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C-uniform distribution of entire functions

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C-Uniform Distribution of Entire Functions.

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SUMMARY - It is proved that under certain conditions on the growth of the entire function $f(z)$ the curve $f(t)$ (for real t) is uniformly distributed modulo 1 in the complex plane. The same is valid for two-dimensional flows $f(s+it)$. Furthermore two uniformly distributed functions, the exponential function $f(t) = \exp[at]$ and the Weierstraß σ-function $f(t) = \sigma(t)$ (not satisfying the growth condition) are investigated.

1. Introduction.

A continuous function $f\colon [0, \infty) \to \mathbb{R}^d$ is said to be uniformly distributed modulo 1 (for short: u.d.) if

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \chi_I(\{f(t)\})\, dt = \lambda(I) \tag{1.1}$$

holds for all boxes $I = [a_1, b_1]\times\ldots\times[a_d, b_d] \subseteq [0, 1)^d$; χ_I is the characteristic function of I and $\lambda(I)$ its Lebesgue measure, $\{f(t)\} = f(t) - [f(t)]$ denotes the componentwise fractional part of $f(t)$. If $\{f(t)\}$ is interpreted as a particle's motion on the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$, definition (1.1) means that the ratio of the particle's stay in any box to the whole time converges to the volume of the box. A quantitative

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measure for the convergence in (1.1) is the discrepancy

$$D_T(f) = \sup_I \left| \frac{1}{T} \int_0^T \chi_I(\{f(t)\})\, dt - \lambda(I) \right|. \tag{1.2}$$

It is well known that $f(t)$ is u.d. if and only if $D_T(f)$ tends to 0 (for $T \to \infty$); cf. [4]. By a famous criterion due to H. Weyl [10] (1.1) is equivalent to

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \exp\left[2\pi i \langle h, f(t)\rangle\right] dt = 0 \tag{1.3}$$

for all integral lattice points $h \neq 0$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$.

As general references for the theory of uniformly distributed functions we propose the monographs by E. Hlawka [5] and by L. Kuipers and H. Niederreiter [6].

In this article we study the distribution behaviour of an (in general complex valued) entire function $f(t)$ considered as a mapping $[0, \infty) \to \mathbb{R}^2$. If all coefficients f_n of the Taylor expansion of $f(z)$ are real we consider f as a function $[0, \infty) \to \mathbb{R}$.

In the special case $f_n \geqslant 0$ (for $n \geqslant n_0$) Satz 8 of E. Hlawka [4] immediately yields

$$D_T(f) = O\left(\frac{1}{T}\right), \tag{1.4}$$

since $f(t)$ is in this case an increasing and convex function (for $t \geqslant t_0$). In section 2 we are interested in entire functions f of very small growth; more precisely we assume

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} < \frac{3}{2}, \tag{1.5}$$

where $M(r) = \max_{|z|\leqslant r} |f(z)|$. We will prove: If f is of type (1.5) and has real Taylor coefficients, then $f(t)$ is u.d. in $\mathbb{R}$. An analogon for complex f_n is an immediate consequence of this result. We remark that similar theorems (with 4/3 instead of 3/2) for the uniform distribution of sequences are due to G. Rauzy [7], G. Rhin [8] and

R. C. Baker [1], [2]. The examples of G. Rauzy [7] and R. C. Baker [1] show also in the case of uniformly distributed functions that the constant 3/2 cannot be replaced by a constant $c > 2$.

In section 3 we discuss the exponential function $\exp[at]$ $(a \in \mathbb{C})$ and the Weierstrass σ-function and give estimates for the discrepancy. In the final section 4 we extend the previous results to the case of two-dimensional flows $f(s + it)$.

