

# RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

DIETER HELD

## **The situation of $Sp_4(4) \cdot 2$ in the sporadic simple group He**

*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 80 (1988), p. 117-125

[http://www.numdam.org/item?id=RSMUP\\_1988\\_\\_80\\_\\_117\\_0](http://www.numdam.org/item?id=RSMUP_1988__80__117_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1988, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

## The Situation of $Sp_4(4) \cdot 2$ in the Sporadic Simple Group He

DIETER HELD (\*)

### 0. Introduction.

The objective of this paper is to prove the following result:

**THEOREM.** The simple group He contains an extension of  $Sp_4(4)$  by the field automorphism as a maximal subgroup.

**REMARK.** Recently, researchers have shown increasing interest in the study of geometries for the sporadic simple groups; see for instance [4]. For such studies, a solid information about the maximal subgroups of the group in question is mandatory. Also for the purpose of existence proofs by coset enumeration methods one needs detailed knowledge of the situation of these subgroups. Although, the assertion of the theorem is contained in an unpublished article by W. Gemmer [3], the proof, there, is not completely without flaws. Therefore, and because the content of the theorem has been used by various authors, it seems worthwhile to present a direct, short and conceptual argument for it.

In what follows, we shall denote He by  $G$ , and shall make use of the information about the local structure of  $G$  contained in [6]. By  $Sp_4(4)$  we denote the projective symplectic 4-dimensional group over  $GF(4)$ .

(\*) Indirizzo dell'A.: Fachbereich Mathematik der Universität, D-6500 Mainz, Rep. Fed. Tedesca.

### 1. A $(B, N)$ -pair of type $Sp_4(4)$ .

The Sylow 2-subgroup  $T$  of  $G$  contains precisely two elementary abelian subgroups  $R_1$  and  $R_2$  of order  $2^6$ . These two subgroups are not conjugate in  $G$ . We have that  $N(R_i)$  splits over  $R_i$ , and that a complement  $M_i$  of  $R_i$  in  $N(R_i)$  has the following properties:  $|\mathbf{O}(M_i)| = 3$ ,  $\mathbf{O}(M_i) \subseteq M'_i$ ,  $M'_i/\mathbf{O}(M_i) \cong A_6$ ,  $M_i/\mathbf{O}(M_i) \cong \Sigma_6$ , and  $\mathbf{C}_{N(R_i)}(\mathbf{O}(M_i)) = M'_i$ . Put  $\langle h_i \rangle = \mathbf{O}(M_i)$ . Then,  $h_i$  acts fixed-point-free on  $R_i$ . It is known that  $R_1 R_2 \subseteq R_i M'_i$ . Thus,  $R_1 R_2 \langle h_1, h_2 \rangle$  is a group of order  $2^8 \cdot 3^2$ . Since  $[h_1, h_2] \in R_1 \cap R_2$ , we may conjugate  $h_2$  and  $M_2$  by an element of  $R_1 \cap R_2$  to get that  $H = \langle h_1, h_2 \rangle$  is an elementary abelian group of order 9. Clearly,  $R_1 R_2 H$  is 2-closed and  $\mathbf{C}_{R_1 R_2}(H) = \langle 1 \rangle$ , since  $h_i$  acts fixed-point-free on  $R_i$  for  $i = 1, 2$ ; note that an involution of  $R_1 R_2$  lies in  $R_1 \cup R_2$ .

Put  $Z = R_1 \cap R_2 = \langle z_1, z_3, \pi, \mu\tau \rangle$ . In  $Z$  there are precisely two four-subgroups consisting of non-central involutions. These are  $Z_1 = \langle z_3\pi, z_1\mu\tau \rangle$  and  $Z_2 = \langle z_1z_3\pi, z_3\mu\tau \rangle$ . Note that  $Z$  possesses precisely 9 central involutions of  $G$ . Clearly,  $H$  normalizes non-trivially  $Z_1$  and  $Z_2$ . Since  $h_1, h_1^2, h_2, h_2^2$  act fixed-point-free on  $Z$  and are all 3-central, we get that all elements of  $H^{\sharp}$  are 3-central in  $G$ ; note that an element of order 3 of  $G$  does not centralize an elementary abelian subgroup of order 8 in  $G$ . It is important to observe that  $Z_1$  and  $Z_2$  are conjugate by the involution  $z_2 \in T$  and that

