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Weight of a compactification and generating sets of functions

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Weight of a Compactification and Generating Sets of Functions (*).

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Introduction.

As usual, let $C^*(X)$ be the ring of bounded continuous real functions on any Tychonoff space X. The embeddings of X in real cubes, defined by subfamilies of $C^*(X)$, give one of the standard methods for the construction and comparison of compactifications of X. It is well known that every compactification of X can be so generated.

In this context, one faces two natural problems: to characterize subsets of $C^*(X)$ which give embeddings, hence which generate compactifications; to establish whether two given subsets of $C^*(X)$ generate equivalent compactifications.

These problems have already been studied by many authors, see, in particular [5], [1], [2], [3]. In this paper some new answers are given to those questions.

In § 1 preliminary definitions and notations are established, and some previous results are recalled.

§ 2 contains a proof of the following statement: a subset F of $C^*(X)$ generates a compactification if and only if the subring generated by F separates points from closed sets. A further result presented there is the following: if two subsets F and G of $C^*(X)$ generate the same subring or have a common closure with respect to uniform con-

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vergence topology, then, if F generates a compactification αX, G also does. Moreover, we show that all the subsets of $C^*(X)$ which generate any fixed compactification have the same density character (in the uniform convergence topology).

The last result of this section asserts that, if all generating subsets of αX are infinite, then every generating set has a generating subset of cardinality equal to the minimum among cardinalities of all generating sets.

§ 3 deals with cardinality of « determining » sets of functions. Following [1], we say that a set of functions F determines αX if αX is the smallest compactification of X to which every element of F extends. In the paper quoted above, it is shown that the least cardinality of determining sets of functions for αX is not greater than $w(\alpha X - X)$, the equality holding if X is locally compact. There an example is also given of a compactification αX determined by a finite set but such that $w(\alpha X - X) = c$. The authors make the conjecture that, if a compactification can be determined by a countable set of functions, then the weight of the remainder cannot exceed c.

We present a sufficient condition in order that the smallest cardinality of determining sets of αX be equal to $w(\alpha X)$ (and therefore to $w(\alpha X - X)$). We also give an example of a space X such that the Stone-Cech compactification of X, βX, is determined by a single function, while $w(\beta X - X) > c$.

In § 4, using generating sets as a tool, we study lattice properties of the set $K_w(X)$ of compactifications having the same weight as X. A proof is given of the fact that $K_w(X)$ is upper complete if and only if it is equal to $K(X)$, the set of all compactifications of X. Then conditions are given for a subset of $K_w(X)$ to have a least upper bound in $K_w(X)$. Finally, we prove that $K_w(X)$ is a lattice if and only if $K(X)$ is. It is to be noted that sufficient conditions for $K(X)$ to be a lattice are given in [4], [9], [10].

1. Definitions and symbols.

By term space, we shall mean Tychonoff space. All compactifications are assumed to be T_2-compactifications.

Let us fix terminology and symbols, in accordance with [5], [1], [2]. $K(X)$ denotes the set of compactifications of X (up to homeomorphism leaving X pointwise fixed).

If $\alpha X \in K(X)$, we set

$$C_\alpha = \{f \in C^*(X) : \exists f^\alpha : \alpha X \to \mathbb{R} \text{ continuous extension of } f\}.$$

For each $f \in C^*(X)$, let I_f be any closed interval containing $f(X)$. Let $F \subset C^*(X)$ and let $e_F : X \to \prod_{f \in F} I_f$ be the diagonal map. We shall say that F generates the compactification αX if e_F is an embedding and $\overline{e_F(X)} = \alpha X$. It is known that C_α generates αX and that if F generates αX then $F \subset C_\alpha$. Moreover, if $\alpha X, \gamma X \in K(X)$ then $\alpha X \leqslant \gamma X$ if and only if $C_\alpha \subset C_\gamma$; if $F, G \subset C^*(X)$ generate respectively αX and γX, then $F \subset G$ implies $\alpha X \leqslant \gamma X$ (where $\leqslant$ means the usual partial order in $K(X)$).

As in [1], we say that a subset F of $C^*(X)$ « determines » αX if $\alpha X = \min\{\gamma X \in K(X) : F \subset C_\gamma\}$. Clearly « F generates αX » implies « F determines αX », but the converse does not hold: for instance if X is locally compact and f is constant then $\{f\}$ determines the one-point compactification.