2. Entire functions of very small growth.

THEOREM 1. *Let $f(z)$ be a (non constant) entire function satisfying* (1.5) *such that all Taylor coefficients of f are real. Then $f(t)$ (considered as a function $[0, \infty) \to \mathbb{R}$) is u.d.*

COROLLARY 1. *Let $f(z)$ be an entire function satisfying* (1.5) *such that either the quotient* $\mathrm{Re}(f_n)/\mathrm{Im}(f_n)$ *is irrational for some Taylor coefficient f_n with $n \geqslant 1$ or there are two Taylor coefficients f_n, f_m $(1 \leqslant m < n)$ with* $\mathrm{Re}(f_m)/\mathrm{Im}(f_m) \neq \mathrm{Re}(f_n)/\mathrm{Im}(f_n)$. *Then $f(t)$ (considered as a function $[0, \infty) \to R^2$) is u.d.*

PROOF OF COROLLARY 1. We apply Theorem 1 to the function

$$g(t) = \sum_{n=0}^{\infty} \big(h_1 \mathrm{Re}(f_n) + h_2 \mathrm{Im}(f_n)\big) \cdot t^n, \qquad h = (h_1, h_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}.$$

Then Weyl's criterion (1.3) immediately yields the result of the corollary.

REMARK 1. In the case of uniformly distributed sequences R. C. Baker [2] has shown that, in general, there is no estimate for the discrepancy of entire functions satisfying (1.5). Since we use similar techniques for the proof of Theorem 1 as have been used for sequences it is not possible to obtain a general estimate for the discrepancy by this method. Nevertheless estimates can be proved for special functions, compare (1.4).

PROOF OF THEOREM 1. We will apply Weyl's criterion and set

$$S(T) = \int_0^T \exp[2\pi i h f(t)]\, dt, \qquad h \in \mathbb{Z} \setminus \{0\}.$$

As in [7] we make use of increasing sequences (n_k), (P_k), (Q_k) tending to infinity and satisfying $P_{k+1} < Q_k$ for sufficiently large k (for details see Lemma 1). In Lemma 3 we will prove that

$$|S(T+P_k) - S(T)| < \varepsilon P_k \tag{2.1}$$

holds for every $\varepsilon > 0$, for all sufficiently large $k \geqslant k_0(\varepsilon)$ and all $T \leqslant Q_k - P_k$. Choosing ε, k such that (2.1) holds and $C = C(\varepsilon) = Q_{k_0(\varepsilon)}$ we obtain by induction

$$|S(T)| < \varepsilon T + C \quad \text{for all } T \in [0, Q_{k-1}]\,. \tag{2.2}$$

Trivially (2.2) is valid for $k = k_0$. Assume that (2.2) holds for some $k \geqslant k_0$. If $T \in [0, Q_k]$ then $T = [T/P_k]P_k + R$ with $R < P_k < Q_{k-1}$ and (2.1) combined with the assumption yields

$$|S(T)| \leqslant |S(T) - S(R)| + |S(R)| < \varepsilon \left[\frac{T}{P_k}\right] P_k + \varepsilon R + C = \varepsilon T + C\,.$$

Thus (2.2) is proved for all $\varepsilon > 0$ and all $k \geqslant k_0(\varepsilon)$.

Since $\lim_{k\to\infty} Q_k = \infty$, we derive from (2.2) for every $\varepsilon > 0$

$$\limsup_{T\to\infty} \left|\frac{S(T)}{T}\right| \leqslant \varepsilon\,;$$

hence $f(t)$ is u.d.

In the following we give a detailed definition of the above sequence (n_k), (P_k), (Q_k) and establish some essential properties. From condition (1.5) we obtain by Cauchy's inequality

$$\lim_{n\to\infty} \frac{\log(1/|f_n|)}{n^c} = +\infty \tag{2.3}$$

for some $c > 3$ (f_n denote the Taylor coefficients of f). If f is a non constant polynomial estimate (1.4) can be applied; hence f is u.d. in this case. In the following we assume that $f_n \neq 0$ for infinitely many n. Set $n_0 = \min\{n: f_n \neq 0\}$. If n_k is defined set $l_k = \log(1/|f_{n_k}|)$ and

$$m_k = \sup\{m: \log(1/|f_n|) \geqslant l_k + m(n - n_k) \text{ for all } n \geqslant n_k\}$$

and define

$$n_{k+1} = \max\{n > n_k : \log(1/|f_n|) = l_k + m_k(n - n_k)\} . \tag{2.4}$$

(By (2.3) the sequences (m_k) and (n_k) are well-defined.) Furthermore we set

$$\begin{cases} p_k = \dfrac{n_k l_k}{n_k^2 - 1}, & P_k = \exp[p_k], \\ M_k = \exp[m_k], & Q_k = \dfrac{4^{n_k+1}}{M_k}. \end{cases} \tag{2.5}$$