$$Z_1 \langle z_2 \tau' \rangle \cong Z_2 \langle z_2 \tau' \rangle \cong D_8.$$

Since  $H$  normalizes  $R_1$  and  $R_2$ , there is an  $H$ -invariant complement  $W_2$  of  $Z$  in  $R_1$  and an  $H$ -invariant complement  $W_1$  of  $Z$  in  $R_2$  by Maschke's theorem. Compute  $[h_2, W_2] \subseteq W_2 \cap R_2 \subseteq R_1 \cap R_2 \cap W_2 = \langle 1 \rangle$  and  $[h_1, W_1] \subseteq W_1 \cap R_1 \subseteq R_1 \cap R_2 \cap W_1 = \langle 1 \rangle$ . We get

$$R_1 R_2 H = R_1(\langle h_1 \rangle \times W_1 \langle h_2 \rangle) \quad \text{and} \quad R_1 R_2 H = R_2(\langle h_2 \rangle \times W_2 \langle h_1 \rangle).$$

Since  $h_i$  acts fixed-point-free on  $R_i$ , we see that  $W_1 \langle h_2 \rangle$  and  $W_2 \langle h_1 \rangle$  are both isomorphic to  $A_4$ .

Now,  $\langle h_1, h_2 \rangle$  acts on  $Z_i$  as an automorphism group of order 3. Hence,  $h_1 h_2$  or  $h_1 h_2^2$  centralizes  $Z_1$ . Interchanging  $h_2$  and  $h_2^2$  if necessary, we may and shall assume that  $[h_1 h_2, Z_1] = \langle 1 \rangle$ . It follows that  $[h_1 h_2^2, Z_2] = \langle 1 \rangle$ . Hence,  $Z_1 \langle h_1 h_2^2 \rangle$  and  $Z_2 \langle h_1 h_2 \rangle$  are both isomorphic

to  $A_4$ . We have obtained

$$\mathbf{C}(Z_1) \supseteq R_1 R_2 \langle h_1 h_2 \rangle \quad \text{and} \quad \mathbf{C}(Z_2) \supseteq R_1 R_2 \langle h_1 h_2^2 \rangle.$$

Clearly,

$$R_1 R_2 \langle h_1 h_2 \rangle = R_1 W_1 \langle h_1 h_2 \rangle \quad \text{and} \quad R_1 R_2 \langle h_1 h_2^2 \rangle = R_2 W_2 \langle h_1 h_2^2 \rangle.$$

Remember that  $Z_1^{z_2} = Z_2$ . We apply now [6; Lemma (3.4)] and get that  $\mathbf{C}(Z_i) \cap \mathbf{N}(R_i)$  is a splitting extension of  $R_i$  by a group isomorphic to  $A_5$ . We put

$$B_1 = \mathbf{C}(Z_1) \cap \mathbf{N}(R_1) \quad \text{and} \quad B_2 = \mathbf{C}(Z_2) \cap \mathbf{N}(R_2).$$

Thus,

$$R_1 W_1 \langle h_1 h_2 \rangle \subseteq B_1 \quad \text{and} \quad R_2 W_2 \langle h_1 h_2^2 \rangle \subseteq B_2.$$

Consider the subgroup  $P_i = B_i \langle h_i \rangle$ . Clearly,  $|P_i| = 2^8 3^2 5$ , and by a result of W. Gaschütz, we get that  $P_i = R_i \langle \tilde{h}_i \rangle \times \tilde{L}_i$ , where  $o(\tilde{h}_i) = 3$  and  $\tilde{L}_i \cong A_5$  for  $i = 1, 2$ . From earlier information,  $\tilde{h}_i$  acts fixed-point-free on  $R_i$  and  $\mathbf{O}(\mathbf{N}(R_i)/R_i) = R_i \langle \tilde{h}_i \rangle / R_i$ . The group generated by all 2-elements and all 5-elements of  $P_i$  is equal to  $R_i \tilde{L}_i$ . It follows  $R_i \tilde{L}_i = B_i$ . We conjugate  $\langle \tilde{h}_i \rangle$  onto  $\langle h_i \rangle$  by an element  $x \in R_i$  so that  $h_i$  centralizes  $\tilde{L}_i^x = L_i$ . Then,

$$R_i L_i = B_i \quad \text{and} \quad P_i = R_i \langle h_i \rangle \times L_i.$$

