $Z \cap T = \emptyset$, Z chain of r conics, T bamboo of degree s, $\dim(Y \cap Q) = 0$, and $h^0(Q, \mathcal{I}_{Y \cap Q, Q}(a, b)) = \max(0, (a+1)(b+1) - 4r - 2s)$.*

PROOF. - From now on, we assume $s = 0$, the general case being similar (or use [1], 6.2). If $s = 0$, it is sufficient to prove 2.1 when $(a+1)(b+1) - 4 < 4r < (a+1)(b+1) + 4$. By the properness of Hilb $(\mathbb{P}^3)$, for any u and any plane H there is a scheme W, with $W_{\mathrm{red}} \subset H$, W_{red} with only ordinary double points, W reduce outside the singular locus of W_{red}, W limit of a family of chains of u conics. Just to fix the notations, we assume $a + b + 1 \equiv 3 \bmod (4)$, the remaining cases being similar. Fix 3 general planes M, N, R. Take limits W, X, D, respectively of chains of $[(a+b+1)/4]$, $[(a+b-1)/4]$, and 2, conics, with $W_{\mathrm{red}} \subset M$, $X_{\mathrm{red}} \subset X$, $D_{\mathrm{red}} \subset R$, $W \cup X \cup D$ intersecting transversally Q, $W \cup X \cup D$ limit of a family of chains of conics, $\mathrm{card}(D \cap Q \cap M) = 2$, $\mathrm{card}(D \cap Q \cap N) = 2$. $\mathrm{card}(X \cap Q \cap M) = 1$. Set

$$M' := M \cap Q\,, \quad N := N \cap Q\,, \quad A := (W \cup X \cup D) \cap Q\,.$$

Any forms of type (a, b) vanishing on A, vanishes on $a + b + 1$ points on M', hence on M'.

Any form of type $(a-1, b-1)$ vanishing on the points of $A \setminus (A \cap M')$, vanishes on $a + b - 1$ points of N', hence on N'. We reduce to an assertion about forms of type $(a-2, b-2)$ (in which we have to consider also the 4 points in $D \cap (Q \setminus (M' \cup N'))$). We continue in the same way, never adding curves intersecting $M' \cup N'$; for the next step we take R as first working plane. At the end in case (i) we reduce to the case $a = 3$ or 4 (plus a few points), in case (ii) to cases with $a < 4$. Consider for instance the case $a = b = 3$ and no point left from the previous construction (the worst case). Take 3 general smooth curves L, L', L'' of type $(1, 1)$ on Q, and let H, H', H'' the planes they span. The following chain $A \cup A' \cup A'' \cup B$ of conics solves our problem. A (resp. A') is a sufficiently general conic in H (resp. H'), A'' is a sufficiently general conic in H'' containing a point of L, B is a conic containing 2 points of L and a point of L', but not intersecting L''. ■

The aim of this section is the proof of the following lemma.

LEMMA 2.2. Fix integers n, x, with $n > 29$, $0 \leq a \leq 2n - 2$. There exists (Y, Z, T) with $Y = Z \cup T \subset \mathbf{P}^3$, $Z \cap T = \emptyset$, Z chain of conics, T bamboo of degree a, $r_{Y,3}(n)$ surjective, $\deg(Y) \geq r(n, 0, 3) - 9 - n/6$ (hence $\dim(\operatorname{Ker}(r_{Y,3}(n))) \leq r(n, 0, 3)/2 + (n^2/6) + 10n + 1$.

PROOF. Let s be the maximal integer with $s \leq n$, $s \equiv n \bmod (4)$, say $n = s + 4t$, $r(s, 0, 3) - 3 \leq (r(n, 0, 3) - a)/2 - (n - s)/3$. By [4], 3.1, there is a bamboo $E \in \mathbf{P}^3$, $\deg(E) = r(s, 0, 3) - 2$, with $r_{E,3}(s)$ surjective. If $a = 0$, set $F := E$, $T = \emptyset$. If $a > 0$, take $F \subset E$, $\deg(F) = \deg(E) - 1$, F union of two disjoint bamboos A, T, $\deg(T) = a$. Then we fix a smooth quadric Q and we apply the so called Horace's method (introduced in [8]) $2t$ times. At the odd (resp. even) steps we add in Q lines of type $(1, 0)$ (resp. $(0, 1)$). We order the lines $L_1, \ldots, L_{\deg(E)}$ of E in such a way that $L_i \cap L_j \neq \emptyset$ if and only if $|i - j| < 2$. Just to fix the notations we assume $s = 6k$, k integer (which, together with $s = 6k + 5$, is the worst case). We add in Q the union U of $4k + 2 = r(6k + 2, 0, 3) - (6k + 2, 0, 3) - 1$ lines A_i, $1 \leq i \leq 4k + 2$, of type $(1, 0)$ with A_i intersecting L_{2i-1} for every i. We claim that $r_{F \cup U,3}(6k + 2)$ is surjective for general F. To prove the claim, it is sufficient to find $S \subset \mathbf{P}^3$,

$$\operatorname{card}(S) = q(6k + 2, 0, 3) + (6k + 2)\left(r(6k + 2, 0, 3) - \deg(F \cup U)\right),$$

with $r_{F \cup U \cup S,3}(6k + 2)$ injective. Take $S = S' \cup S''$, with

$$\operatorname{card}(S') = q(6k + 2, 0, 3) + (6k)\left(r(6k + 2, 0, 3) - \deg(F)\right),$$

in $\mathbf{P}^3\setminus Q$, $S'' \cap S' = \emptyset$ S'' general in Q. Fix $f \in H_0(\mathbf{P}^3, \mathcal{J}_{F\cup U\cup S}(6k+2))$.