LEMMA 1.

(i) $\log(1/|f_n|) \geqslant l_k + m_k(n - n_k)$ *for all* $n \geqslant n_k$

(ii) $l_{k+1} = l_k + m_k(n_{k+1} - n_k)$

(iii) $n_k < n_{k+1}, \quad m_k < m_{k+1}$

(iv) $\lim\limits_{k\to\infty} \dfrac{m_k}{n_k^{c-1}} = \infty, \quad \lim\limits_{k\to\infty} \dfrac{l_k - p_k}{n_k^c} = \infty$

(v) $\lim\limits_{k\to\infty} \dfrac{m_k - p_{k+1}}{n_{k+1}^{c-2}} = \infty$

(vi) $\lim\limits_{k\to\infty} \dfrac{m_k + l_k - (n_k + 1)p_k}{n_k^{c-2}} = \infty$

(vii) $\lim\limits_{k\to\infty} \dfrac{p_k - (l_k/n_k)}{n_k^{c-3}} = \infty$

(viii) $P_{k+1} < Q_k$ *for every* $k \geqslant k_0$

(ix) $\lim\limits_{k\to\infty} 2^{n_k+2}(|f_{n_k}| P_k^{n_k+1})/M_k = 0$

(x) $\lim\limits_{k\to\infty} (|f_{n_k}|/2)^{-1/n_k} P_k^{-1} = 0 .$

The proof of (i) to (vii) can be given by verbally the same arguments as in [7]. Properties (viii), (ix), (x) are immediate consequences of the former ones.

In order to show (2.1) we make use of the following lemma.

LEMMA 2. *Let* $p(t) = at^N + a_1 t^{N-1} + \dots + a_{N-1}t + a_N$ *be a polynomial of degree* N *with real coefficients. Then for all* $A < B$

$$\left|\int_A^B \exp\,[2\pi i p(t)]\,dt\right| \leqslant \frac{26}{|a|^{1/N}}\,.$$

PROOF. Using the substitution $u = t(N|a|)^{1/N}$ for $N \geqslant 2$ we have to prove

$$\left|\int_\alpha^\beta \exp\,[2\pi i q(u)]\,du\right| \leqslant 26 \tag{2.6}$$

for any α, β and a polynomial $q(u) = (1/N)u^N + b_1 u^{N-1} + \dots + b_N$. Applying Theorem 3.4.1 of R. P. Boas [3] we have $|q'(u)| \leqslant K$ for every $K > 0$ and all $u \in S$, where the set S is the union of at most $(N-1)$ intervals and its measure is $\leqslant 12K^{1/(N-1)}$.

Therefore there are at most N intervals (contained in $[\alpha, \beta]$) where q is monotone and $|q'| > K$. Hence the second mean value theorem yields on such an interval I

$$\left|\int_I \exp\,[2\pi i q(u)]\,du\right| \leqslant \frac{2}{K}\,.$$

Combining this with the trivial bound

$$\left|\int_S \exp\,[2\pi i q(u)]\,du\right| \leqslant 12K^{1/(N-1)}$$

and choosing $K = N$ yields (2.6). Thus the proof of Lemma 2 is finished, since the case $N = 1$ is trivial.

To complete the proof of Theorem 1 it remains to show estimate (2.1). This is worked out in the following Lemma.

