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Nonadmissible Genealogical Trees.

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In the following, K denotes a field, A denotes the free algebra $K\langle x_1, ..., x_m\rangle$ in $m \geq 2$ non commutative variables, and we always use the term « module » to mean left module. With these hypotheses, we fix the definitions and notations used throughout the paper.

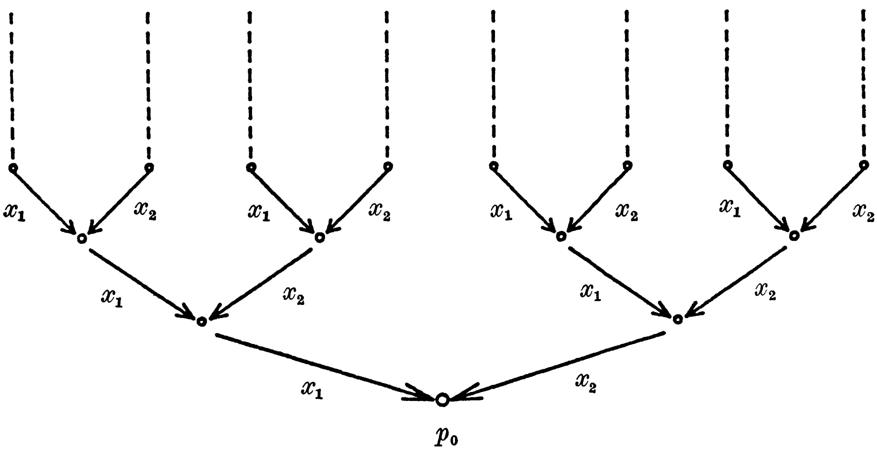
First of all, let T_∞ denote the « genealogical » oriented tree completely determined by the following conditions:

(i) T_∞ has countably many vertices $\{p_n : n \in N\}$ and countably many arrows of m different types, denoted by $x_1, ..., x_m$.

(ii) There is no arrow with starting point p_0, and there is exactly one arrow with starting point p_n for any $n > 0$.

(iii) For any $j = 1, ..., m$ and any $n \in N$, there is exactly one arrow of type x_j with ending point p_n.

If $m = 2$, then T_∞ is of the following form.



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Next, let M_∞ denote the A-module defined as follows:

(a) The underlying K-vector space of M_∞ is K^N.

(b) If v is an element of M_∞ of the form $v = (k_n)_{n\in N}$, and $j = 1, \ldots, m$, then $x_j(v) = (k_{j,n})_{n\in N}$, where, for any n,

$$k_{j,n} = k_i \quad \text{if there is an arrow of the form } \underset{p_i}{\circ} \xrightarrow{x_j} \underset{p_n}{\circ}\,.$$

If v is an element of M_∞ of the form $v = (k_n)_{n\in N}$, then we denote by $\operatorname{supp}(v)$ the set of all $n \in N$ such that $k_n \neq 0$.

For any vertex p of T_∞, we define an element $w(p) \in A$ as follows: first of all $w(p_0) = 1$; secondly, if $p \neq p_0$ and the path along T_∞ from p to p_0 is of the form

$$\underset{p_0}{\circ} \underbrace{\xleftarrow{z_r} \circ \ldots \circ \xleftarrow{z_1}}_{r \text{ arrows}} \underset{p}{\circ}\,, \qquad \text{then } w(p) = z_r \ldots z_1\,.$$

Keeping the notation of [1], and using terminology suggested by [2], we say that a sequence $W = (l_n)_{n\in N}$, with $l_n \in \{x_1, \ldots, x_m\}$ for any n, is a *word* in the letters $x_1, \ldots, x_m$.

We say that an infinite subtree of T_∞ of the form

$$\underset{p_0}{\circ} \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \ldots$$

is a *branch* of T_∞. Moreover, if W is the word $(l_n)_{n\in N}$ and the branch B of T_∞ is of the form

$$\underset{p_0}{\circ} \xleftarrow{l_0} \circ \xleftarrow{l_1} \circ \xleftarrow{l_2} \circ \xleftarrow{l_3} \circ \ldots,$$

then we say that B is *the branch of T_∞ corresponding to W.*

Finally, if T is a subtree of T_∞ obtained by «glueing together branches of T_∞», that is with the property that any vertex of T belongs to a branch of T_∞ contained in T, then we briefly say that T is a *genealogical tree.* For any genealogical tree T, we denote by $M(T)$ the A-submodule of M_∞ defined by the formula

$$M(T) = \{v \in M_\infty \colon p_n \text{ is a vertex of } T \text{ for any } n \in \operatorname{supp}(v)\}\,.$$

We say that a genealogical tree T is *admissible*, if soc $M(T)$ is essential and isomorphic to the simple module $A/\langle x_1, \ldots, x_m \rangle$.

