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Some Sporadic Groups as Galois Groups.

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In recent years a large number of finite simple groups have been proved to be Galois groups over the field $\mathbb{Q}(t)$ or $\mathbb{Q}^{ab}(t)$ of rational functions over $\mathbb{Q}$ or its maximal abelian extension $\mathbb{Q}^{ab}$ (see e.g. [1], [4], [5], [7], [8], [11]). In [6] Matzat studies the problem for composite groups (see also [7]). He defines a GAR-realization of a finite group G with trivial center $Z(G)$ for an algebraic function field $K = k(t_1, ..., t_r)$ of finite transcendence degree over k to be a regular field extension N/K with Galois group G with two additional properties:

(A) $\mathrm{Aut}(N/k)$ has a subgroup $A = \mathrm{Aut}(G)$ and K is the fixed field of the group corresponding to $\mathrm{Inn}(G)$.

(R) Any regular field extension R/N^A with $\bar{k}R = \bar{k}(t_1, ..., t_r)$ is a rational function field over k. (Here $\bar{k}$ is the algebraic closure of k and N^A is the fixed field of A.) Matzat shows (see [6]) that if all the composition factors of a finite group G have GAR-realizations over $k(t)$, k a Hilbert field (just as $\mathbb{Q}$ or $\mathbb{Q}^{ab}$) then G can be realized as a Galois group over k. He also shows ([6], Satz 6) that

a) the sporadic simple groups M_{11}, M_{12}, M_{22}, J_1, J_2, HS, Sz, ON, Co_3, Co_2, Co_1, Fi_{23}, Fi'_{24}, F_5, F_3, F_2, F_1 have GAR-realizations over $\mathbb{Q}(t)$ and

b) all sporadic simple groups with the possible exceptions He, Fi_{22} and J_4 have GAR-realizations over $\mathbb{Q}^{ab}(t)$.

The purpose of this note is to prove

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THEOREM. The sporadic simple groups He, Fi_{22} and J_4 have GAR-realizations over $\mathbf{Q}(t)$.

By extending the field of constants with $\mathbf{Q}^{ab}$ one obtains from the GAR-realizations over $\mathbf{Q}(t)$ GAR-realizations over $\mathbf{Q}^{ab}(t)$. Thus every sporadic simple group has a GAR-realization over $\mathbf{Q}^{ab}(t)$ and so, in particular, is a Galois group over $\mathbf{Q}^{ab}$.

The method of proof is quite similar to that of [5]. For conjugacy classes $C_1, \dots, C_n$ of a finite group G let a class structure be

$$\mathfrak{C} = (C_1, \dots, C_n) = \{(g_1, \dots, g_n) : g_i \in C_i\},$$

$l^i(\mathfrak{C})$ be the number of orbits of $\operatorname{Inn}(G)$ on

$$\{(g_1, \dots, g_n) \in \mathfrak{C} : g_1 \dots g_n = 1, \langle g_1, \dots, g_n \rangle = G\},$$

and

$$n(\mathfrak{C}) = \frac{1}{|G|} \left|\{(g_1, \dots, g_n) \in \mathfrak{C} : g_1 \dots g_n = 1\}\right| .$$

The point is, that $n(\mathfrak{C})$ (usually called « normalized structure constant ») can be computed from the character table of G:

$$n(\mathfrak{C}) = \sum_{i=1}^{h} \frac{|G|^{n-2}}{\chi_i(1)^{n-2}} \prod_{j=1}^{n} \frac{\chi_i(g_j)}{|C_G(g_j)|},$$

where $\{\chi_1, \dots, \chi_h\} = \operatorname{Irr}(G)$ is the set of complex irreducible characters of G, $g_j \in C_j$, and $C_G(g_j)$ is the centralizer of g_j, as usual. Obviously $l^i(\mathfrak{C}) \leqslant n(\mathfrak{C})$ and in order to compute $l^i(\mathfrak{C})$, which is relevant to the problem at hand one usually invokes information on the maximal subgroups of G. A special case of a theorem of Matzat and Thompson ([5], [11]) says, that if a finite group G with $Z(G) = \{1\}$ has a rational class structure $\mathfrak{C}$ (i.e. a class structure with all conjugacy classes C_i rational) with $l^i(\mathfrak{C}) = 1$, then there is a regular field extension $N/\mathbf{Q}(t)$ with Galois group G. Hence the Theorem follows from the following Lemma as in [6].

