

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

JOHN LENNOX

FEDERICO MENEGAZZO

HOWARD SMITH

JAMES WIEGOLD

**Groups with finite automorphism classes of subgroups**

*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 79 (1988), p. 87-96

[http://www.numdam.org/item?id=RSMUP\\_1988\\_\\_79\\_\\_87\\_0](http://www.numdam.org/item?id=RSMUP_1988__79__87_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1988, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## Groups with Finite Automorphism Classes of Subgroups.

JOHN LENNOX - FEDERICO MENEGAZZO  
HOWARD SMITH - JAMES WIEGOLD (\*)

### 1. Introduction.

The *automorphism class* of a subgroup  $H$  of a group  $G$  is the orbit  $\{H^\alpha: \alpha \in \text{Aut } G\}$  of  $H$  under the action of  $\text{Aut } G$ . One of the main aims of [7] was to classify those groups for which the automorphism classes are boundedly finite, the answer being as follows.

**THEOREM A.** *The following properties of a group  $G$  are equivalent.*

- (i) *The automorphism classes of subgroups of  $G$  are boundedly finite.*
- (ii) *The automorphism classes of abelian subgroups of  $G$  are boundedly finite.*
- (iii) *Either (a)  $\text{Aut } G$  is finite,*

(\*) Indirizzo degli AA.: J. LENNOX: Department of Pure Mathematics, University College, Cardiff CF11XL, Wales; F. MENEGAZZO: Istituto di Algebra e Geometria, Università di Padova, Via Belzoni 7, I-35100 Padova (Italia); H. SMITH: Department of Mathematics, Michigan State University, East Lansing, Michigan, USA; J. WIEGOLD: Department of Pure Mathematics, University College, Cardiff CF11XL, Wales.

The fourth author thanks the Istituto di Algebra e Geometria dell'Università di Padova for its warm and generous hospitality during the period that work on this paper was carried out, and Consiglio Nazionale delle Ricerche for its financial support.

Lavoro eseguito con il contributo del M.P.I.

- or (b) *there is a direct decomposition  $G = G_1 \times G_2$ , where  $G_1$  is a locally cyclic torsion group,  $G_2$  is a finite central extension of a direct product of finitely many groups of type  $Z_{p^\infty}$  for different primes  $p$ , and  $G_1$  and  $G_2$  do not contain elements of the same prime order.*

The problem of determining the groups in which the automorphism classes are merely finite was left open in [7]. We shall prove here that the two classes are the same:

**THEOREM B.** *The following properties of a group  $G$  are equivalent.*

- (i) *The automorphism classes of subgroups of  $G$  are boundedly finite.*
- (ii) *The automorphism classes of abelian subgroups of  $G$  are finite.*
- (iii) *Either (a)  $G$  is not periodic and  $\text{Aut } G$  is finite,*  
or (b)  *$G$  is periodic and of the structure described in (iii) (b) of Theorem A.*

To prove this, we need only show that a group with finite automorphism classes of abelian subgroups is of the type mentioned in (iii) of Theorem B, and then apply Theorem A.

The analogous results for conjugacy of subgroups and of abelian subgroups are due to B. H. Neumann [6] and Eremin [2]; groups with finite classes of abelian subgroups are just the centre-by-finite groups. We shall make much use of this result in our proof, which is, surprisingly, a good deal harder than that of the Neumann-Eremin theorem.

By an *abus de langage* we call the set  $\{H^\alpha: \alpha \in \text{End } G\}$  of endomorphic images of a subgroup  $H$  of a group  $G$  the endomorphism class of  $H$ ; note that endomorphism classes need not be disjoint. The following theorem is an analogue of Theorems A and B for endomorphism classes:

**THEOREM C.** *The following properties of a group  $G$  are equivalent.*

- (i) *The endomorphism classes of subgroups of  $G$  are boundedly finite.*
- (ii) *The endomorphism classes of abelian subgroups of  $G$  are finite.*
- (iii)  *$G$  is a finite central extension of a direct product of finitely many groups of type  $Z_{p^\infty}$  for different primes  $p$ .*

Finally, we have the following curious result:

**THEOREM D.** *Let  $G$  be a group in which the endomorphism classes of elements are boundedly finite. Then  $G$  is finite.*

This generalizes a theorem of Baer [1] stating that a group is finite if its endomorphism set is finite. See [7] for groups in which the automorphism classes of elements are boundedly finite. The boundedness condition in Theorem D is essential, as a glance at the direct product of infinitely many finite groups of coprime orders shows.