LEMMA 3. *For every* $\varepsilon > 0$ *and sufficiently large* $k \geqslant k_0(\varepsilon)$ *we have*

$$|S(T + P_k) - S(T)| < \varepsilon P_k \qquad \textit{for } T \leqslant Q_k - P_k\,.$$

PROOF. Set

$$g(t) = \sum_{j=0}^{n_k} \frac{(t-T)^j}{j!} f^{(j)}(T),$$

$$\alpha_k = \frac{f^{(n_k)}(T)}{n_k!}, \qquad S' = \int_T^{T+P_k} \exp[2\pi i h g(t)]\,dt \qquad (h \in \mathbb{Z}\setminus\{0\}).$$

By Lemma 1 (i) we obtain for $n \geqslant n_k$

$$|f_n| \leqslant |f_{n_k}| M_k^{n_k - n};$$

therefore

$$\left|\frac{f^{(n_k+1)}(t)}{(n_k+1)!}\right| \leqslant \left|\sum_{n=n_k+1} \binom{n}{n_k+1} f_n t^{n-(n_k+1)}\right| \leqslant \frac{|f_{n_k}|}{M_k}\left(\frac{1}{1-Q_k/M_k}\right)^{n_k+2} \tag{2.7}$$

for $t \in [0, Q_k]$. Hence by Taylor's formula and $Q_k \leqslant M_k/2$

$$|f(t) - g(t)| \leqslant 2^{n_k+2} \frac{f_{n_k}}{M_k} P_k^{n_k+1} \tag{2.8}$$

for $t \in [T, T+P_k]$. We have

$$|\alpha_k - f_{n_k}| = \frac{1}{n_k!}|f^{(n_k)}(T) - f^{(n_k)}(0)| \leqslant (n_k+1)\,2^{n_k+1}\frac{|f_{n_k}|Q_k}{M_k} \leqslant \frac{|f_{n_k}|}{2}.$$

Therefore we get by Lemma 1 and (2.8)

$$|S(T+P_k) - S(T)| \leqslant |S(T+P_k) - S(T) - S'| + |S'| \leqslant$$
$$\leqslant \left(2^{n_k+2}\frac{|f_{n_k}|}{M_k}P_k^{n_k+1} + 26\left(\frac{|f_{n_k}|}{2}\right)^{-1/n_k} P_k^{-1}\right)P_k.$$

Thus, applying Lemma 1 (ix) and (x), the proof of Lemma 3 is complete.

3. Some special entire functions.

As a first example (not satisfying (1.5)) we want to consider the entire function

$$f(t) = \exp[at] \qquad (a \in \mathbb{C},\ \mathrm{Re}(a) > 0). \tag{3.1}$$

If $\mathrm{Im}(a) = 0$ we can apply Satz 8 of [4] and obtain estimate (1.4) for the discrepancy $D_T(f)$. In the case $\mathrm{Im}(a) \neq 0$ we will apply the inequality of Erdös-Turan (for complex functions) $f(t) = f_1(t) + if_2(t)$

$$D_T(f) \leqslant c\left(\frac{1}{H} + \sum_{0<\|h\|\leqslant H}\bigl(\max(|h_1|, 1)\max(|h_2|, 1)\bigr)^{-1} \cdot \left|\frac{1}{T}\int_0^T \exp[2\pi i(h_1 f_1(t) + h_2 f_2(t))]\,dt\right|\right) \quad (H \text{ an arbitrary positive integer}) \tag{3.2}$$

with an absolute constant $c > 0$; note that $\|h\| = \max(|h_1|, |h_2|)$. We set $a = \alpha + i\beta$ and obtain for some γ

$$h_1 \operatorname{Re}(\exp[at]) + h_2 \operatorname{Im}(\exp[at]) = \sqrt{h_1^2 + h_2^2}\exp[\alpha t]\sin(\beta t + \gamma) =: g(t)\,. \tag{3.3}$$