LEMMA. *a*) For the rational class structure $\mathfrak{C} = (2B, 6C, 30A)$ of $\operatorname{Aut}(He)$ one has $l^i(\mathfrak{C}) = n(\mathfrak{C}) = 1$.

b) For the rational class structure

$$\mathfrak{C} = (2A, 18E, 42A) \quad \text{of} \quad \operatorname{Aut}(Fi_{22})$$

one has $l^i(\mathfrak{C}) = n(\mathfrak{C}) = 1$.

c) For the rational class structure $\mathfrak{C} = (2A, 4C, 11A)$ of J_4 one has $l^i(\mathfrak{C}) = 1$.

Here we use the notation of the ATLAS [2]; in particular $2B$, $6C$, $30A$ are rational conjugacy classes in $\mathrm{Aut}(He) = He.2$ of elements of order 2, 6 and 30, respectively. From the Lemma and the result of Matzat and Thompson cited above it follows that there is a regular field extension $N/Q(t)$ with Galois group $\mathrm{Aut}(He)$ (or $\mathrm{Aut}(Fi_{22})$). The fixed field of $He \simeq \mathrm{Inn}(He)$ (resp. Fi_{22}) is a rational function field. So one obtains a GAR-realization of He (resp. Fi_{22}), since condition (R) is fulfilled by [6], Bemerkung 4.

PROOF. a) Let $G = \mathrm{Aut}(He)$ and $\mathfrak{C} = (2B, 2C, 30A)$. From the character table of G (see e.g. [2]) it follows that $n(\mathfrak{C}) = 1$.

Let $g_1 \in 2B$ and $g_2 \in 6C$ be such that $g_1 g_2 \in 30A$, so $(g_1, g_2, (g_1 g_2)^{-1}) \in \mathfrak{C}$ and let $U = \langle g_1, g_2 \rangle$.

We will show that $U = G$, thereby proving that $l^i(\mathfrak{C}) = n(\mathfrak{C})$. The maximal subgroups of G with order divisible by 30 are (cf. [2]) He and

$$PSP(4,4)\cdot 4, \qquad 3\cdot S_7 \times 2,$$

$$2^2\cdot PSL(3,4)\cdot D_{12}, \qquad (S_5 \times S_5):2.$$

$$5^2:4S_4,$$

The first two subgroups in this list have relatively small indices in G (2058 and 8330, respectively); so it is very easy to find the corresponding permutation characters (the CAS-system, cf. [10], does this automatically). One finds that the permutation characters vanish on the class $30A$, so that U cannot be contained in one of these subgroups. Also $5^2:4S_4$ has no elements of order 15, because the 3-Sylow subgroups of $GL(2,5)$ have regular orbits on the non-zero vectors of the standard module.

In order to exclude $U \leqslant 3\cdot S_7 \times 2$ and

$$U \leqslant (S_5 \times S_5):2 = S_5 \sim S_2,$$

the character tables of these groups are computed and the fusion of these groups into G is determined as in [10]. The character tables of these groups are included in the appendix. The relevant parts of the fusions are

class of $3\cdot S_7\times 2$	$2C$	$6F$	$6J$	$30A$	fuses to
class of Aut(He)	$2B$	$6C$	$6C$	$30A$	

and

class of $(S_5\times S_5){:}2$	$2B$	$6D$	$30A$	fuses to
class of Aut(He)	$2B$	$6C$	$30A$.	

Using the character tables one finds that

$$n(2C, 6F, 30A) = n(2C, 6J, 30A) = 0 \quad \text{in } 3\cdot S_7\times 2$$

and

$$n(2B, 6D, 30A) = 0 \quad \text{in } (S_5\times S_5){:}2\,.$$

Hence U cannot be contained in $3\cdot S_7\times 2$ or $(S_5\times S_5){:}2$ either, so $U = G$.

b) We consider the rational class structure $\mathfrak{C} = (2A, 18E, 42A)$ of $G = \mathrm{Aut}(Fi_{22})$, again refering to the ATLAS [2] for the notation of the classes (in the notation of the CAS-library, cf. [10], it would be $(2A, 18D, 42A)$). From the character table one computes $n(\mathfrak{C}) = 1$.

Let $g_1 \in 2A$ and $g_2 \in 18D$ be such that $g_1 g_2 \in 42A$ and $U = \langle g_1, g_2\rangle$. Again we have to show that $U = G$.