All our methods are elementary and notation standard. We use without further comment the fact that groups of the type under discussion are centre-by-finite and thus have finite commutator subgroups. Further, if  $\alpha$  is any endomorphism of a group  $G$  into the centre of  $G$  such that  $\alpha^2 = 0$ , then the map  $1 + \alpha$  is an automorphism of  $G$ . [4] is a good reference for facts on abelian groups that we use.

We thank Joachim Neubüser for pointing out a serious error in a first version of Theorem C.

## 2. Proof of Theorem B: the periodic case.

In this section,  $G$  is a periodic group in which the orbits of abelian subgroups are finite, and we show that  $G$  has the structure defined in (iii) (b) of Theorem A. The proof is accomplished in several stages.

1)  $G$  does not have a divisible  $p$ -subgroup of rank 2, for any prime  $p$ .

Suppose that  $G$  contains a subgroup  $U \times V$ , where  $U \cong V \cong \mathbf{Z}_{p^\infty}$ . Since  $G$  is central-by-finite,  $U$  and  $V$  are central; and since  $G'$  is finite,  $UVG'/VG' \cong \mathbf{Z}_{p^\infty}$ . Thus  $G'/VG' = UVG'/VG' \times R/VG'$  for some  $R \leq G$ , and maps  $\varphi$  such that  $UVG'/VG' \xrightarrow{\varphi} V$ ,  $R/VG' \rightarrow 1$  give rise to endomorphisms  $\alpha: G \rightarrow Z(G)$  such that  $\alpha^2 = 0$ . But

$$U^{1+\alpha} = \{u(uVG')^\varphi: u \in U\};$$

since there are  $2_{\mathfrak{s}}$  different choices for the complement of  $U$  in  $U \times V$ , this means that the  $\text{Aut } G$ -orbit of  $U$  is infinite, a contradiction.

2) *The reduced part of every Sylow  $p$ -subgroup of the centre of  $G$  is finite.*

If not, then  $G/G'$  has the same property, and we can write  $G/G' = \langle aG' \rangle \times X/G'$ , where  $\langle aG' \rangle$  is a non-trivial cyclic  $p$ -group: since  $|G:X| < \infty$ ,  $X \cap Z(G)$  must have an infinite elementary abelian subgroup  $Y$ , because a basic subgroup of an abelian  $p$ -group is finite if and only if it is a direct factor and thus the reduced part is finite. Any homomorphism  $\varphi: \langle aG' \rangle \rightarrow Y$ , together with the trivial map  $X/G' \rightarrow 1$ , gives rise to an endomorphism  $\alpha$  of  $G$  such that  $\alpha^2 = 0$ , and  $\langle a \rangle^{1+\alpha} = \langle a \cdot (aG')^\varphi \rangle$ ; since  $Y$  is infinite,  $\langle a \rangle$  has infinite  $\text{Aut } G$ -orbit.

A consequence of 1 and 2 is:

3) *Every infinite Sylow  $p$ -subgroup of  $Z(G)$  is of the form  $Z_{p^\infty} \times F_p$ , where  $F_p$  is finite.*

For every  $p > |G'|$ , the Sylow  $p$ -subgroups of  $G$  are central. We use this fact to prove:

4) *The subgroup  $S = \langle F_p : p > |G'| \rangle$  is a direct factor of  $G$ .*

Every subgroup  $F_p G' / G'$  is a direct factor of  $G/G'$ , since it is finite and the reduced part of a Sylow  $p$ -subgroup, so that  $S G' / G'$  is a direct factor of  $G/G'$ , say  $G/G' = S G' / G' \times L / G'$ . But then  $G = S G' L = S L$  and  $S G' \cap L = G'$ , so that  $S \cap G' L = 1$  and  $S$  is a direct factor.