According to this definition, a word W is an *admissible word* in the sense of [1] if and only if the branch of T_∞ corresponding to W is an admissible genealogical tree.

The last two definitions used in the sequel deal with nonadmissible words. We say that a word $W = (l_n)_{n \in N}$ is a *strongly nonadmissible word*, if any word U of the form

$$U = (l_{-r}, \ldots, l_{-1}, l_0, l_1, l_2, l_3, \ldots),$$

with $r \geqslant 1$ and $l_{-i} \in \{l_n : n \in N\}$ for any $i = 1, \ldots, r$, is a nonadmissible word. We say that a word W is a *weakly nonadmissible word*, if W is neither admissible nor strongly nonadmissible.

In section 1, we show that a genealogical tree T is not admissible if and only if T contains a sequence of distinct vertices $(q_n)_{n \in N}$ which are starting points of « connected paths », that is with the property that $w(q_{n+1}) \in Aw(q_n)$ for any n. As we shall see, this characterization of the nonadmissible genealogical trees is the obvious « two-dimensional » version of a characterization, deduced from [1], of the nonadmissible branches of T_∞. Using this result, we give an example of a nonadmissible genealogical tree formed by countably many admissible branches.

In section 2, we first determine all the strongly nonadmissible words. Roughly speaking, we can say that a word is strongly nonadmissible if and only if it is as « chaotic » as might be expected. Next, we prove that there exist as many as possible admissible words, strongly nonadmissible words and weakly nonadmissible words. Finally, we construct an admissible genealogical tree with $2^{\aleph_0}$ branches.

In section 3, we investigate the structure of the A-module $M(T)$ for what is probably the easiest choice of a nonadmissible genealogical tree, namely that of a tree with exactly one branch corresponding to a word W of the form $W = (x, x, x, x, x, \ldots)$ for some letter x. In this case, $M(T)$ is the direct sum of $|M(T)|$ indecomposable A-modules, running through all the indecomposable injective $K[x]$-modules.

A first example of a nonadmissible genealogical tree with all admissible branches was announced at the LMS Durham Symposium on Representations of Algebras (July 1985), and I would like to thank the organizers – and in particular Prof. S. Brenner – for the opportunity of taking part in the meeting.

1. We begin with a result on words.

LEMMA 1 ([1] Theorem 1). *A word* $W = (l_n)_{n\in N}$ *is admissible if and only if there exists some* $d \in N$ *such that* $l_0 \ldots l_d \neq l_{n-d} \ldots l_n$ *for any* $n > d$.

Next we formulate a negative version of Lemma 1.

LEMMA 2. *A word* $W = (l_n)_{n\in N}$ *is not admissible if and only if there exists a strictly increasing sequence of natural numbers* $(d_n)_{n\in N}$ *such that* $l_0 \ldots l_{d_{n+1}} \in Al_0 \ldots l_{d_n}$ *for any* n.

Using the terminology fixed in the introduction, we can restate Lemmas 1 and 2 in the following form.

(*) A branch B of T_∞ is admissible if and only if B contains a vertex $q \neq p_0$ such that $w(p) \notin Aw(q)$ for any vertex p of B different from q.

(**) A branch B of T_∞ is not admissible if and only if B contains a sequence of distinct vertices $(q_n)_{n\in N}$ such that $w(q_{n+1}) \in Aw(q_n)$ for any n.

We shall see at the end of this section that the existence of a special vertex q as in (*) does not characterize the admissible genealogical trees. However the next theorem shows that the existence of a sequence of vertices $(q_n)_{n\in N}$ as in (**) actually characterizes the nonadmissible genealogical trees.

THEOREM 3. *Let* T *be a genealogical tree. Then the following conditions are equivalent*:

(i) T *is a nonadmissible genealogical tree.*

(ii) *There exists a sequence* $(q_n)_{n\in N}$ *of distinct vertices of* T *such that* $w(q_{n+1}) \in Aw(q_n)$ *for any* n.

PROOF. (i) $\Rightarrow$ (ii). The hypothesis that T is a nonadmissible genealogical tree enables us to find a nonzero vector $v \in M(T)$ such that $(1, 0, 0, 0, 0, 0, \ldots) \notin Av$. Consequently, if $f \in A$, then either $f(v) = 0$ or $f(v)$ has infinite support. We claim that, if $n \in \operatorname{supp}(v)$, then there exist infinitely many $i \in \operatorname{supp}(v)$ such that $w(p_i) \in Aw(p_n)$. Indeed, since $w(p_n)(v) \neq 0$, it follows that $w(p_n)(v)$ has infinite support. This implies that the set $\{i \in \operatorname{supp}(v)\colon w(p_i) \in Aw(p_n)\}$ is infinite, as claimed. Hence we may immediately construct, by induction, a sequence of distinct vertices $(q_n)_{n\in N}$ of T with the property that $q_n \in \in \{p_i : i \in \operatorname{supp}(v)\}$ and that $w(q_{n+1}) \in Aw(q_n)$ for any n. Therefore (ii) holds.