The maximal subgroups of Aut(Fi_{22}) with orders divisible by 7 are (cf. [2], and a list of corrections and additions to [2] issued by the authors)

$$2\cdot PSU(6,2)\cdot 2\,, \qquad 2^{10}{:}M_{22}{:}2\,,$$

$$G_2(3){:}2\,, \qquad 2^7{:}PSp(6,2)\,,$$

$$PSO^+(8,2){:}S_3\times 2\,, \qquad S_3\times PSU(4,3)\cdot 2^2\,.$$

and, of course, Fi_{22}.

The groups $PSU(6,2)$, $G_2(3)$, M_{22} and $PSp(6,2)$ have no elements of order 21. So U cannot be contained in one of the corresponding extensions.

The subgroup $S_3 \times PSU(4,3) \cdot 2^2$ contains elements of $2A$ and $42A$ but none of the class $18E$, as the permutation character shows. Of course, it is not feasible to list all candidates for a permutation character of G of this degree (1 647 360). So at first the fusion of $S_3 \times PSU(4,3)$ into G was determined as in [10], and hence the permutation character of this subgroup. This gives very strong restrictions for the irreducible constituents of the permutation character of $S_3 \times PSU(4,3) \cdot 2^2$, so that this can easily be found. The character table of $H = PSO^+(8,2){:}S_3 \times 2$ is known (see e.g. [9]) and it is not difficult to obtain the fusion of H into G. One finds that $18E \cap H$ and $42A \cap H$ are contained in the normal subgroup $N = PSO^+(8,2){:}A_3 \times 2$, whereas $2A \cap N = \emptyset$. Hence $U \not\leq H$ and so $U = G$.

c) Let $\mathfrak{C}$ be the rational class structure $(2A, 4C, 11A)$ of $G = J_4$ in the notation of the ATLAS [2] or the CAS-library, cf. [10]. From the charactertable one computes $n(\mathfrak{C}) = \frac{3}{2}$.

Any maximal subgroup of J_4 with order divisible by 11 is conjugate to one of the following (cf. [2], and the list of corrections and additions to [2] issued by the authors):

$$H_1 = PSL(2,23){:}2\,, \qquad H_5 = PSU(3,11){:}2\,,$$

$$H_2 = PSL(2,32){:}5\,, \qquad H_6 = 2^{11}{:}M_{24}\,,$$

$$H_3 = 11_+^{1+2}{:}(5 \times 2S_4)\,, \qquad H_7 = 2_+^{1+12} \cdot 3M_{22}{:}2\,.$$

$$H_4 = M_{22}{:}2\,,$$

H_2 has no elements of order 4. The subgroups H_1, H_4, or H_5 contain no elements of the class $4C$ of J_4. This is so, since in J_4 the elements of $4C$ are not squares of elements, whereas all elements of order 4 in H_1, H_4, and H_5 are, in fact, squares as can be seen from the powermaps in the character tables, cf. [2]. It is quite obvious that in H_3 a product of an involution with an element of order 4 does not have order 11. Furthermore $2^{11}{:}M_{24}$ does not contain elements of the class $11A$. This can be seen e.g. by computing the only candidate for a permutation character of degree 173 067 389 (which is the index of H_6 in J_4) of J_4; this character vanishes on the class $11A$. Actually it is not even necessary to compute this permutation character θ, for a glance at the character table of J_4 shows that all possible constituents of θ have non-negative values at $11B$, so that $2^{11}{:}M_{24}$ must contain $11B$ and, since all elements of order 11 are conjugate in M_{24}, not $11A$.

The character table of H_7 has been computed by B. Fischer [3] together with the fusion into J_4: The relevant part of the fusion is

$$
\begin{array}{ll}
2A,\ 2B,\ 2D,\ 2F,\ 2H & \text{of } H_7 \text{ fuse into } 2A \text{ of } J_4, \\
4C,\ 4G,\ 4L,\ 4O,\ 4R,\ 4U,\ 4X & \text{of } H_7 \text{ fuse into } 4C \text{ of } J_4, \\
11A & \text{of } H_7 \text{ fuses into } 11A \text{ of } J_4.
\end{array}
$$

Computing the structure constants of H_7 one finds that $n(C_1, C_2, C_3) = 0$ for all conjugacy classes C_1, C_2, C_3 of H_7 fusing into $2A$, $4C$, $11A$ of J_4, respectively, except for

$$n(2H, 4U, 11A) = \tfrac{1}{2}.$$

($2H$ corresponds to the outer involution class $2B$ of $M_{22}{:}2$, and $4U$ to the class $4C$.)

Altogether this shows that the number of triples (x, y, z) with $x \in 2A$, $y \in 4C$, $xy = z^{-1} \in 11A$ in J_4 which generate a proper subgroup of J_4 is at most (in fact equal to) $\frac{1}{2}|J_4|$. This implies $l^i(\mathfrak{C}) = 1$, since J_4 has trivial center.