The next step is proved in a very similar way to 4 and we omit the proof.

5) *For every  $p > |G'|$ , a divisible  $p$ -subgroup of  $G$  is a direct factor of  $G$ .*

For our final step, note that  $S$  is the direct product of the  $F_p$ .

6) *Almost all the  $F_p$  are cyclic.*

It is enough to show that almost all the  $F_p$  with  $p > |G'|$  are cyclic. Suppose that infinitely many of them are non-cyclic, so that  $F_p$  has a direct factor of the form  $\langle a_p, b_p : a_p^{n_p} = b_p^{m_p} = [a_p, b_p] = 1 \rangle$ ,  $n_p \geq m_p \geq 1$ , for  $p$  lying in an infinite set  $\pi$ . Consider an automorphism  $\varphi$  of  $G$  that extends the automorphisms  $a_p \rightarrow a_p b_p^{\lambda_p}$ ,  $b_p \rightarrow b_p$ ,  $p \in \pi$ , where  $(\lambda_p, p) = 1$ . The image of the abelian subgroup  $A = \langle a_p : p \in \pi \rangle$  under  $\varphi$  is  $\langle a_p b_p^{\lambda_p} : p \in \pi \rangle$ , and it is clear that suitable choices of the  $\lambda_p$  give rise to infinitely many  $\text{Aut } G$ -images of  $A$ .

A quick glance now shows that  $G$  has the structure required, and the proof of the periodic case of Theorem B is complete.

**3. Proof of Theorem B: the nonperiodic case.**

Throughout this section,  $G$  is a non-periodic group in which the orbits of abelian subgroups are finite and  $T$  its periodic part (which is a subgroup since  $G$  is an  $FC$ -group). Again we proceed in several stages to the proof that  $\text{Aut } G$  is finite.

1)  $T$  has no subgroups isomorphic to a  $Z_{p^\infty}$ .

Let  $A = \langle a_1, a_2, \dots \rangle$  be such a subgroup, with  $a_1^p = 1$ ,  $a_{i+1}^p = a_i$  for  $i \geq 1$ ; as before,  $A$  is central.

If  $(G/T)^p \neq G/T$ , take  $x \in G$  such that  $xT \notin (G/T)^p$ . For  $k \geq 1$ , set  $X_k/T = (G/T)^{p^k}$ ; then  $xX_k$  has maximal order  $p^k$  in  $G/X_k$ , so that  $G/X_k = \langle xX_k \rangle \times Y_k/X_k$  for some subgroup  $Y_k$  of  $G$ . If  $\sigma_k$  is the endomorphism of  $G$  defined by  $x^{\sigma_k} = a_k$ ,  $Y_k^{\sigma_k} = 1$ , then  $1 + \sigma_k \in \text{Aut } G$  and  $x^{1+\sigma_k} = xa_k$ ; but this gives the contradiction that  $\langle x \rangle$  has infinitely many images under  $\text{Aut } G$ .