(ii) $\Rightarrow$ (i). Let $(q'_n)_{n\in N}$ be a subsequence of $(q_n)_{n\in N}$ such that $\deg w(q'_{n+1}) > 2 \deg w(q'_n)$ for any n. Next let u be an element of $M(T)$ such that $\operatorname{supp}(u) = \{i \in N : p_i = q'_n \text{ for some } n\}$. Then we may write u as an infinite sum of the form $u = \sum_{n\in N} u_n$, where, for any n, the support of u_n has exactly one element s_n and $s_i < s_j$ if $i < j$. We want to show that $(1, 0, 0, 0, 0, 0, \ldots) \notin Au$. To see this, fix any $f \in A$ such that $f(u) \neq 0$. Evidently we can write $f(u)$ as an infinite sum of the form $f(u) = \sum_{n\in N} f(u_n)$. At this point, let $i = \min\{n \in N : f(u_n) \neq 0\}$ and choose some $j \geqslant i$ such that $\deg w(q'_j) > \deg f$. Then all the vectors $f(u_n)$ with $n \geqslant j$ have nonempty and pairwise disjoint supports. Hence $f(u)$ has infinite support, and so $(1, 0, 0, 0, 0, 0, \ldots) \notin Au$. This proves that T is a nonadmissible genealogical tree, as asserted in (i). ■

As an immediate consequence of Theorem 3, we obtain the following corollary.

COROLLARY 4. *Let T be a genealogical tree formed by finitely many admissible branches. Then T is an admissible genealogical tree.*

The next corollary shows that we cannot weaken the hypotheses of Corollary 4.

COROLLARY 5. *There exists a nonadmissible genealogical tree formed by countably many admissible branches.*

PROOF. Let x and y denote two distinct letters from $x_1, \ldots, x_m$. Next let $(q_n)_{n\in N}$ denote the sequence of vertices of T_∞ defined inductively by the formula

$$w(q_n) = \begin{cases} x^2 & \text{if } n = 0 \\ xy^n w(q_{n-1}) & \text{if } n > 0 . \end{cases}$$

Finally, for any n, let B_n denote the branch of T_∞ uniquely determined by the following conditions:

(i) q_n is a vertex of B_n.

(ii) If $n > 0$, then any path along B_n arriving at q_n consists of all arrows denoted by x, while any path along B_0 arriving at q_0 consists of all arrows denoted by y.

Hence the branch B_0 is of the form

$$\underset{p_0}{\circ} \xleftarrow{x} \circ \xleftarrow{x} \underset{q_0}{\circ} \xleftarrow{y} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \ldots ,$$

while, for any $n > 0$, the branch B_n is of the form

$$\underset{p_0}{\circ} \xleftarrow{x} \circ \underbrace{\xleftarrow{y} \circ \ldots \circ \xleftarrow{y}}_{n \text{ arrows}} \circ \ldots \underset{q_n}{\circ} \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \ldots .$$

At this point, let T denote the genealogical tree obtained by glueing together all the branches B_n's. Since $(q_n)_{n\in N}$ is a sequence of vertices of T satisfying condition (ii) of Theorem 3, it follows that T is a non-admissible genealogical tree. On the other hand, let B be a branch of T. Then either $B = B_n$ for some n, or $B = B_\infty$, where B_∞ is the following branch of T_∞:

$$\underset{p_0}{\circ} \xleftarrow{x} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \ldots .$$

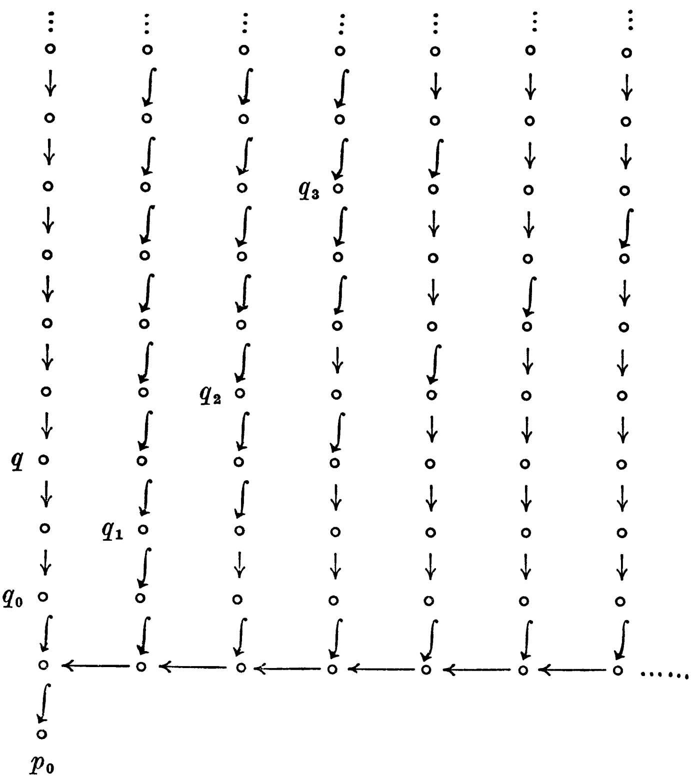
In both cases, Lemma 1 guarantees that B is an admissible branch. This completes the proof of the corollary. ∎