Remark. J_4 is not rigid in the sense of Thompson [11], that is contains no class structure $\mathfrak{C}$ with $n(\mathfrak{C}) = l^i(\mathfrak{C}) = 1$. If $\mathfrak{C} = (C_1, C_2, C_3)$ is a class structure in J_4 with $n(\mathfrak{C}) = 1$, and $x \in C_1$, $y \in C_2$, $xy \in C_3$ then $\langle x, y\rangle$ is a dihedral group or $\mathfrak{C} = (2A, 4A, 11B)$ and $\langle x, y\rangle \leqslant 2^{11}{:}M_{24}$.

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APPENDIX

(S5XS5):2

	2	7	6	7	4	3	3	1	4	3	1	5	4	5	3	3	3
	3	2	1	.	2	2	1	.	1	.	1	2	1	.	2	2	1
	5	2	1	.	1	.	2	2	.	1	1	.	.	.	.	.	.
		1A	2A	2B	3A	3B	5A	5B	6A	10A	15A	2C	4A	4B	6B	6C	12A
	2P	1A	1A	1A	3A	3B	5A	5B	3A	5A	15A	1A	2A	2B	3A	3B	6A
	3P	1A	2A	2B	1A	1A	5A	5B	2A	10A	5A	2C	4A	4B	2C	2C	4A
	5P	1A	2A	2B	3A	3B	1A	1A	6A	2A	3A	2C	4A	4B	6B	6C	12A
SUBGROUP FUSION INTO HE.2																	
		1	2	3	4	4	9	9	10	16	21	2	6	7	10	10	18
	IND																
X.1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.3	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.4	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.5	+	2	2	2	2	2	2	2	2	2	2	-2	-2	-2	-2	-2	-2
X.6	+	8	4	.	5	2	3	-2	1	-1	.	4	2	.	1	-2	-1
X.7	+	8	4	.	5	2	3	-2	1	-1	.	4	2	.	1	-2	-1
X.8	+	8	4	.	5	2	3	-2	1	-1	.	-4	-2	.	-1	2	1
X.9	+	8	4	.	5	2	3	-2	1	-1	.	-4	-2	.	-1	2	1
X.10	+	10	6	2	4	-2	5	.	.	1	-1	2	.	-2	2	2	.
X.11	+	10	6	2	4	-2	5	.	.	1	-1	2	.	-2	2	2	.
X.12	+	10	6	2	4	-2	5	.	.	1	-1	-2	.	2	-2	-2	.
X.13	+	10	6	2	4	-2	5	.	.	1	-1	-2	.	2	-2	-2	.
X.14	+	12	4	-4	6	.	7	2	-2	-1	1	.	.	.	.	.	.
X.15	+	12	4	-4	6	.	7	2	-2	-1	1	.	.	.	.	.	.
X.16	+	16	.	.	4	1	-4	1	.	.	-1	4	.	.	-2	1	.
X.17	+	16	.	.	4	1	-4	1	.	.	-1	4	.	.	-2	1	.
X.18	+	16	.	.	4	1	-4	1	.	.	-1	4	.	.	-2	1	.
X.19	+	16	.	.	4	1	-4	1	.	.	-1	4	.	.	-2	1	.
X.20	+	25	5	1	-5	1	.	.	-1	.	.	1	-1	1	1	1	-1
X.21	+	25	5	1	-5	1	.	.	-1	.	.	1	-1	1	1	1	-1
X.22	+	25	5	1	-5	1	.	.	-1	.	.	1	-1	1	1	1	-1
X.23	+	25	5	1	-5	1	.	.	-1	.	.	1	-1	1	1	1	-1
X.24	+	32	.	.	8	2	-8	2	.	.	-2	-6	.	.	4	-2	.
X.25	+	36	-12	4	.	.	6	1	.	-2	.	.	.	.	.	.	.
X.26	+	36	-12	4	.	.	6	1	.	-2	.	.	.	.	.	.	.
X.27	+	40	4	.	1	-2	-5	.	1	-1	1	4	-2	.	1	-2	1
X.28	+	40	4	.	1	-2	-5	.	1	-1	1	4	-2	.	1	-2	1
X.29	+	40	4	.	1	-2	-5	.	1	-1	1	-4	2	.	-1	2	-1
X.30	+	40	4	.	1	-2	-5	.	1	-1	1	-4	2	.	-1	2	-1
X.31	+	48	-8	.	6	.	-2	-2	-2	2	1	.	.	.	.	.	.
X.32	+	48	-8	.	6	.	-2	-2	-2	2	1	.	.	.	.	.	.
X.33	+	50	10	2	-10	2	.	.	-2	.	.	-2	2	-2	-2	-2	2
X.34	+	60	-4	-4	-6	.	5	.	2	1	-1	.	.	.	.	.	.
X.35	+	60	-4	-4	-6	.	5	.	2	1	-1	.	.	.	.	.	.