Thus we may assume that  $G/T$  is  $p$ -divisible. Let  $X/G'$  be a complement of the quasicyclic subgroup  $AG'/G'$  in  $G/G'$ ; then  $G = AX$ ,  $X \cap A = \langle a_m \rangle$  is finite,  $X/(T \cap X)$  is  $p$ -divisible and there is a subgroup  $Y \leq X$  with  $T \cap X \leq Y$  such that  $X/Y = \langle c_1 Y, c_2 Y, \dots \rangle \cong A$ , the isomorphism being given by  $a_i = (c_i Y)^\varphi$  for all  $i$ . For a  $p$ -adic integer  $\alpha$ , let  $\gamma_\alpha$  be the automorphism of  $G$  that is 1 on  $A$  and such that  $x^{\gamma_\alpha} = x(xY)^{\varphi^\alpha}$  for  $x \in X$ . Then  $X^{\gamma_\alpha} = \langle a_1^\alpha c_1, a_2^\alpha c_2, \dots \rangle Y$  and  $X^{\gamma_\alpha} \cap A = X \cap A$ . We claim that  $X^{\gamma_\alpha} \neq X^{\gamma_\beta}$  if  $\alpha \neq \beta$ . Indeed, if  $X^{\gamma_\alpha} = X^{\gamma_\beta}$  we have  $a_i^\beta c_i \in X^{\gamma_\alpha}$ , so that  $a_i^{\alpha-\beta} \in X^{\gamma_\alpha} \cap A = \langle a_m \rangle$ , and so  $p^m(\alpha - \beta) = 0$ . But then  $\alpha = \beta$ , so our claim is justified and  $\{X^\gamma : \gamma \in \text{Aut } G\}$  is infinite; this is not yet a contradiction, since  $X$  is non-abelian if  $G$  is. However, a very easy argument now shows that the abelian subgroup  $X^n$ ,  $n = |G : Z(G)|$ , is such that  $(X^n)^\alpha \neq (X^n)^\beta$  if  $\alpha \neq \beta$ , and this is the required contradiction.

2) Every Sylow  $p$ -subgroup of  $T$  is finite.

Clearly, since there are no divisible  $p$ -subgroups by 1, it is enough to show that there are no infinite elementary abelian  $p$ -subgroups. By way of contradiction, assume that  $E$  is one. Since  $G'$  is finite, the  $p$ -component  $T_p/G'$  of  $T/G'$  is not divisible, so that it has a cyclic direct factor  $\langle aG' \rangle$  of order  $p^k > 1$ , and of course we may choose  $a$  to be of order  $p^h$ ,  $h \geq k$ . Write  $G/G' = \langle aG' \rangle \times B/G'$ . Then the inter-

section  $E_0 = E \cap Z(G) \cap B$  is still infinite, since  $B \cap Z(G)$  is of finite index; for each  $z \in E_0$ , the identity map of  $B$  extends to an automorphism  $\varphi_z$  of  $G$  sending  $a$  to  $az$ , and again we have the contradiction that the  $\text{Aut } G$ -orbit of  $\langle a \rangle$  is infinite.

3)  $T$  is finite.

Let  $\omega$  be the set of primes  $p$  for which  $Z(G)$  has a  $p$ -element. Our task is to show that  $\omega$  is finite, so we assume that it is infinite, and without loss of generality that the index  $n$  of  $Z(G)$  in  $G$  is coprime to  $p$ , if  $p \in \omega$ .

CASE 1. *There exist an element  $g$  of infinite order and an infinite subset  $\omega_0 \subseteq \omega$  such that, for every  $p \in \omega_0$ ,  $g$  fails to be infinitely  $p$ -divisible.*

Enumerate the elements of  $\omega_0$  in some way,  $\omega_0 = \{p_1, p_2, \dots\}$ , and for each  $p_i \in \omega_0$  define  $k_i$  to be the largest integer such that  $x_i^{p_i^{k_i}} = g$  for some  $x \in G$ . Without loss of generality we may assume that  $g \in Z(G)$ . Set  $y_0 = g$ , and suppose that  $y_1^{p_1^{k_1}} = y_0$ . The  $p_2$ -characteristic of  $y_1$  is again  $k_2$ , and we choose  $y_2$  such that  $y_2^{p_2^{k_2}} = y_1$ . Proceed in this way; once  $y_0, y_1, \dots, y_r$  have been chosen, the  $p_{r+1}$ -characteristic of  $y_r$  is  $k_{r+1}$ , and we choose  $y_{r+1}$  such that  $y_{r+1}^{p_{r+1}^{k_{r+1}}} = y_r$ , where  $s$  is short for  $p_{r+1}^{k_{r+1}}$ . By construction, the subgroup  $Y = \langle y_0, y_1, \dots, y_r, \dots \rangle$  is locally cyclic and torsion-free, and moreover  $y_i \notin G^{p_i}$  for  $i > 0$ .