We know that  $h_1$  centralizes  $W_1 \langle h_1 h_2 \rangle$  and that  $W_1 \langle h_1 h_2 \rangle \subseteq \mathbf{N}(R_1) \cap \mathbf{C}(Z_1)$ . Clearly,  $W_1 \langle h_1 h_2 \rangle \subseteq \mathbf{C}(h_1) \cap B_1 = L_1$ . Analogously,

$$W_2 \langle h_1 h_2^2 \rangle \subseteq \mathbf{C}(h_2) \cap B_2 = L_2.$$

Remember that  $L_i \cong A_5$ . There is an involution  $r_i$  in  $L_i$  such that

$$(h_1 h_2)^{r_1} = h_1^2 h_2^2 \quad \text{and} \quad (h_1 h_2^2)^{r_2} = h_1^2 h_2.$$

By choice, we have  $[h_1, r_1] = [h_2, r_2] = 1$ . We get that  $\langle r_1, r_2 \rangle \subseteq \mathbf{N}(H)$ .

Let us describe the action of  $\langle r_1, r_2 \rangle$  on  $H$ :

$$r_1: h_1 \rightarrow h_1, h_1 h_2 \rightarrow h_1^2 h_2^2, \quad r_2: h_2 \rightarrow h_2, h_1 h_2^2 \rightarrow h_1^2 h_2.$$

Thus,

$$r_2: h_1 \rightarrow h_1^2 h_2^2, \quad r_1: h_2 \rightarrow h_1 h_2^2.$$

It follows

$$r_1 r_2: h_1 \rightarrow h_1^2 h_2^2 \rightarrow h_1^2 \rightarrow h_1 h_2 \rightarrow h_1, \quad r_1 r_2: h_2 \rightarrow h_1^2 h_2 \rightarrow h_2^2 \rightarrow h_1 h_2^2 \rightarrow h_2.$$

Put  $N = H\langle r_1, r_2 \rangle$ . Since  $r_2 r_1: h_1 \rightarrow h_1 h_2$ , we see that  $\langle r_1, r_2 \rangle$  induces a non-abelian automorphism group of  $H$  and  $(r_1 r_2)^4 \in \mathbf{C}(H)$ . By a result of [2], we have  $\mathbf{C}(H) = H \times \hat{V}$ , where  $\hat{V}$  is a four-group having only non-central involutions and  $\mathbf{C}(\hat{V})/\hat{V} \cong L_3(4)$ . Since elements of order 8 in  $G$  are roots of central involutions of  $G$ , we see that  $r_1 r_2$  cannot have order 8. The case  $o(r_1 r_2) = 2^3 3$  cannot occur, since in  $G$  there are no elements of order 24. It follows that  $(r_1 r_2)^4 \in H$ . This implies that  $N/H \cong D_8$  and  $\mathbf{C}_N(H) = H$ . In what follows we put  $N/H = W$ ; note that  $W$  is the Weyl-group and  $H$  is a Cartan-subgroup of  $Sp_4(4)$ . Compute:

$$\begin{aligned} r_1: Z_2 \langle h_1 h_2 \rangle &\rightarrow Z_2^{r_1} \langle h_1^2 h_2^2 \rangle = Z_2^1 \langle h_1 h_2 \rangle, \\ r_2: Z_1 \langle h_1 h_2^2 \rangle &\rightarrow Z_1^{r_2} \langle h_1^2 h_2 \rangle = Z_1^2 \langle h_1 h_2^2 \rangle. \end{aligned}$$

We get further:

$$r_1: [Z_2^{r_1}, h_2] \rightarrow [Z_2, h_1 h_2^2] = \langle 1 \rangle, \quad r_2: [Z_1^{r_2}, h_1] \rightarrow [Z_1, h_1^2 h_2^2] = \langle 1 \rangle.$$