By 2.1 and the generality of S'', for general F we may assume $f|Q = 0$. Thus f is divided by the equation q of Q. Since f/q vanishes on $F \cup S'$, we have $f/q = 0$, hence the claim is proved. Then we deform the lines A_i to lines A_i' with the following rule. Let U' be the union of the lines A_i'. We assume that U' intersects transversally Q. A_i' intersects L_j if and only if A_i intersects L_i. A_1' intersects Q at a point on a line B_1 of type $(0, 1)$ intersecting L_2. Inductively, we impose that $A_j' \cap Q$, $j > 1$, has a point on a line B_j of type $(0, 1)$ intersecting L_{2j} and a point on the line B_{j-1} intersecting L_{2j-2} and A_{j-1}'. The lines B_i are called «good» secants to $F \cup U'$. Then we repeat the Horace's method in the following way. To $F \cup U'$ we add in $Q4k+4 = r(6k+4, 0, 3) - r(6k+2, 0, 3)$ lines B_i of type $(0, 1)$, among them the «good» secants previously constructed, B_{6k+3} linked to L_1, B_{6k+4} linked to $L_{\deg(F)}$: Then we continue (after smoothing the reducible conics obtained, if you prefer). In the step from $m-2$ to m we add to a curve $X(m-2)$ in $Qr(m, 0, 3) - r(m-2, 0, 3)$ lines if $q(m, 0, 3) \geqslant q(m-2, 0, 3)$, $r(m, 0, 3) - r(m-2, 0, 3) - 1$ lines if $q(m, 0, 3) < q(m-2, 0, 3)$ (i.e. if $m \equiv 0$ or $5 \bmod (6)$). In the odd step from $m-2$ to m we add a certain number, say x, of lines of type $(1, 0)$, and creates x «good» secants. In the even step from m to $m+2$ we add to $X(m)$ the x «good» secants creates in the previous step and one or two lines (always of type $(0, 1)$) linked either to a free conic of $X(m)$ or to a free line in a bamboo of $X(m)$. We use that after 6 steps the lines added in the 3 even steps are exactly 4 more than the lines added in the 3 odd steps. This explain the term «$-(n-s)/3$» in the choice of s. ∎

Consider the following assertion $T(n, a, b)$, defined for all integers n, a, b. $T(n, a, b)$: There is (Y, Z, T) with $Y = Z \cup T \subset \mathbf{P}^3$, $Z \cap T = \emptyset$, Z chain of a conics, T bamboo of degree b, and with $r_{Y,3}(n)$ surjective.

In the following section, for the proof of theorems 1, 2 we will need the assertion $T(n, a, b)$ for the values of n, a, b, listed in 2.3.

LEMMA 2.3. – The assertion $T(n, a, b)$ is true if (n, a, b) has one of the following values: (2, 1, 0), (2, 0, 2), (3, 2, 0), (3, 2, 1), (4, 3, 0), (4, 2, 2), (5, 4, 0), (5, 3, 2), (6, 4, 2), (6, 3, 4), (7, 2, 8), (7, 5, 3), (8, 3, 10), (8, 5, 5), (9, 9, 2), (9, 7, 4), (10, 8, 5), (10, 5, 12), (11, 9, 7), (11, 5, 14), (12, 11, 6), (12, 7, 15), (13, 14, 4), (14, 14, 8), (15, 16, 9), (16, 18, 9), (17, 22, 6), (18, 16, 24), (18, 22, 11), (19, 18, 24), (19, 25, 11), (20, 27, 12), (21, 20, 33), (21, 32, 8), (22, 33, 12), (22, 25, 29), (23, 35, 15), (23, 27, 30), (24, 29, 13), (24, 30, 32), (25, 45, 8), (26, 45, 15), (27, 48, 17), (28, 52, 16), (29, 59, 10), (30, 48, 41).

Sketch of proof. The cases with $n < 24$ can be done using several times the Horace's costruction applied not to quadrics but to planes: it is easier; if $n < 10$, we do not need any nilpotent, if $n > 9$ we use nilpotents as in [8]; only the cases with $n > 21$ are more difficult; however they can be handle also using quadrics as in the proof of 2.2. If $n > 23$, the proof of 2.2 works verbatim, and gives indeed stronger results; for (24, 39, 13) start taking in the proof of 2.2 $s = 12$; for the remaining (n, a, b) start from $s = n - 8$. ■

3. In this section we show how to modify the proofs in [4], to prove theorems 1, 2. The proof of the case « P^4 » given in [4], § 8, cannot be adapted, but the results proven here in § 2 are sufficient to prove this case and the inductive assertions of [4] needed for the proofs in $\mathbf{P}^N$, $N > 5$. We will use the numbers $y(k, N)$, ..., introduced in § 0.