2. Sets of functions generating a given compactification.

As it is well known, any subset of $C^*(X)$ which separates points from closed sets generates a compactification, but, in general, the converse does not hold. In fact, a subset of $C^*(X)$ generates a compactification if and only if it generates a subring which separates points from closed sets. We present, in the following, a proof of this statement.

If $F \subset C^*(X)$, $\langle F \rangle$ will denote the (non unitary) subring of $C^*(X)$ generated by F.

LEMMA 2.1. Let X be a compact space and $F \subset C^*(X)$. Then F separates points of X if and only if $\langle F \rangle$ separates points from closed sets.

PROOF. Let F be a subset of $C^*(X)$ which separates points of X, A a closed set of X and $x_0 \in X - A$. For every $y \in A$, let $f_y \in F$ be such that $f_y(x_0) \neq f_y(y)$ and let U_y be an open neighbourhood of y such that $f_y(x_0) \notin f_y(U_y)$. If $\{U_{y_i} : i = 1, \dots, n\}$ is a subfamily of

$\{U_y\}_{y\in A}$ which still covers A, we set, for every $i = 1, \dots, n$, $f_i = f_{y_i}$, $a_i = f_i(x_0)$ and $g_i = (f_i - a_i)^2$. If $g = \sum_{i=1}^{n} g_i$, we have $g(x_0) = 0$ and $g(y) > 0$ for all $y \in A$. Let $r = \min\{g(y) : y \in A\}$. Now, if $g_1 = g - \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} f_i^2 - 2\sum_{i=1}^{n} a_i f_i$ then $d(g_1(x_0), g_1(A)) = r$, where d is the Euclidean distance. For every $i = 1, \dots, n$, let

$$M_i = \max\{|f_i(x)| : x \in X\}$$

and $p_i/q_i \in Q$ such that $|a_i - p_i/q_i| < r/6nM_i$. If we set

$$g_2 = \sum_{i=1}^{n} f_i^2 - 2\sum_{i=1}^{n} (p_i/q_i) f_i ,$$

we have, for all $x \in X$,

$$|g_1(x) - g_2(x)| = 2\left|\sum_{i=1}^{n} f_i(x)(a_i - p_i/q_i)\right| \leqslant 2\sum_{i=1}^{n} |f_i(x)|\,|a_i - p_i/q_i| \leqslant r/3 .$$

Hence, g_2 also separates x_0 from A and, if $g_3 = \left(\prod_{i=1}^{n} q_i\right) g_2$, then $g_3 \in \langle F \rangle$ and separates x_0 from A.

Conversely, it is clear that if F does not separate points of X, nor $\langle F \rangle$ does.

Now, let X be any (Tychonoff) space. For the proof of the following theorem, it is useful to recall the following known facts: if $\alpha X \in K(X)$ then the extension map $f \mapsto f^\alpha$ is a ring-isomorphism between C_α and $C(\alpha X)$; $F \subset C_\alpha$ generates αX if and only if $F^\alpha = \{f^\alpha : f \in F\}$ separates points of αX (see [2]).

THEOREM 2.2. Let $F \subset C^*(X)$. Then F generates a compactification if and only if $\langle F \rangle$ separates points from closed sets of X.

PROOF. If F generates αX, then F^α separates points of αX. By lemma 2.1 $\langle F^\alpha \rangle = \langle F \rangle^\alpha$ separates points from closed subsets of αX and hence $\langle F \rangle$ separates points from closed subsets of X.

Conversely, if $\langle F\rangle$ separates points from closed subsets of X, then $\langle F\rangle$ generates a compactification αX. Therefore $\langle F\rangle^\alpha = \langle F^\alpha\rangle$ separates points of αX and hence F^α separates points of αX as well. We conclude that F generates αX.

We observe that, when F contains sufficiently many constants, theorem 2.2 can be obtained from Stone-Weierstrass theorem (see [2], th. 3.1).

In [1] the authors give some conditions for subsets of $C^*(X)$ to determine the same compactification.

This is the case when two subsets generate the same subring, or have the same closure with respect to the uniform convergence topology. These results are consequences of the fact that every C_α is a closed ring.

We are going to prove an analogous result at the level of « generation » instead of « determination » of compactifications.

COROLLARY 2.3. Let $F, G \subset C^*(X)$. If F generates αX and $\langle F\rangle = \langle G\rangle$, then G generates αX.

PROOF. It is a direct consequence of theorem 2.2.

From now on, we shall consider $C^*(X)$ endowed with the uniform convergence topology, with respect to which C_α and $C(\alpha X)$ are naturally homeomorphic.