5	5	5	5	4	3	4	2	2	3	2	1	4	4	2	1	3	3	2
2	1	1	.	2	2	1	2	1	1	.	1	1	.	1	.	1	.	1
1	.	1	.	1	.	.	.	1	.	1	1	1	.	.	1	.	.	.
2D	2E	4C	4D	6D	6E	6F	6G	10B	12B	20A	30A	2F	4E	6H	10C	4F	8A	12C
1A	1A	2A	2A	3A	3A	3A	3B	5A	6A	10A	15A	1A	2B	3B	5B	2C	4B	6C
2D	2E	4C	4D	2D	2D	2E	2D	10B	4C	2CA	10B	2F	4E	2F	10C	4F	8A	4F
2D	2E	4C	4D	6D	6E	6F	6G	2D	12B	4C	6D	2F	4E	6H	2F	4F	8A	12C
27	27	28	28	29	30	30	30	35	36	40	43	2	7	10	16	28	34	36
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	2	4	.	3	3	-1	.	1	1	-1	-2	.	.	.	.	.	.	.
-6	-2	-4	.	-3	-3	1	.	-1	-1	1	2	.	.	.	.	.	.	.
2	-2	4	.	5	-1	1	2	-3	1	-1	.	.	.	.	.	.	.	.
-2	2	-4	.	-5	1	-1	-2	3	-1	1	.	.	.	.	.	.	.	.
6	2	4	.	6	.	2	.	1	-2	-1	1	.	.	.	.	.	.	.
-6	-2	-4	.	-6	.	-2	.	-1	2	1	-1	.	.	.	.	.	.	.
4	.	6	2	4	-2	.	-2	-1	.	1	-1	.	.	.	.	.	.	.
-4	.	-6	-2	-4	2	.	2	1	.	-1	1	.	.	.	.	.	.	.
6	-2	6	-2	6	.	-2	.	1	.	1	1	.	.	.	.	.	.	.
-6	2	-6	2	-6	.	2	.	-1	.	-1	-1	.	.	.	.	.	.	.
6	.	.	.	-4	2	.	-1	-2	.	.	1	4	.	1	-1	2	.	-1
8	.	.	.	-4	2	.	-1	-2	.	.	1	-4	.	-1	1	-2	.	1
-8	.	.	.	4	-2	.	1	2	.	.	-1	4	.	1	-1	-2	.	1
-8	.	.	.	4	-2	.	1	2	.	.	-1	-4	.	-1	1	2	.	-1
5	1	-5	-1	5	-1	1	-1	.	1	.	.	5	1	-1	.	1	-1	1
5	1	-5	-1	5	-1	1	-1	.	1	.	.	-5	-1	1	.	-1	1	-1
-5	-1	5	1	-5	1	-1	1	.	-1	.	.	5	1	-1	.	-1	1	-1
-5	-1	5	1	-5	1	-1	1	.	-1	.	.	-5	-1	1	.	1	-1	1
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
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14	2	-4	.	-1	-1	-1	2	-1	-1	1	-1	.	.	.	.	.	.	.
-14	-2	4	.	1	1	1	-2	1	1	-1	1	.	.	.	.	.	.	.
6	2	4	.	-9	-3	-1	.	1	1	-1	1	.	.	.	.	.	.	.
-6	-2	-4	.	9	3	1	.	-1	-1	1	-1	.	.	.	.	.	.	.
12	-4	.	.	-6	.	2	.	2	.	.	-1	.	.	.	.	.	.	.
-12	4	.	.	6	.	-2	.	-2	.	.	1	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	-2	-6	2	6	.	-2	.	1	.	-1	1	.	.	.	.	.	.	.
-6	2	6	-2	-6	.	2	.	-1	.	1	-1	.	.	.	.	.	.	.