Remark. We can now justify the observation preceding Theorem 3. Indeed, let T be the genealogical tree constructed in the proof of Corollary 5, and let q be the vertex of T completely determined by the property that $w(q) = x^2y^2$. We claim that, if p is a vertex of T different from q, then $w(p) \notin Ax^2y^2$. To see this, we proceed by induction on the index n of the branch B_n containing p. If either $n = 0$ or $n = 1$, then the assertion is obvious. Hence we may assume that p is a vertex of the branch B_n with $n \geqslant 2$, and that the assertion holds for all the vertices of the branch B_{n-1}. Now let $(l_n)_{n\in N}$ denote the word corresponding to the branch B_{n-1}, and let $(l_n^*)_{n\in N}$ denote the word corresponding to the branch B_n. Then, by the inductive hypothesis, $l_n l_{n+1} l_{n+2} l_{n+3} \neq x^2y^2$ for any n. On the other hand, by the definition of B_n, we may write

$$(l_n^*)_{n\in N} = (x, \underbrace{y, \ldots, y}_{n}, l_0, l_1, l_2, l_3, \ldots) .$$

This implies that $l^*_n l^*_{n+1} l^*_{n+2} l^*_{n+3} \neq x^2 y^2$ for any n. Hence $w(p) \notin \notin Ax^2y^2$, and so the assertion holds for all the vertices of the branch B_n. Consequently, $w(p) \notin Ax^2y^2$ for any vertex p of T different from q. Since T is a nonadmissible genealogical tree, we conclude that the obvious generalization of (∗) does not determine all the admissible genealogical trees.

With the convention that $\circ \leftarrow\!\!\!\rightharpoondown \circ$ stands for $\circ \xleftarrow{x} \circ$, and that $\circ \longleftarrow \circ$ stands for $\circ \xleftarrow{y} \circ$, we may visualize the structure of T as follows.



2. The next theorem characterizes strongly nonadmissible words.

THEOREM 6. *Let $W = (l_n)_{n \in N}$ be a word. Then the following conditions are equivalent*:

(i) *W is a strongly nonadmissible word.*

(ii) *If r is a positive integer and w is a monomial of degree r in the letters $\{l_n : n \in N\}$, then there exists some n such that $w = l_n \dots l_{n+r-1}$.*

PROOF. (i) $\Rightarrow$ (ii). Let W be a strongly nonadmissible word, and assume, contrary to (ii), that there is a monomial w of the form $w = l_{-r} \dots l_{-1}$, with $r \geqslant 1$ and $l_{-i} \in \{l_n : n \in N\}$ for any $i = 1, \dots, r$, such that $w \neq l_n \dots l_{n+r-1}$ for any n. To find a contradiction, let $U = (l'_n)_{n \in N}$ denote the word

$$U = (l'_n)_{n \in N} = (l_{-r}, \dots, l_{-1}, l_0, l_1, l_2, l_3, \dots).$$

Then $l'_0 \dots l'_{r-1} = l_{-r} \dots l_{-1} = w$, while $l'_{n-r+1} \dots l'_n = l_{n-2r+1} \dots l_{n-r} \neq w$ for any $n \geqslant 2r - 1$. By Lemma 2, this implies that U cannot be a nonadmissible word. Hence W cannot be a strongly nonadmissible word. This contradiction shows that there is some n such that $w = l_n \dots l_{n+r-1}$, and so (ii) holds.

(ii) $\Rightarrow$ (i). Assume that W satisfies condition (ii). Now let $U = (l'_n)_{n \in N}$ be a word of the form

$$U = (l'_n)_{n \in N} = (l_{-r}, \dots, l_{-1}, l_0, l_1, l_2, l_3, \dots),$$

with $r \geqslant 1$ and $l_{-i} \in \{l_n : n \in N\}$ for any $i = 1, \dots, r$. We claim that U is a nonadmissible word. To see this, let d be a natural number, and let w denote the monomial $w = l'_0 \dots l'_d$. Then, by (ii), we can find some n such that $w = l_n \dots l_{n+d}$. Since $l_n \dots l_{n+d} = l'_{n+r} \dots l'_{n+r+d}$, it follows that $w = l'_0 \dots l'_d = l'_{n+r} \dots l'_{n+r+d}$ with $n + r \geqslant r > 0$. Consequently, by Lemma 1, U is a nonadmissible word, as claimed. Hence (i) holds, and the theorem is proved. ■

The following corollary gives a « quantitative » result on words.