It follows

$$Z_2^{r_1} \subseteq \mathbf{C}(h_2) \cap R_1, \quad Z_1^{r_2} \subseteq \mathbf{C}(h_1) \cap R_2,$$

since  $r_i \in \mathbf{N}(R_i)$  and  $R_1 \cap R_2 = Z_1 \times Z_2$ . From the structures of the centralizers of elements of order 3 in  $G$  we get

$$Z_2^{r_1} = W_2 \quad \text{and} \quad Z_1^{r_2} = W_1.$$

If  $r_1$  would lie in  $\mathbf{N}(R_2)$ , then  $r_1$  would normalize  $R_1 R_2$  and also  $R_1 \cap R_2 = Z$  which is against  $Z \cap Z_2^{r_1} = Z \cap W_2 = \langle 1 \rangle$ . Thus,  $r_1 \notin \mathbf{N}(R_2)$ . Analogously, we get  $r_2 \notin \mathbf{N}(R_1)$  and  $Z \cap Z_1^{r_2} = Z \cap W_1 = \langle 1 \rangle$ . One

obtaines

$$R_1 = W_1^{r_1} W_2^{r_1} W_2 \quad \text{and} \quad R_2 = W_1^{r_1} W_2^{r_1} W_1.$$

Amongst other facts, we have proved the following result:

The subgroups  $W_1^{r_1}(=Z_1)$  and  $W_2^{r_1}(=Z_2)$  lie in  $Z = R_1 \cap R_2$ . Further,  $[W_1^{r_1}, r_1] = \langle 1 \rangle$  and  $[W_2^{r_1}, r_2] = \langle 1 \rangle$ . One has  $R_1 R_2 = W_1^{r_1} W_2^{r_1} W_2 W_1$ ; here  $R_1 = W_1^{r_1} W_2^{r_1} W_2$  and  $R_2 = W_1^{r_1} W_2^{r_1} W_1$ . Also  $N = H \langle r_1, r_2 \rangle$ , where  $N/H = W \cong D_8$ ,  $C_N(H) = H$ .

Put  $B = R_1 R_2 H$ . We shall show that the following conditions of [8] are satisfied:

(i')  $B \cup B r_i B$  is a subgroup of  $G$  for  $i = 1, 2$ ;

(iv) If  $l(r_i w) \geq l(w)$  for some  $w \in W$  in the generators  $r_1, r_2$  then  $B^w \subseteq B^{r_i w} B$  for  $i = 1, 2$ .

We have  $B^{r_1} \cap B = R_1 H$ ; note that  $H$  acts transitively on  $R_1 R_2 / R_1$  and that  $r_1 \notin N(B)$  as  $r_1 \notin N(R_1 R_2)$  and  $O_2(B) = R_1 R_2$ . Similarly, one gets  $B^{r_2} \cap B = R_2 H$ . Therefore, the number of left cosets of  $B$  in  $B r_i B$  is equal to  $[B : B \cap B^{r_i}] = 2^8 3^2 : 2^6 3^2 = 4$ . It follows that  $B \cup B r_i B = P_i = R_i (\langle h_i \rangle \times L_i)$  for  $i = 1, 2$ . We have shown that (i') holds.

Note that  $w \in \{r_2, r_2 r_1, r_2 r_1 r_2\}$  if  $i = 1$ , and that  $w \in \{r_1, r_1 r_2, r_1 r_2 r_1\}$  if  $i = 2$ .

Case 1. Here,  $i = 1$  and  $w = r_2$ . Compute:

$$B = W_1^{r_1} W_2^{r_1} W_1 W_2 H, \quad B^{r_2} = W_1 W_2^{r_1} W_1^{r_1} W_2^{r_1} H, \\ B^{r_1 r_2} = W_1 W_2^{r_1} W_1^{r_1 r_2} W_2^{r_1} H.$$

It follows  $B^{r_2} \subseteq B^{r_1 r_2} B$ .

Case 2. Here,  $i = 1$  and  $w = r_2 r_1$ . Compute:

$$B^{r_2 r_1} = W_1^{r_1} W_2 W_1^{r_1} W_2^{r_1 r_1} H, \quad B^{r_1 r_2 r_1} = W_1^{r_1} W_2^{r_1 r_1} W_1^{r_1 r_2 r_1} W_2 H.$$

It follows  $B^{r_2 r_1} \subseteq B^{r_1 r_2 r_1} B$ .