3.S7X2

2	5	4	5	4	4	2	4	3	2	1	4	3	1	1	1	5	5	4	3	2	2	3
3	3	3	2	2	2	2	1	1	1	1	2	2	1	1	1	1	1	1	1	1	.	1
5	1	1	.	.	.	.	.	.	1	1	.	.	.	.	.	1	.	.	.	.	1	.
7	1	1	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.
	1A	3A	2A	6A	3B	3C	4A	12A	5A	15A	6B	6C	7A	21A	21B	2B	2C	4B	6D	6E	10A	12B
2P	1A	3A	1A	3A	3B	3C	2A	6A	5A	15A	3B	3B	7A	21B	21A	1A	1A	2A	3B	3C	5A	6B
3P	1A	1A	2A	2A	1A	1A	4A	4A	5A	5A	2A	2A	7A	7A	7A	2B	2C	4B	2B	2C	10A	4B
5P	1A	3A	2A	6A	3B	3C	4A	12A	1A	3A	6B	6C	7A	21A	21B	2B	2C	4B	6D	6E	2B	12B
7P	1A	3A	2A	6A	3B	3C	4A	12A	5A	15A	6B	6C	1A	3A	3A	2B	2C	4B	6D	6E	10A	12B
SUBGROUP FUSION INTO HF.2																						
	1A	3A	2A	6A	3A	3B	4A	12B	5A	15A	6A	6A	7A	21A	21B	2A	2B	4A	6A	6B	10A	12B
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
X.3	6	6	2	2	3	.	.	.	1	1	-1	-1	-1	-1	-1	4	.	2	1	.	-1	-1
X.4	6	6	2	2	3	.	.	.	1	1	-1	-1	-1	-1	-1	-4	.	-2	-1	.	1	1
X.5	14	14	2	2	2	-1	.	.	-1	-1	2	2	.	.	.	6	2	.	.	-1	1	.
X.6	14	14	2	2	2	-1	.	.	-1	-1	2	2	.	.	.	-6	-2	.	.	1	-1	.
X.7	14	14	2	2	-1	2	.	.	-1	-1	-1	-1	.	.	.	4	.	-2	1	.	-1	1
X.8	14	14	2	2	-1	2	.	.	-1	-1	-1	-1	.	.	.	-4	.	2	-1	.	1	-1
X.9	15	15	-1	-1	3	.	-1	-1	.	.	-1	-1	1	1	1	5	-3	1	-1	.	.	1
X.10	15	15	-1	-1	3	.	-1	-1	.	.	-1	-1	1	1	1	-5	3	-1	1	.	.	-1
X.11	20	20	-4	-4	2	2	.	.	.	.	2	2	-1	-1	-1	.	.	.	.	.	.	.
X.12	21	21	1	1	-3	.	-1	-1	1	1	1	1	.	.	.	1	-3	-1	1	.	1	-1
X.13	21	21	1	1	-3	.	-1	-1	1	1	1	1	.	.	.	-1	3	1	-1	.	-1	1
X.14	35	35	-1	-1	-1	-1	1	1	.	.	-1	-1	.	.	.	5	1	-1	-1	1	.	-1
X.15	35	35	-1	-1	-1	-1	1	1	.	.	-1	-1	.	.	.	-5	-1	1	1	-1	.	1
X.16	12	-6	4	-2	.	.	.	.	2	-1	4	-2	-2	1	1	.	.	.	.	.	.	.
X.17	30	-15	6	-3	.	.	2	-1	.	.	.	.	2	-1	-1	.	.	.	.	.	.	.
X.18	30	-15	-2	1	.	.	-2	1	.	.	4	-2	2	-1	-1	.	.	.	.	.	.	.