Consider the following assertions:

$Y(k, N)$, $k > 0$, $n > 3$: there exists a chain Y of $y(k, N)$ smooth conics in $\mathbf{P}^N$ with $r_{Y,N}(k)$ surjective; if either $k > 6$ or $k > 2$, $N > 4$, or $N > 6$, there is such a Y which is contained in an integral hypersurface of degree k.

$Z(k, N)$, $k > 0$, $N > 3$: there exists a chain Y of $c(k, N) + k - 1$ smooth conics in $\mathbf{P}^N$ with $r_{Y,N}(k)$ injective.

$W(k, a, N, j)$, $k > 2$, $0 \leqslant a < 2k - 1$, $N > 3$, $1 < j < 2N + 3$: for every subset $S \subset \mathbf{P}^N$, card $(S) = j$, S in linear general position, and every $A, B \in S$, $A \neq B$, there is a curve $Y \subset \mathbf{P}^N$ such that:

(a) $Y \cap S = \{A, B\}$, $r_{Y \cup S,N}(k)$ is surjective and a general hypersurface of degree k containing $Y \cup S$ is irreducible;

(b) $Y = J \cup T$ with $J \cap T = \emptyset$; J is a chain of $y'(k, N) - a - 1$ conics; $T = \emptyset$ if $a = 0$; if $a > 0$, T is a bamboo of degree $2a$.

$H(k, N)$, $k > 0$, $N > 3$: there exists a curve $Y = Z \cup T \subset \mathbf{P}^N$ such that:

(a) Z is a canonical trigonal (resp. bielliptic) curve of degree $2(x(k, N) - j(k, N))$ and genus $x(k, N) - j(k, N) + 1$;

(b) T is a bamboo of degree $2j(k, N)$ intersecting Z exactly at a point, say P, and quasi-transversally; P is a point in a final line of T;

(c) $r_{Y,N}(k)$ is bijective.

$W(k, a, N, j)$ and $H(k, N)$ are slight modifications of the assertions of [4] with the same name. In [4] $Y(k, N)$ and $Z(k, N)$ were proved for $N > 4$. The same method (Horace's construction using a hyperplane) gives $Y(k, 4)$, $Z(k, 4)$, using 2.2 if $k > 30$ (plus a numerical lemma: « $2c(k, 4) - 2 > r(k, 0, 3)/2 + (k^2/6) + 10k$ if $k > 30$ » whose proof is left to the reader), using 2.3 if $k < 31$.

In the same way we get the « new » assertions $W(k, a, 4, j)$, $H(k, 4)$. Then the proofs of the « new » $W(k, a, N, j)$, $H(k, N)$, $N > 4$, are done by induction as in [4]; the cases with $N = 4$ simplify the discussion of the cases with low k for $N = 5$ given in [4], 6.4. Then theorem 1 is proved in $\mathbb{P}^N$, $N > 3$, in the same way the corresponding theorem is proved for $\mathbb{P}^N$, $N > 4$, in [4], end of § 7. The same proof works for theorem 0.2, although a simpler one could be done in this case, adding in a hyperplane irreducible hyperelliptic curves.

4. After this paper was typed, we read [6]. It is elementary to show how the results of [6], th. 4, 5, about sygygies of general canonical curves can be adapted to give results about syzygies of general k-gonal curves for suitable k. We have:

Proposition 4.1. Let X be a non-hyperlliptic k-gonal genus n curves. Assume $K_{p,2}(X) = 0$ for an integer p with $1 \leqslant p \leqslant n - 3$ $p \leqslant k - 3$. Then

(a) If C is a general k-gonal curve of genus $n + p + 1$, then $K_{p,2}(C) = 0$.

(b) If C is a general k-gonal curve of genus m, where $m \equiv n \bmod (p + 1)$ and $m \leqslant n$, then $K_{p,2}(C) = 0$.

For the proof of 4.1, it is sufficient to take in the proof of [6], th. 4, as divisor $q_1 + q_2 + \ldots + q_{p+2}$ a divisor contained in a g^1_k on X (strictly contained by the assumption $k > p + 2$) and as smoothing of $X \cup Y$ an admissible k-cover. Then from 4.1 we get verbatim the following improved version of [6], th. 5:

PROPOSITION 4.2. Let C be a general k-gonal curve of genus g.

(a) $K_{2,2}(C) = 0$ if $g \geqslant 7$ and either $k > 5$ or $g \equiv 1, 2 \bmod (3)$ and $k = 5$.

(b) $K_{3,2}(5) = 0$ if $g \geqslant 9$ and either $k > 6$ or $g \equiv 1, 2 \bmod (4)$ and $k = 6$.

(c) $K_{4,2}(C) = 0$ if $g \geqslant 11$, $k > 6$, and $g \equiv 1, 2 \bmod (5)$.

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Manoscritto pervenuto in redazione il 16 ottobre 1987.