Now let  $t_i$  be an element of order  $p_i$  in  $Z(G)$  such that  $\langle t_i \rangle G^{p_i} \neq YG^{p_i} = \langle y_i \rangle G^{p_i}$ , if one exists, and let  $\varphi_i$  be the usual automorphism induced from a homomorphism  $G/G^{p_i} \rightarrow \langle t_i \rangle$  with kernel a complement of  $YG^{p_i}/G^{p_i}$  containing  $\langle t_i \rangle G^{p_i}$ : If no such  $t_i$  exists, then  $G = \langle u_i \rangle \times K_i$ , where  $\langle u_i \rangle$  is the  $p_i$ -part  $S_{p_i}(G)$  of  $G$ , which is of course the Sylow  $p_i$ -subgroup of  $Z(G)$  since  $p_i \nmid n$ . In that case, let  $\varphi_i$  be the identity on  $K_i$  and a non-trivial power on  $\langle u_i \rangle$  (which must exist except in the case  $\langle u_i \rangle \cong Z_2$ , and we can ignore this). In both cases,  $Y \cdot Y^{\varphi_i} \cap S_{p_i}(G) \neq 1$  and  $Y^{\varphi_i} \leq Y \times S_{p_i}(G)$ , and this shows that our subgroup  $Y$  has infinite  $\text{Aut } G$ -orbit.

CASE 2. *Every element of infinite order is infinitely  $p$ -divisible for almost all  $p$ .*

Assume first that  $G/T$  has infinite rank. Then there is a countable independent subset of  $G/T$  which we may index by elements of  $\omega: \{x_p T\}_{p \in \omega}$ , and we can suppose that each  $x_p$  is in  $Z(G)$ . For each  $p \in \omega$ , choose  $a_p \in S_p(G)$ , the Sylow  $p$ -subgroup of  $G$ , of maximum

order, and an integer  $s_p$  such that  $s_p \neq 1$  and  $ps_p \equiv p$ , where the congruences are modulo the exponent of  $S_p(G)$ . Note that  $S_p(G)$  is a direct factor of  $G$  since  $S_p(G) \cap G' = 1$  ( $p$  does not divide the index of  $Z(G)$  in  $G$  and therefore does not divide  $|G'|$  [8]) and  $S_p(G)G'/G'$  is a direct factor of  $G/G'$ . Let  $C_p$  be a complement of  $S_p(G)$  in  $G$ , and define an automorphism  $\varphi_p$  of  $G$  that is the identity on  $C_p$  and the  $s_p$ -th power automorphism on  $S_p(G)$ . With  $X = \langle a_p x_p^p : p \in \omega \rangle$ , we have  $X^{\varphi_p} \neq X$ ,  $X^{\varphi_p} \leq X \times S_p(G)$ , and thus  $X$  has infinite  $\text{Aut } G$ -orbit.

Thus, we may assume that  $G/T$  has finite rank and is  $p$ -divisible for almost all primes in  $\omega$ . Choose a prime  $q \in \omega$ , so that  $q \nmid |G'| \cdot |G : Z(G)|$ , as before, and  $G/T$  is  $q$ -divisible. Then  $G = S_p(G) \times B$ , where  $B$  is  $q$ -divisible,  $|B : Z(B)| \mid n$ ,  $Z(B)$  is  $q$ -divisible and has no  $q$ -torsion, by 1. (Recall that  $n = |G : Z(G)|$ ). If  $r$  is the order of  $q \pmod n$ , then the mapping  $\alpha : B \rightarrow B$  given by  $b^\alpha = b^{q^r}$  is an automorphism of  $B$  (this because  $q^r \equiv 1 \pmod{|B : Z(B)|}$ ) and  $\alpha$  induces an automorphism on  $Z(B)$ , which clearly extends to an automorphism of  $G$ . Every element of infinite order in  $B$  has infinitely many images under  $\langle \alpha \rangle$ .

4) For every  $p \in \omega$ ,  $|G : G^p|$  is finite.