X.19	42	-21	2	-1	.	.	-2	1	2	-1	-4	2	.	.	.	.	.	.	.	.	.	.
X.20	42	-21	-6	3	.	.	2	-1	2	-1	.	.	.	.	.	.	.	.	.	.	.	.
X.21	48	-24	.	.	.	.	.	.	-2	1	.	.	-1	A	B	.	.	.	.	.	.	.
X.22	48	-24	.	.	.	.	.	.	-2	1	.	.	-1	B	A	.	.	.	.	.	.	.
X.23	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.24	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
X.25	6	6	2	2	3	.	.	.	1	1	-1	-1	-1	-1	-1	4	.	2	1	.	-1	-1
X.26	6	6	2	2	3	.	.	.	1	1	-1	-1	-1	-1	-1	-4	.	-2	-1	.	1	1
X.27	14	14	2	2	2	-1	.	.	-1	-1	2	2	.	.	.	6	2	.	.	-1	1	.
X.28	14	14	2	2	2	-1	.	.	-1	-1	2	2	.	.	.	-6	-2	.	.	1	-1	.
X.29	14	14	2	2	-1	2	.	.	-1	-1	-1	-1	.	.	.	4	.	-2	1	.	-1	1
X.30	14	14	2	2	-1	2	.	.	-1	-1	-1	-1	.	.	.	-4	.	2	-1	.	1	-1
X.31	15	15	-1	-1	3	.	-1	-1	.	.	-1	-1	1	1	1	5	-3	1	-1	.	.	1
X.32	15	15	-1	-1	3	.	-1	-1	.	.	-1	-1	1	1	1	-5	3	-1	1	.	.	-1
X.33	20	20	-4	-4	2	2	.	.	.	.	2	2	-1	-1	-1	.	.	.	.	.	.	.
X.34	21	21	1	1	-3	.	-1	-1	1	1	1	1	.	.	.	1	-3	-1	1	.	1	-1
X.35	21	21	1	1	-3	.	-1	-1	1	1	1	1	.	.	.	-1	3	1	-1	.	-1	1
X.36	35	35	-1	-1	-1	-1	1	1	.	.	-1	-1	.	.	.	5	1	-1	-1	1	.	-1
X.37	35	35	-1	-1	-1	-1	1	1	.	.	-1	-1	.	.	.	-5	-1	1	1	-1	.	1
X.38	12	-6	4	-2	.	.	.	.	2	-1	4	-2	-2	1	1	.	.	.	.	.	.	.
X.39	30	-15	6	-3	.	.	2	-1	.	.	.	.	2	-1	-1	.	.	.	.	.	.	.
X.40	30	-15	-2	1	.	.	-2	1	.	.	4	-2	2	-1	-1	.	.	.	.	.	.	.
X.41	42	-21	2	-1	.	.	-2	1	2	-1	-4	2	.	.	.	.	.	.	.	.	.	.
X.42	42	-21	-6	3	.	.	2	-1	2	-1	.	.	.	.	.	.	.	.	.	.	.	.
X.43	48	-24	.	.	.	.	.	.	-2	1	.	.	-1	A	B	.	.	.	.	.	.	.
X.44	48	-24	.	.	.	.	.	.	-2	1	.	.	-1	B	A	.	.	.	.	.	.	.

```
W = EXP ( 2.PI.I / N )

N = 21   A = /A = 1 +W -2W2 +W4 +W7 -W8 -W9 -W11
              = (1 + SQRT(21)) / 2
N = 21   B = /B = -W +2W2 -W4 -W7 +W8 +W9 +W11
              = (1 - SQRT(21)) / 2
```