In order to estimate the integrals in (3.2) we apply the second mean value theorem on at most $|\beta T|/(2\pi) + 2$ intervals I_j where $g(t)$ is strictly monotone and

$$|g'(t)| \geqslant \varepsilon \exp[j2\pi\alpha/|\beta|]\sqrt{h_1^2 + h_2^2}$$

(for an $\varepsilon > 0$ which is chosen later). Observe that the length of the remaining intervals in $[0, T]$ is $O(\varepsilon T)$. Hence we have

$$\int_0^T \exp[2\pi i(h_1 f_1(t) + h_2 f_2(t))]\,dt = O\left(\varepsilon T + \frac{1}{\varepsilon\sqrt{h_1^2 + h_2^2}}\right).$$

Applying (3.2) and choosing $\varepsilon = T^{-\frac{1}{2}}(h_1^2 + h_2^2)^{-\frac{1}{4}}$ and $H = [T^{\frac{1}{2}}]$ yields

$$D_T(f) = O\left(\frac{1}{\sqrt{T}}\right). \tag{3.4}$$

In the following we consider the Weierstrass σ-function. Let $\Omega[\omega_1, i\omega_2]$ be a lattice in the complex plane generated by two positive real numbers ω_1, ω_2. Then $\sigma(z)$ is an entire function and can be defined by

$$\sigma(z) = z\prod_{\omega\in\Omega\setminus\{0\}}\left(1 - \frac{z}{\omega}\right)\exp\left[\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right], \tag{3.5}$$

which is real valued for real z. We also assume that the real constant

$$\delta_1 = \frac{\sigma'(z+\omega_1)}{\sigma(z+\omega_1)} - \frac{\sigma'(z)}{\sigma(z)} \tag{3.6}$$

(cf. [9]) is positive. In order to establish an estimate for the discrepancy of $\sigma(t)$, $t \in [0, \infty)$ we use the functional equation

$$\sigma(z+\omega_1) = -\sigma(z)\exp\left[\delta_1\left(z+\frac{\omega_1}{2}\right)\right]. \tag{3.7}$$

We consider intervals $J_k = [k\omega_1, (k+1)\omega_1]$, $k = 0, 1, 2, \ldots$. Since the $\wp$-function has at most two zeroes in J_k, the Weierstrass ζ-function (with $\zeta' = -\wp$) has at most 3 zeroes in J_k (note that all involved functions are real under the above assumptions). Since $\zeta(t) = \sigma'(t)/\sigma(t)$, $\sigma(t)$ consists of at most 4 strictly monotone pieces on J_k. Applying (3.7), the inequality of Erdös-Turan and the second mean value theorem, we obtain as in the previous example

$$D_T(\sigma) = O\left(\frac{1}{\sqrt{T}}\right). \tag{3.8}$$

4. Two dimensional flows.

It is also of some intrest to consider the distribution behaviour of two dimensional flows $f(z)$, $z = s + it$. Generalizing definition (1.1) we call such a (complex valued) flow u.d. (mod 1) if

$$\lim_{S,T\to\infty} \frac{1}{ST}\int_0^S\int_0^T \chi_I(\{f(s+it)\})\,dt\,ds = \lambda(I) \tag{4.1}$$

holds for all two dimensional intervals $I \subseteq [0, 1)^2$; note that S, T tend independently to infinity. By similar arguments as in section 2 the following result can be established.

THEOREM 2. *Let $f(z)$ be an non constant entire function satisfying* (1.5) *such that f_n is real or the quotient* $\mathrm{Re}(f_n)/\mathrm{Im}(f_n)$ *is irrational for almost all $n \geqslant 1$. Then the flow $f(s+it)$ is u.d. mod* 1.

REMARK 2. For some applications it might be useful to consider a two dimensional flow as u.d. mod 1 if

$$\lim_{S,T\to\infty} \frac{1}{4ST} \int_{-S}^{S} \int_{-T}^{T} \chi_I(\{f(s+it)\})\,dt\,ds = \lambda(I) \tag{4.2}$$

holds for all $I \subseteq [0,1]^2$. Obviously, Theorem 2 is true for this notion of uniform distribution, too.

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