If  $|G : G^p|$  is infinite, then  $|G : G^p G' Q_p|$  is infinite, where  $Q_p$  is the Sylow  $p$ -subgroup of  $Z(G)$ . Take any subgroup  $N$  of index  $p$  containing  $G^p G' Q_p$ , and let  $\varphi$  be a homomorphism of  $G$  into  $Z(G)$  with kernel  $N$ . Write  $Z(G) = Q_p \times X$ . Then  $X \cap X^{1+\varphi} = X \cap N$ ; and, since there are infinitely many  $N$ , there must be infinitely many  $X \cap N$  since  $|G : X|$  and  $|G : N|$  are (boundedly) finite. Thus there are infinitely many  $X^{1+\varphi}$ .

We come now to the final stage of the proof. Let  $R$  be a transversal for  $Z(G)$  in  $G$ ; if  $K$  is any subgroup of  $G$  containing  $R$ , then  $C_G(K) = Z(G)$ . Set  $\Gamma = \text{Aut } G$ .

For a finite subset  $F$  of  $G$ , we define the closure  $H(F)$  of  $F$  as follows:

$$H(F) = \{x \in G : x^k \in \langle f^\gamma : f \in F \cup R, \gamma \in \Gamma \rangle \text{ for some } k \neq 0\}.$$

Clearly  $H(F)$  is a characteristic subgroup,  $G/H(F)$  is torsion-free, and  $Z(H(F)) \leq Z(G)$ . From here on we assume that  $\text{Aut } G$  is infinite and find a contradiction.

5) For every finite subset  $F$  of  $G$ ,  $C_\Gamma(H(F))$  has finite index in  $\Gamma$ , and  $C_\Gamma(H(F)) \cap C_\Gamma(G/H(F))$  is finite.



Clearly,  $\Omega(F) := \{f^\gamma : f \in F, \gamma \in \Gamma\} \cup \{r^\gamma : r \in R, \gamma \in \Gamma\} \cup T$  is a finite  $\Gamma$ -invariant set. Thus the kernel  $K$  of the restriction  $\Gamma \rightarrow \text{Sym}(\Omega(F))$  has finite index;  $K$  is the centraliser in  $\Gamma$  of  $L := \langle f^\gamma : f \in F \cup R, \gamma \in \Gamma \rangle T$ . To establish the first claim we need to show that  $|K : C_R(H(F))|$  is finite. Every element  $\alpha$  of  $K$  centralises  $H(F)/T$  and  $L$ , so it induces a homomorphism  $H(F)/L \rightarrow T \cap Z(L)$ , namely  $hL \mapsto [h, \alpha]$ ; it is important here to notice that  $Z(L) \leq Z(G)$  since  $L \supseteq R$ . However,  $C_R(H(F))$  is just the kernel of this map, and thus has finite index since its image is in  $T$  and therefore finite.

Next,  $C_R(H(F)) \cap C_R(G/H(F))$  is isomorphic to  $\text{Hom}(G/H(F), Z(H(F))) \leq \text{Hom}(G/H(F), Z(G))$  via the obvious map. There is no homomorphism  $G/H(F) \rightarrow Z(H(F))$  with non-periodic image, else there would be elements  $a \in G$ ,  $c \in Z(H(F))$  of infinite order and  $\gamma \in C_R(H(F)) \cap C_R(G/H(F))$  such that  $a^\gamma = ac$ , and then  $\{\langle a^{\gamma^i} \rangle = \langle ac^i \rangle : i \in \mathbb{N}\}$  would be an infinite set of images of  $\langle a \rangle$ . Thus  $\text{Hom}(G/H(F), Z(H(F)))$  is in fact  $\text{Hom}(G/H(F), Z(H(F)) \cap T)$ , which is finite by 3 and 4.

6) For every finite subset  $F$  of  $G$ , there exist a cyclic subgroup  $\langle aH(F) \rangle$  of  $G/H(F)$  with  $a \in Z(G)$ , and  $\gamma \in C_R(H(F))$  such that  $\langle aH(F) \rangle \not\leq \langle a^\gamma H(F) \rangle$ .