Case 3. Here,  $i = 1$  and  $w = r_2 r_1 r_2$ . Compute:

$$B^{r_2 r_1 r_2} = W_1^{r_1 r_2} W_2^{r_1} W_1 W_2^{r_2 r_1 r_2} H, \quad B^{r_1 r_2 r_1 r_2} = W_1^{r_1 r_2} W_2^{r_2 r_1 r_2} W_1^{r_1 r_2 r_1 r_2} W_2^{r_1} H.$$

It follows  $B^{r_1 r_2} \subseteq B^{r_1 r_2 r_1} B$ .

*Case 4.* Here,  $i = 2$  and  $w = r_1$ . Compute:

$$B^{r_1} = W_1^r W_2 W_1^r W_2^r H, \quad B^{r_1 r_1} = W_1^r W_2 W_1^r W_2^r H.$$

It follows  $B^{r_1} \subseteq B^{r_1 r_1} B$ .

*Case 5.* Here,  $i = 2$  and  $w = r_1 r_2$ . Compute:

$$B^{r_1 r_2} = W_1 W_2^r W_1^r W_2^r H, \quad B^{r_1 r_2 r_1} = W_1^r W_2^r W_1 W_2^r H.$$

It follows  $B^{r_1 r_2} \subseteq B^{r_1 r_2 r_1} B$ .

*Case 6.* Here,  $i = 2$  and  $w = r_1 r_2 r_1$ : Compute:

$$B^{r_1 r_2 r_1} = W_1^r W_2^r W_1^r W_2^r H, \quad B^{r_1 r_2 r_1 r_1} = W_1^r W_2^r W_1^r W_2^r H.$$

It follows  $B^{r_1 r_2 r_1} \subseteq B^{r_1 r_2 r_1 r_1} B$ .

Application of [8] yields that  $U = BNB$  is a subgroup of  $G$ . Note that  $B \cap N = H$ , since  $C_B(H) = H$  and  $B$  is 2-closed. A further application of [8] yields that  $w = w'$  is forced by  $BwB = Bw'B$  for  $w, w' \in W$ . It follows

$$|U| = \sum |BwB| = |B| \sum [B: B^w \cap B],$$

where  $w$  runs through all elements of  $W$ .

Note that  $W \cong D_8$ , and so, we have to compute  $[B: B^w \cap B]$  for the eight elements of  $W$ . Obviously,  $[B: B^1 \cap B] = 1$ . Since  $C(H) \cap R_1 R_2 = \langle 1 \rangle$ , we get that an  $H$ -invariant subgroup of  $R_1 R_2$  has order  $2^i$ , where  $i$  is even; see [5; V.3.15]. We know already that  $[B: B^{r_1} \cap B] = 2^2$ . It is clear that  $B^w \cap B$  is normalized by  $H$ , since  $H^w = H$  for each  $w \in W$ . Thus,  $[B: B^w \cap B] = 2^i$  with  $i$  even. For the order of  $U$  we get the following equation:

$$|U| = 2^8 3^2 (1 + 2^2 + 2^2 + 2^{i_4} + \dots + 2^{i_8}),$$

where  $0 \leq i_j \leq 8$  as  $|R_1 R_2| = 2^8$ . By our above remark, we have that  $i_j$  is even.

We had proved earlier that  $r_i \notin N(B)$  for  $i = 1, 2$ . It follows together with [8] that the conditions of definition (3.24) of [7] are satisfied. In particular, theorem (3.28) of [7] holds. This implies that

$N_G(P) = P$  for any subgroup  $P$  with  $U \supset P \supset B$ . Hence,  $N_G(B) = B$ , and so  $B^w \cap B \subset B$  for any  $w \in W^f$ . It follows  $2^{i_j} \geq 4$  for  $j \in \{4, \dots, 8\}$ ; remember that  $i_j$  is even.