COROLLARY 7. *There exist $2^{\aleph_0}$ admissible words, strongly nonadmissible words and weakly nonadmissible words.*

PROOF. Let x and y denote two distinct letters from $x_1, \dots, x_m$. We divide the proof in three steps.

Step 1. Let $\alpha = (a_n)_{n \in N}$ be a strictly increasing sequence of positive integers, and let W_α denote the word

$$W_\alpha = (\underbrace{x, \dots, x}_{a_0}, y, \underbrace{x, \dots, x}_{a_1}, y, \underbrace{x, \dots, x}_{a_2}, y, \dots).$$

More precisely, let $W_\alpha = (l_n)_{n\in N}$, where

$$l_n = \begin{cases} y & \text{if } n = i + \sum_{j=0}^{i} a_j \quad \text{for some } i \in N \\ x & \text{otherwise}. \end{cases}$$

Now let $d = a_0 + a_1 + 1$; then $l_0 \dots l_d = x^{a_0} y x^{a_1} y$. Since α is a strictly increasing sequence, we have $l_0 \dots l_d \neq l_{n-d} \dots l_n$ for any $n > d$. Hence, by Lemma 1, W_α is an admissible word. Since the map $\alpha \mapsto W_\alpha$ is injective, there exist $2^{\aleph_0}$ admissible words.

Step 2. As in Step 1, let $\alpha = (a_n)_{n\in N}$ be a strictly increasing sequence of positive integers. Next, let $\alpha^* = (a_n^*)_{n\in N}$ be the «periodic» sequence defined as follows. First of all, if $n \in N$, then $a^*_{2^{n+1}-2} = a_n$ and $2^{n+1} - 2 = \min\{i \in N: a_i^* = a_n\}$; secondly, we choose $a_1^* = a_0$. Assume now, by induction, that, for some $n \geqslant 1$, we have already defined all the elements a_i^* with $0 \leqslant i \leqslant 2^{n+1} - 2$. Then we define the elements a_i^* with $2^{n+1} - 1 \leqslant i \leqslant 2^{n+2} - 3$ by means of the equality $(a^*_{2^{n+1}-1}, \dots, a^*_{2^{n+2}-3}) = (a_0^*, \dots, a^*_{2^{n+1}-2})$. Now let W_{α^*} denote the word

$$W_{\alpha^*} = (\underbrace{x, \dots, x}_{a_0^*}, y, \underbrace{x, \dots, x}_{a_1^*}, y, \underbrace{x, \dots, x}_{a_2^*}, y, \dots),$$

that is let $W_{\alpha^*} = (l_n)_{n\in N}$, where

$$l_n = \begin{cases} y & \text{if } n = i + \sum_{j=0}^{i} a_j^* \\ x & \text{otherwise}. \end{cases}$$

Since $l_n l_{n+1} \neq y^2$ for any n, we deduce from Theorem 6 that W_{α^*} cannot be a strongly nonadmissible word. We claim that W_{α^*} is not admissible. Indeed, fix any $d \in N$. Then the definition of α^* enables us to find two natural numbers r and n with $r \geqslant d$ and $n \geqslant 1$ such that $l_0 \dots l_d \dots l_r = x^{a_0^*} y \dots y x^{a^*_{2^{n+1}-2}}$. Since

$$(a_0^*, \dots, a^*_{2^{n+1}-2}) = (a^*_{2^{n+1}-1}, \dots, a^*_{2^{n+2}-3}),$$

there exists some $s > 0$ such that $l_0 \dots l_d \dots l_r = l_s \dots l_{s+d} \dots l_{s+r}$. Consequently $l_0 \dots l_d = l_s \dots l_{s+d}$ with $s > 0$. Hence, by Lemma 1, W_{α^*}

is a nonadmissible word. Therefore W_{α^*} is a weakly nonadmissible word. Since the map $\alpha^* \mapsto W_{\alpha^*}$ is injective, we obtain $2^{\aleph_0}$ weakly nonadmissible words.

Step 3. Let σ be a sequence of the form $\sigma = (w_n)_{n \in N}$, where w_n runs through all the monomials of positive degree in the letters x and y. For any n, let d_n denote the degree of w_n. Next, let

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ d_0 + \ldots + d_{n-1} & \text{if } n > 0 \,, \end{cases}$$

and let $b_n = d_0 + \ldots + d_n - 1$ for any n. Finally, let W_σ denote the word $(l_n)_{n \in N}$ defined by glueing together all the monomials w_n, as illustrated in the following picture.

$$\boxed{w_0}\,\boxed{w_1}\,\boxed{w_2}\,\boxed{w_3}\;\ldots \rightsquigarrow W_\sigma \,.$$

More precisely, let W_σ denote the word $(l_n)_{n \in N}$ uniquely determined by the condition that $l_{a_n} \ldots l_{b_n} = w_n$ for any n. Then W_σ obviously satisfies condition (ii) of Theorem 6, and so W_σ is a strongly nonadmissible word. Since the map $\sigma \mapsto W_\sigma$ is injective, there exist $2^{\aleph_0}$ strongly nonadmissible words.