If not, then  $\langle gH(F) \rangle = \langle g^\gamma H(F) \rangle$  for all  $g \in G$  (since some power of every element is central) and thus the automorphism group induced by  $C_R(H(F))$  on  $G/H(F)$  would be just  $\langle -1 \rangle$ ; however, that is impossible since  $C_R(H(F)) / (C_R(H(F)) \cap C_R(G/H(F)))$  is infinite.

Next, we define inductively a sequence  $\gamma_1, \gamma_2, \dots$  of elements of  $\Gamma$  and a sequence  $a_1, a_2, \dots$  of elements of  $Z(G)$  as follows:

$\gamma_1$  is any non-trivial element of  $\Gamma$  (other than  $-1$  if  $G$  is abelian);

$a_1$  is any element of  $Z(G)$  such that  $\langle a_1^{\gamma_1} \rangle \neq \langle a_1 \rangle$ ;

if  $\gamma_1, \gamma_2, \dots, \gamma_n$  and  $a_1, a_2, \dots, a_n$  have been chosen, use 6 to choose  $\gamma_{n+1} \in C_R(H(a_1, \dots, a_n))$ , and  $a_{n+1} \in Z(G)$  such that  $\langle a_{n+1}^{\gamma_{n+1}} H(a_1, \dots, a_n) \rangle \not\leq \langle a_{n+1} H(a_1, \dots, a_n) \rangle$ .

Set  $X = \langle a_1, \dots, a_n, \dots \rangle$ ;  $X$  is abelian.

7) For every  $n$ ,  $X \cap H(a_1, \dots, a_n) \leq \langle a_1, \dots, a_n \rangle T$ .

Assume that there is an element  $g \notin T$  such that  $g \in H(a_1, \dots, a_n) \cap \langle a_{n+1}, a_{n+2}, \dots \rangle$ , say  $gT = a_{n+1}^{k_1} \dots a_{n+m}^{k_m} T$  with the obvious notation, and  $k_m \neq 0$ . Then  $a_{n+m}^{k_m} \in H(a_1, \dots, a_n, \dots, a_{n+m-1})$  so that  $a_{n+m} \in H(a_1, \dots, a_{n+m-1})$ , which contradicts the choice of  $a_{n+m}$ . Thus

$$\begin{aligned} X \cap H(a_1, \dots, a_n) &\leq (\langle a_1, \dots, a_n \rangle T \langle a_{n+1}, \dots \rangle) \cap H(a_1, \dots, a_n) = \\ &= \langle a_1, \dots, a_n \rangle T (\langle a_{n+1}, \dots \rangle \cap H(a_1, \dots, a_n)) = \langle a_1, \dots, a_n \rangle T. \end{aligned}$$

We are now ready for the final step. Let  $i, j$  be integers such that  $1 < i < j$ . Then

$$\begin{aligned} X^{j_i} \cap H(a_1, \dots, a_i) &= (X \cap H(a_1, \dots, a_i))^{j_i} = \\ &= X \cap H(a_1, \dots, a_i) \leq \langle a_1, \dots, a_i \rangle T, \end{aligned}$$

by 7. On the other hand,  $a_i^{j_i} \in X^{j_i} \cap H(a_1, \dots, a_i)$ , whereas  $a_i^{j_i} \notin \langle a_1, \dots, a_i \rangle T$  by our choice of  $j_i$ . Thus  $X^{j_i} \cap H(a_1, \dots, a_i) \neq X^{j_i} \cap H(a_1, \dots, a_i)$ , so that  $X^{j_i} \neq X^{j_j}$  and  $X$  has infinitely many images under  $\Gamma$ .

#### 4. Proofs of Theorems C and D.

To prove Theorem C, we note that a group  $G$  in which the endomorphism classes of abelian subgroups are finite must have one of the structures described in (iii) of Theorem B. Since it is clear that  $G$  cannot be the direct product of infinitely many nontrivial groups, if it is periodic it must be of the type required in Theorem C.