Put  $n = (1 + 4 + 4 + 2^{i_4} + \dots + 2^{i_8})$ . Then,  $|U| = 2^8 3^2 \cdot n$ . We know that 5 divides  $n$ . Since  $|U|$  divides  $|G|$ , we get that  $n$  divides  $3 \cdot 5^2 \cdot 7^3 \cdot 17 = 437\,325$ . A little computer program gives the solution:

$$i_4 = 4, \quad i_5 = 4, \quad i_6 = 6, \quad i_7 = 6, \quad i_8 = 8.$$

Thus

$$|U| = 2^8 3^2 \cdot 425 = 2^8 3^2 5^2 17 = |Sp_4(4)|.$$

We want to show that  $U \cong Sp_4(4)$ . By the result of [1] it is enough to prove that  $U$  is a simple group all of whose involution centralizers are 2-constrained. Since in  $G$  there is no odd order subgroup normalized by  $R_1 R_2$ , we get that  $O(U) = \langle 1 \rangle$ . Since an element of order 17 does not centralize an involution of  $G$ , we get that  $O_2(U) = \langle 1 \rangle$ . Denote by  $K$  a minimal normal subgroup of  $U$ . Then, 17 divides  $|K|$ , since otherwise we would get a contradiction to the fact that a Sylow 17-subgroup is selfcentralizing; note that  $K$  cannot be solvable. Since  $17^2$  does not divide  $|G|$ , we get that  $K$  is a simple group.

Denote by  $S$  a Sylow 17-subgroup of  $K$ . Then, by Frattini's argument, we have  $U = K \cdot N_G(S)$ . Since  $K$  is simple and non-abelian, we must have  $N_G(S) \supset S$  by a transfer lemma due to Burnside. Hence,  $[U : K]$  divides 4, since  $[N_G(S) : S] = 8$ . But  $R_1 R_2$  has no elements of order 8. This forces  $[U : K] \in \{1, 2\}$ . Assume that  $[U : K] = 2$ . Then, we have  $|K| = 2^7 3^2 5^2 17$ . Let  $R$  be a Sylow 2-subgroup of  $K$  which lies in  $R_1 R_2$ . Evidently,  $K \cap R_1 R_2 = R$ ,  $|R| = 2^7$ , and so,  $R$  is normalized by  $H$  which—however—is against  $C(H) \cap R_1 R_2 = \langle 1 \rangle$ . It follows  $[U : K] = 1$ , and this implies that  $U$  is a simple group.

We have still to show that the centralizer of every involution of  $U$  is 2-constrained using easily accessible information from [6]. From the structure of  $C_G(z_1)$ , we get  $C_U(z_1) = R_1 R_2$ ; obviously  $R_1 R_2$  is 2-constrained. In  $G$ , the involutions  $z_3 \pi$  and  $z_1 z_3 \pi$  are conjugate via  $z_2$ . We know that  $C_G(z_3 \pi) \cap N_G(R_1) = B_1$  and that  $C_G(z_1 z_3 \pi) \cap N_G(R_2) = B_2$ ; also  $|B_i| = 2^8 \cdot 3 \cdot 5$ . From the order of  $U$  and the structure of  $C_G(z_3 \pi)$ , we get that  $C_U(z_3 \pi) = B_1$  and  $C_U(z_1 z_3 \pi) = B_2$ . Clearly,  $B_i$  is 2-constrained for  $i = 1, 2$ . If  $|C_G(z_1 z_3 \pi) \cap N_G(R_1)| = 2^8 \cdot 3 \cdot 5$ , then we would get  $B_1 = B_2$ , since  $B_2$  would be the unique normal subgroup of index 3 in  $N_G(R_1)$ . But this is not possible, as 5



does not divide  $|\mathbf{C}_G(z_1)|$ . Thus,  $|\mathbf{C}(z_1 z_3 \pi) \cap \mathbf{N}_U(R_1)| = 2^3 \cdot 3$ . It follows that there are precisely three  $U$ -classes of involutions passing through  $R_1$ . Representatives for these classes are  $z_1$ ,  $z_3 \pi$ , and  $z_1 z_3 \pi$ . The same result holds for the  $U$ -classes of involutions passing through  $R_2$ . Since every involution of  $U$  is conjugate in  $U$  to an involution of  $R_1 \cup R_2$ , we have obtained that the centralizer in  $U$  of every involution of  $U$  is 2-constrained. Moreover, one observes that  $U$  has precisely three classes of involutions. We have proved that  $U \cong Sp_4(4)$ .