5	4	5	4	4	2	4	3	2	1	4	3	1	1	1	5	5	4	3	2	2	3
3	3	2	2	2	2	1	1	1	1	2	2	1	1	1	1	1	1	1	1	.	1
1	1	.	.	.	.	.	.	1	1	.	.	.	.	.	1	.	.	.	.	1	.
1	1	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.
2D	6F	2E	6G	6H	6I	4C	12C	10B	30A	6J	6K	14A	42A	42B	2F	2G	4D	6L	6M	10C	12D
1A	3A	1A	3A	3B	3C	2A	6A	5A	15A	3B	3B	7A	21B	21A	1A	1A	2A	3B	3C	5A	6B
2D	2D	2E	2F	2D	2D	4C	4C	10B	10B	2F	2F	14A	14A	14A	2F	2G	4D	2F	2G	10C	4D
2D	6F	2E	6G	6H	6I	4C	12C	2D	6F	6J	6K	14A	42A	42B	2F	2G	4D	6L	6M	2F	12D
2D	6F	2E	6G	6H	6I	4C	12C	10B	30A	6J	6K	2D	6F	6F	2F	2G	4D	6L	6M	10C	12D
															FUSION INTO HF.2						
2C	6C	2C	6D	6D	6E	4D	12C	10B	30A	6C	6D	14C	42A	42B	2C	2C	4D	6D	6F	10B	12C
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
6	6	2	2	3	.	.	.	1	1	-1	-1	-1	-1	-1	4	.	2	1	.	-1	-1
6	6	2	2	3	.	.	.	1	1	-1	-1	-1	-1	-1	-4	.	-2	-1	.	1	1
14	14	2	2	2	-1	.	.	-1	-1	2	2	.	.	.	6	2	.	.	-1	1	.
14	14	2	2	2	-1	.	.	-1	-1	2	2	.	.	.	-6	-2	.	.	1	-1	.
14	14	2	2	-1	2	.	.	-1	-1	-1	-1	.	.	.	4	.	-2	1	.	-1	1
14	14	2	2	-1	2	.	.	-1	-1	-1	-1	.	.	.	-4	.	2	-1	.	1	-1
15	15	-1	-1	3	.	-1	-1	.	.	-1	-1	1	1	1	5	-3	1	-1	.	.	1
15	15	-1	-1	3	.	-1	-1	.	.	-1	-1	1	1	1	-5	3	-1	1	.	.	-1
20	20	-4	-4	2	2	.	.	.	.	2	2	-1	-1	-1	.	.	.	.	.	.	.
21	21	1	1	-3	.	-1	-1	1	1	1	1	.	.	.	1	-3	-1	1	.	1	-1
21	21	1	1	-3	.	-1	-1	1	1	1	1	.	.	.	-1	3	1	-1	.	-1	1
35	35	-1	-1	-1	-1	1	1	.	.	-1	-1	.	.	.	5	1	-1	-1	1	.	-1
35	35	-1	-1	-1	-1	1	1	.	.	-1	-1	.	.	.	-5	-1	1	1	-1	.	1
12	-6	4	-2	.	.	.	.	2	-1	4	-2	-2	1	1	.	.	.	.	.	.	.
30	-15	6	-3	.	.	2	-1	.	.	.	.	2	-1	-1	.	.	.	.	.	.	.
30	-15	-2	1	.	.	-2	1	.	.	4	-2	2	-1	-1	.	.	.	.	.	.	.
42	-21	2	-1	.	.	-2	1	2	-1	-4	2	.	.	.	.	.	.	.	.	.	.
42	-21	-6	3	.	.	2	-1	2	-1	.	.	.	.	.	.	.	.	.	.	.	.
48	-24	.	.	.	.	.	.	-2	1	.	.	-1	A	B	.	.	.	.	.	.	.
48	-24	.	.	.	.	.	.	-2	1	.	.	-1	B	A	.	.	.	.	.	.	.
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1
-6	-6	-2	-2	-3	.	.	.	-1	-1	1	1	1	1	1	-4	.	-2	-1	.	1	1
-6	-6	-2	-2	-3	.	.	.	-1	-1	1	1	1	1	1	4	.	2	1	.	-1	-1
-14	-14	-2	-2	-2	1	.	.	1	1	-2	-2	.	.	.	-6	-2	.	.	1	-1	.
-14	-14	-2	-2	-2	1	.	.	1	1	-2	-2	.	.	.	6	2	.	.	-1	1	.
-14	-14	-2	-2	1	-2	.	.	1	1	1	1	.	.	.	-4	.	2	-1	.	1	-1
-14	-14	-2	-2	1	-2	.	.	1	1	1	1	.	.	.	4	.	-2	1	.	-1	1
-15	-15	1	1	-3	.	1	1	.	.	1	1	-1	-1	-1	-5	3	-1	1	.	.	-1
-15	-15	1	1	-3	.	1	1	.	.	1	1	-1	-1	-1	5	-3	1	-1	.	.	1
-20	-20	4	4	-2	-2	.	.	.	.	-2	-2	1	1	1	.	.	.	.	.	.	.
-21	-21	-1	-1	3	.	1	1	-1	-1	-1	-1	.	.	.	-1	3	1	-1	.	-1	1
-21	-21	-1	-1	3	.	1	1	-1	-1	-1	-1	.	.	.	1	-3	-1	1	.	1	-1
-35	-35	1	1	1	1	-1	-1	.	.	1	1	.	.	.	-5	-1	1	1	-1	.	1
-35	-35	1	1	1	1	-1	-1	.	.	1	1	.	.	.	5	1	-1	-1	1	.	-1
-12	6	-4	2	.	.	.	.	-2	1	-4	2	2	-1	-1	.	.	.	.	.	.	.
-30	15	-6	3	.	.	-2	1	.	.	.	.	-2	1	1	.	.	.	.	.	.	.
-30	15	2	-1	.	.	2	-1	.	.	-4	2	-2	1	1	.	.	.	.	.	.	.
-42	21	-2	1	.	.	2	-1	-2	1	4	-2	.	.	.	.	.	.	.	.	.	.
-42	21	6	-3	.	.	-2	1	-2	1	.	.	.	.	.	.	.	.	.	.	.	.
-48	24	.	.	.	.	.	.	2	-1	.	.	1	-A	-B	.	.	.	.	.	.	.
-48	24	.	.	.	.	.	.	2	-1	.	.	1	-B	-A	.	.	.	.	.	.	.

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