The corollary now follows from Steps 1, 2 and 3. ■

Finally, we give an example of a very large admissible genealogical tree.

COROLLARY 8. *There exists an admissible genealogical tree formed by* $2^{\aleph_0}$ *branches.*

PROOF. Let T be the genealogical tree obtained by glueing together all the branches of T_∞ corresponding to the words W_α constructed in Step 1 of the proof of Corollary 7. Hence any W_α is of the form

$$W_\alpha = (\underbrace{x, \ldots, x}_{a_0}, y, \underbrace{x, \ldots, x}_{a_1}, y, \underbrace{x, \ldots, x}_{a_2}, \ldots) \,,$$

where $\alpha = (a_n)_{n \in N}$ is a strictly increasing sequence of positive integers. We claim that T is an admissible genealogical tree. Suppose the contrary, and let $(q_n)_{n \in N}$ be a sequence of vertices of T satisfying condition (ii) of Theorem 3. Then we can find two natural numbers i

and j with $i > 0$ such that $w(q_n) \in Ayx^iyA$ for any $n \geqslant j$. This implies that the set

$$S = \{p : p \text{ vertex of } T,\ w(p) \in Ayx^iy\}$$

is infinite. To see that this is impossible, fix any $p \in S$. Then, by the definition of T, there exists a strictly increasing sequence of positive integers $\alpha = (a_n)_{n\in N}$ such that $i = a_r$ for some $r \in N$ and $w(p) = x^{a_0}y \dots yx^{a_r}y$. Since $r + 1 \leqslant i$, it follows that

$$\underbrace{a_0 + \dots + a_r}_{r+1} + r + 1 \leqslant \underbrace{i + \dots + i}_{i} + i = i(i+1)\,.$$

and so $\deg w(p) \leqslant i(i+1)$. This means that S is finite, and this is the desired contradiction. This contradiction shows that T is an admissible genealogical tree, as claimed. Since T has $2^{\aleph_0}$ branches, the corollary is proved. ∎

REMARK. The above proof shows that the assertion of Corollary 8 holds for $m = 2$, and so for any $m \geqslant 2$. However, if $m \geqslant 3$, then it is even easier to construct an admissible genealogical tree with $2^{\aleph_0}$ branches. In fact, let T be the genealogical tree obtained by glueing together all the branches B of T_∞ corresponding to words W of the form $W = (l_n)_{n\in N}$ with $l_0 = x_3$ and $l_n \in \{x_1, x_2\}$ for any $n > 0$. Then evidently T does not satisfy condition (ii) of Theorem 3, and so T is an admissible genealogical tree with $2^{\aleph_0}$ branches.

3. We do not know the structure of an A-module of the form $M(T)$ with T a nonadmissible genealogical tree. However, the next proposition shows that, if T is a nonadmissible genealogical tree, then $M(T)$ may be very far from being indecomposable.

PROPOSITION 9. *There exists a genealogical tree T such that $M(T)$ is the direct sum of $|M(T)|$ indecomposable A-modules, running through all the indecomposable injective $K[x]$-modules.*

PROOF. Let x be a letter from $x_1, \dots, x_m$, and let T be the genealogical tree of the form

$$\underset{p_0}{\circ} \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \dots .$$

More precisely, let T be the branch of T_∞ corresponding to the word

$(l_n)_{n\in N}$ with $l_n = x$ for any n. We claim that T satisfies the hypotheses of Proposition 9. To see this, we divide the proof in three steps. Throughout the proof, we denote by M the module $M(T)$ and by $t(M)$ the torsion submodule of M.

Step 1. *M is an injective $K[x]$-module.*

PROOF. Let f be an element of $K[x]$ of the form $f = x^i + a_{i-1}\cdot x^{i-1} + \ldots + a_1 x + a_0$ for some $i > 0$, and let v be a vector of M of the form $v = (k_n)_{n\in N}$. Now let $\bar{v}$ denote the vector $\bar{v} = (\bar{k}_n)_{n\in N}$ defined inductively by the formula

$$\bar{k}_n = \begin{cases} 0 & \text{if } 0 \leqslant n \leqslant i-1\,, \\ k_{n-i} - (a_{i-1}\bar{k}_{n-1} \ldots + a_0\bar{k}_{n-i}) & \text{if } n \geqslant i\,. \end{cases}$$

Then, for any $n \geqslant i$, we obtain

$$\bar{k}_n + a_{i-1}\bar{k}_{n-1} + \ldots + a_0\bar{k}_{n-i} =$$
$$= \big(k_{n-i} - (a_{i-1}\bar{k}_{n-1} + \ldots + a_0\bar{k}_{n-i})\big) + a_{i-1}\bar{k}_{n-1} + \ldots + a_0\bar{k}_{n-i} = k_{n-i}\,.$$

It follows that $f(\bar{v}) = v$. Therefore M is a divisible $K[x]$-module, and so, by ([3] Theorem 2.8), M is an injective $K[x]$-module.