## 2. The maximal subgroup isomorphic to $Sp_4(4)$ extended by the field-automorphism.

In the last section we have proved the existence of a subgroup  $U$  of  $G$  such that  $U \cong Sp_4(4)$ . We know that  $U$  is generated by the subgroups  $P_1$  and  $P_2$  of  $U$ , as  $r_i \in P_i$  for  $i = 1, 2$ , and also  $B \subseteq P_1 \cap P_2$ . Consider the element  $z_2 \tau'$ . Clearly,  $z_2 \tau'$  normalizes  $R_1 R_2$  but is not contained in  $R_1 R_2$ . We know also that  $Z_1 \langle z_2 \tau' \rangle$  and  $Z_2 \langle z_2 \tau' \rangle$  are both isomorphic to  $D_8$ . Thus,  $\langle B_1, z_2 \tau' \rangle$  is equal to  $\mathbf{C}(z_3 \pi) \cap \mathbf{N}_G(R_1)$  and  $\langle B_2, z_2 \tau' \rangle$  is equal to  $\mathbf{C}(z_1 z_3 \pi) \cap \mathbf{N}_G(R_2)$ . Application of [6; Lemma (3.2)] yields that  $\langle B_i, z_2 \tau' \rangle$  is a splitting extension of  $R_i$  by  $\Sigma_5$ ; in particular, we get that  $z_2 \tau'$  normalizes  $B_i$  for  $i = 1, 2$ . Clearly,  $[h_1, z_2 \tau'] \in R_1 \langle h_1 \rangle$  and  $[h_2, z_2 \tau'] \in R_2 \langle h_2 \rangle$ . It follows that both  $P_1$  and  $P_2$  are normalized by  $z_2 \tau'$ . Therefore,  $U \langle z_2 \tau' \rangle$  is a subgroup of  $G$  which obviously is an extension of  $Sp_4(4)$  by the field-automorphism.

Put  $G_0 = U \langle z_2 \tau' \rangle$ . We show finally that  $G_0$  is a maximal subgroup of  $G$ . Sylow's theorem gives that  $|\mathbf{N}_{G_0}(S)| = 2^3 \cdot 17$ , where  $S$  is a Sylow 17-subgroup of  $U$ . Note that  $\mathbf{N}_G(S)$  is the full normalizer of  $S$  in  $G$ . Let  $M$  be a subgroup of  $G$  such that  $G \supseteq M \supset G_0$ . Then,  $[M : G_0]$  is equal to a divisor  $t$  of  $2 \cdot 3 \cdot 7^3$  such that  $t \equiv 1 \pmod{17}$ . The only possibility is  $t = 2058$ . Thus,  $M = G$  and  $G_0$  is a maximal subgroup of  $G$ . The theorem is proved.

## REFERENCES

- [1] B. BEISIEGEL, *Über einfache endliche Gruppen mit Sylow 2-Gruppen der Ordnung höchstens  $2^{10}$* , Commun. Algebra, **5** (1977), pp. 113-170.
- [2] G. BUTLER, *The maximal subgroups of the sporadic simple group of Held*, J. Alg., **69** (1980), pp. 67-81.

- [3] W. GEMMER, *Eine Kennzeichnung der einfachen Gruppe von Held*, Diplomarbeit, Universität Mainz, 1974.
- [4] E. GERNER, *Ein Apartment für eine Geometrie der sporadischen einfachen Gruppe He*, J. Algebra, to appear.
- [5] D. GORENSTEIN, *Finite groups*, Harper and Row, New York, 1968.
- [6] D. HELD, *The simple groups related to  $M_{24}$* , J. Alg., **13** (1969), pp. 253-296.
- [7] M. SUZUKI, *Group theory I*, Springer-Verlag, Berlin, 1982.
- [8] J. TITS, *Théorème de Bruhat et sous-groupes paraboliques*, C. R. Acad. Sci. Paris, **254** (1962), pp. 2910-2912.

Manoscritto pervenuto in redazione il 21 agosto 1987.