To show that  $G$  is periodic, proceed like this. With  $n := |G:Z(G)|$ , the transfer map to the centre is just  $g \mapsto g^n$  [5, 10.1.3]. It follows that every endomorphism of the abelian group  $G^n$  extends to one of  $G$ . If  $G$  has an element of infinite order, so does  $G^n$ ; if  $a$  is such an element, the endomorphism class of  $\langle a \rangle$  under the powers of the endomorphism  $x \mapsto x^2$  of  $G^n$ , and thus of its extension to an endomorphism of  $G$ , is infinite.

Conversely, suppose that  $G$  is a finite central extension of a direct product  $C$  of groups of type  $Z_{p,\infty}$  for different primes  $p$ , let  $H$  be a subgroup of  $G$  and  $\alpha$  an endomorphism of  $G$ . Then  $H^\alpha C/C$  is one of boundedly finitely many subgroups of  $G/C$ , and  $H^\alpha C/C \cong H^\alpha/H^\alpha \cap C$ ; but  $H^\alpha \cap C \geq (H \cap C)^\alpha$ , and since  $C$  is fully invariant and a direct

product of different  $Z_{p^\infty}$ 's,  $(H \cap C)^\alpha$  is one of boundedly many subgroups. Thus  $H^\alpha$  is one of boundedly many subgroups, as required.

Finally, we prove Theorem D. Let  $G$  be a group in which the endomorphism classes have cardinal at most  $k$  (finite). In particular, for any elements  $x, y$  of  $G$  there is an integer  $r \leq k$  such that  $x^r \in C(y)$ , and it follows that  $G/Z(G)$  has finite exponent  $m_1$ , say. Further,  $G'$  is finite since the conjugacy classes of elements are boundedly finite [6], of order  $m_2$  say. Let  $m$  be any positive integer divisible by  $m_1$  and  $m_2$ . For all  $x, y \in G$  we have  $(xy)^m = x^m y^m c$  for some  $c \in G' \cap Z(G)$ , so that  $(xy)^{m^2} = x^{m^2} y^{m^2}$  and the map  $x \rightarrow x^{m^2}$  is an endomorphism of  $G$ . It follows that  $G$  has finite exponent.

Now every periodic group with finite commutator subgroup is locally finite, and thus it has an infinite abelian subgroup if it is infinite. One could refer to [3] for this result, though it is very easy to prove it directly in our case. Suppose that our group  $G$  is infinite and let  $A$  be an infinite abelian subgroup. Then  $A$  contains an infinite elementary abelian  $p$ -subgroup  $P$  say, whence  $G/G'$  must have a non-trivial cyclic direct factor  $\langle xG' \rangle$  of  $p$ -power order. For each  $a \in P$ , the map  $x \mapsto a$  extends to an endomorphism of  $G$ , and thus  $x$  has infinite endomorphism class. This contradiction proves that  $G$  is finite, as required.

#### REFERENCES

- [1] R. BAER, *Finite extensions of abelian groups with minimum condition*, Trans. Amer. Math. Soc., **79** (1955), pp. 521-540.
- [2] I. I. EREMIN, *Groups with finite classes of conjugate abelian subgroups*, Mat. Sb., **47** (1959), pp. 45-54.
- [3] P. HALL - C. R. KULATILAKA, *A property of locally finite groups*, J. London Math. Soc., **39** (1964), pp. 235-239.
- [4] L. FUCHS, *Infinite Abelian Groups*, New York and London, 1970.
- [5] B. HUPPERT, *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg-New York, 1967.
- [6] B. H. NEUMANN, *Groups with finite classes of conjugate subgroups*, Math. Z., **63** (1954), pp. 235-248.
- [7] D. J. S. ROBINSON - J. WIEGOLD, *Groups with boundedly finite automorphism classes*, Rend. Sem. Mat. Univ. Padova, **71** (1984), pp. 273-286.
- [8] J. WIEGOLD, *Multiplicators and groups with finite central factor-groups*, Math. Z., **89** (1965), pp. 345-347.

Manoscritto pervenuto in redazione il 5 marzo 1987.