Step 2. $\operatorname{soc} t(M) \cong \bigoplus_p K[x]/(p)$ *with p running through all the monic and irreducible polynomials of $K[x]$.*

PROOF. Let f be an element of $K[x]$ of the form $f = x^i + a_{i-1}\cdot x^{i-1} + \ldots + a_1 x + a_0$ for some $i > 0$, and let $V_f = \{v \in t(M): f(v) = 0\}$. We shall prove that V_f is a cyclic $K[x]$-module isomorphic to $K[x]/(f)$. To this end, we first note that, if $v \in V_f$ and v is of the form $v = (k_n)_{n\in N}$, then

$$k_{n+i} + a_{i-1}k_{n+i-1} + \ldots + a_1 k_{n+1} + a_0 k_n = 0 \qquad \text{for any } n\,.$$

Consequently, for any element $(c_0, \ldots, c_{i-1}) \in K^i$ there exists a unique element $(c_n^*)_{n\in N} \in V_f$ satisfying $c_n^* = c_n$ for any $n = 0, \ldots, i-1$. This means that the canonical projection $\pi\colon M \to K^i$, such that

$$\pi\big((k_n)_{n\in N}\big) = (k_0, \ldots, k_{i-1}) \qquad \text{for any } (k_n)_{n\in N} \in M\,,$$

induces a K-vector space isomorphism between V_f and K^i. Next, let v_f denote the element of V_f uniquely determined by the condition that $\pi(v_f) = (0, \ldots, 0, 1)$. Then $\{\pi(x^j(v_f)) : j = 0, \ldots, i-1\}$ is obviously a base of the K-vector space K^i. This proves that $\{x^j(v_f) : j = 0, \ldots, i-1\}$ is a base of the K-vector space V_f; hence $V_f = K[x]v_f$. Since $\dim_K(V_f) = i$ and $f \in \mathrm{ann}_{K[x]}(v_f)$, it follows that $\mathrm{ann}_{K[x]}(v_f) = (f)$. Thus V_f is isomorphic to $K[x]/(f)$, as claimed. We also note that $t(M) = \sum_f V_f$ with f running through all the monic polynomials of $K[x]$ of positive degree. Therefore

$$(*) \qquad |t(M)| = \max\{|K|, \aleph_0\} = |K(x)|$$

and $\mathrm{soc}\, t(M) = \sum_p V_p = \bigoplus_p V_p$, where p ranges over all the monic and irreducible polynomials of $K[x]$.

Step 3. M has a decomposition of the form $M = t(M) \oplus V_0$, where V_0 is a vector space over $K(x)$ and $\dim_{K(x)}(V_0) = |M|$.

PROOF. The existence of a $K(x)$-vector space V_0 such that $M = t(M) \oplus V_0$ follows from Step 1 and ([3] Theorem 4.4 and Corollary to Theorem 2.32). Now let v be the following element of M:

$$v = (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots).$$

More precisely, let $v = (k_n)_{n \in N}$, where

$$k_n = \begin{cases} 1 & \text{if either } n = 0 \text{ or } n = \sum_{j=2}^{i} j \text{ for some } i \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

We claim that the vectors $\{x^n(v) : n \in N\}$ are K-linearly independent. Indeed, suppose, by contradiction, that this is not true. Then, for some $i \in N$, we may write $x^{i+1}(v) = \sum_{j=0}^{i} t_j x^j(v)$ with $t_j \in K$ for any $j = 0, \ldots, i$. On the other hand, by the definition of v, we can find some n such that $n + i + 1 \in \mathrm{supp}(v)$ and $n + j \notin \mathrm{supp}(v)$ for any $j = 0, \ldots, i$. Consequently $n \in \mathrm{supp}(x^{i+1}(v))$ and $n \notin \mathrm{supp}(x^j(v))$ for any $j = 0, \ldots, i$. Hence $x^{i+1}(v) \neq \sum_{j=0}^{i} t_j x^j(v)$, contrary to the hypothesis.

This contradiction shows that the vectors $\{x^n(v): n \in N\}$ are K-linearly independent. Therefore $V_0 \neq 0$, and we deduce from (*) that

$$(**) \qquad |V_0| = |K(x)| \dim_{K(x)}(V_0) = |t(M)| \dim_{K(x)}(V_0).$$

To end the proof, we distinguish two cases.

Suppose first that $|t(M)| < |M|$. Then we clearly have $|V_0| = |M|$. Moreover, by (**), $|V_0| = \dim_{K(x)}(V_0)$. Consequently $\dim_{K(x)}(V_0) = |M|$, as desired.

Assume now that $|t(M)| = |M|$. In the case, we first note that, by (*), we have

$$(1) \qquad |M| = |K| > \aleph_0.$$

Next, let $\mathcal{B}$ denote a base of the $K(x)$-vector space V_0, and let E denote the smallest subfield of K with the property that, if $v \in \mathcal{B}$ and $v = (k_n)_{n\in N}$, then $k_n \in E$ for any n. Then evidently

$$(2) \qquad |E| \leqslant \max\{\dim_{K(x)}(V_0), \aleph_0\}.$$

Suppose, by contradiction, that $|E| < |K|$. Then, by (1), $\operatorname{tr deg}(K/E)$, the transcendence degree of K over E, is infinite. Hence we may choose a vector $v^* \in M$ of the form $v^* = (k_n^*)_{n\in N}$ such that

$$(3) \qquad k_{n+1}^* \quad \text{is not algebraic over} \quad E(k_0^*, \ldots, k_n^*) \quad \text{for any } n.$$

To find a contradiction, we write v^* in the form $v^* = v' + v''$ with $v' = (k_n')_{n\in N} \in t(M)$ and $v'' = (k_n'')_{n\in N} \in V_0$. Then there exist $f', f'' \in K[x]$, of the form

$$f' = x^i + a_{i-1}x^{i-1} + \ldots + a_1 x + a_0$$

and

$$f'' = x^j + b_{j-1}x^{j-1} + \ldots + b_1 x + b_0$$

with $i, j > 0$, such that $f'(v') = 0$ and $f''(v'') = \sum_{i=1}^{r} f_i(v_i)$ with $r \geqslant 1$, $f_i \in K[x]$ and $v_i \in \mathcal{B}$ for any $i = 1, \ldots, r$. Let $c_1, \ldots, c_h$ denote the coefficients of $f_1, \ldots, f_r$, and let F denote the following subfield of K:

$$F = E(a_0, \ldots, a_{i-1}, b_0, \ldots, b_{j-1}, c_1, \ldots, c_h).$$

At this point, the hypothesis that $f'(v') = 0$ guarantees that

$$k'_{n+i} + a_{i-1}k'_{n+i-1} + \dots + a_1 k'_{n+1} + a_0 k'_n = 0 \quad \text{for any } n\,.$$

Consequently

$$(4) \qquad k'_n \in F(k'_0, \dots, k'_{i-1}) \quad \text{for any } n\,.$$

On the other hand, the hypothesis that $f''(v'') = \sum_{i=1}^{r} f_i(v_i)$ implies that

$$k''_{n+j} + b_{j-1}k''_{n+j-1} + \dots + b_1 k''_{n+1} + b_0 k''_n \in F \quad \text{for any } n\,.$$

Therefore

$$(5) \qquad k''_n \in F(k''_0, \dots, k'_{j-1}) \quad \text{for any } n\,.$$

Putting (4) and (5) together, we conclude that

$$(6) \qquad k^*_n = k'_n + k''_n \in F(k'_0, \dots, k'_{i-1}, k''_0, \dots, k''_{j-1}) \quad \text{for any } n\,.$$

Finally, let F^* denote the field $F^* = F(k^*_n : n \in N)$. Then (6) implies that $\operatorname{tr\,deg}(F^*/F)$ is finite. Since $\operatorname{tr\,deg}(F/E)$ is obviously finite, it follows that also $\operatorname{tr\,deg}(F^*/E)$ is finite, contrary to the hypothesis that v^* satisfies (3). This contradiction shows that $|E| = |K|$. By (1) and (2), this implies that $\dim_{K(x)}(V_0) = |M|$.

Combining Steps 1, 2 and 3, we see that the genealogical tree T satisfies the hypotheses of Proposition 9. ∎

The A-module M constructed in the proof of Proposition 9 gives a « concrete » example of an injective cogenerator for the category of all $K[x]$-modules. We also note that the indecomposable summands of M are, in a sense, the « smallest » possible indecomposable summands of an A-module of the form $M(B)$ with B a branch of T_∞. In fact, we have the following corollary.

COROLLARY 10. *Let B be a branch of T_∞ and let $z = x_1 + \dots + x_m$. Then $M(B)$, regarded as a $K[z]$-module, is the direct sum of $|M(B)|$ injective $K[z]$-modules.*

PROOF. Replace x by z in the proof of Proposition 9. ∎

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