THEOREM 2.4. Let $F, G \subset C^*(X)$. If $\bar{F} = \bar{G}$ and F generates αX, then G generates αX.

PROOF. It will be sufficient to show that, if $\bar{F}$ generates αX then F generates αX. Since $\overline{\langle F\rangle}$ is a ring, one has $\langle F\rangle \subset \langle \bar{F}\rangle \subset \overline{\langle F\rangle}$. By 2.2, $\langle \bar{F}\rangle$ separates points from closed sets, then $\langle F\rangle$, being dense in $\langle \bar{F}\rangle$, also separates points from closed sets. Applying again 2.2, we get that F generates αX.

We find it convenient to slightly modify the notations and definitions adopted in [1], § 4, for the following cardinal invariants.

If $\alpha X \in K(X)$, we set $\delta(\alpha X) = \min\{|F| : F \subset C^*(X),\ F$ determines $\alpha X\} + \aleph_0$, $\varepsilon(\alpha X) = \min\{|F| : F \subset C^*(X),\ F$ generates $\alpha X\} + \aleph_0$ and $\sigma(\alpha X) = \min\{|F| : F \subset C^*(X),\ F$ generates αX and F separates points from closed sets$\} + \aleph_0$. The equalities between these cardinal invariants, which appear in the following lemma, are essentially known (see also [1], th. 4.2). Therefore, we shall give only some hints for their proof.

By $w(Y)$ and $d(Y)$ we mean the weight and density character of Y, respectively.

LEMMA 2.5. Let $\alpha X \in K(X)$, then

$$\varepsilon(\alpha X) = \sigma(\alpha X) = w(\alpha X) = d(C_\alpha)\,.$$

PROOF. The first equality is a consequence of theorem 2.2. The second one follows from the well known fact that every space of weight $\mathfrak{m}$ can be embedded in a real $\mathfrak{m}$-cube. Finally $w(\alpha X) = d(C_\alpha)$ follows from Stone-Weierstrass theorem.

PROPOSITION 2.6. Let $F \subset C^*(X)$ generate αX. Then $d(F) = d(C_\alpha)$.

PROOF. Since C_α is a metric space, then $d(F) \leqslant d(C_\alpha)$. Let G be a dense subset of F. Then, by 2.4, G generates αX. Therefore $|G| + \aleph_0 \geqslant \varepsilon(\alpha X) = d(C_\alpha)$, hence $d(F) \geqslant d(C_\alpha)$.

PROPOSITION 2.7. Let $F \subset C^*(X)$. If F generates αX, then there exists $G \subset F$ such that G generates αX and $|G| + \aleph_0 = \varepsilon(\alpha X)$. Furthermore, if F also separates points from closed sets, then G can be chosen such that it separates points from closed sets.

PROOF. From 2.6 and 2.5, $d(F) = \varepsilon(\alpha X)$, hence F contains a dense subset G with $|G| + \aleph_0 = \varepsilon(\alpha X)$. From 2.4, G generates αX. If F separates points from closed sets, then so does G, being dense in F (in this case 2.4 is not necessary).

3. About cardinality of « determining » sets of functions.

We recall that « generating » implies « determining », hence clearly $\delta(\alpha X) \leqslant \varepsilon(\alpha X) = w(\alpha X)$. Let us remark that one can have a strict inequality: e.g. if X is locally compact and ωX is the one point compactification, then $\delta(\omega X) = \aleph_0$ but $w(\omega X) = w(X)$. Now we want to give a condition on αX granting that $\delta(\alpha X) = w(\alpha X)$.

LEMMA 3.1 ([6], 3.5F). If $\alpha X \in K(X)$ then there exists $\gamma X \in K(X)$ such that $\gamma X \leqslant \alpha X$ and $w(\gamma X) = w(X)$.

THEOREM 3.2. If $\alpha X \in K(X)$ and $w(\alpha X) > w(X)$ then $\delta(\alpha X) = w(\alpha X)$.

PROOF. Let us suppose that $F \subset C_\alpha$ determines αX and that $|F| + \aleph_0 < w(\alpha X)$. From 3.1, there exists $\gamma X < \alpha X$ such that $w(\gamma X) < w(\alpha X)$. Then, from 2.5, there exists $G \subset C_\gamma$ such that G separates points from closed subsets of X and $|G| + \aleph_0 < w(\alpha X)$. Since $C_\gamma \subset C_\alpha$ one has $F \cup G \subset C_\alpha$ and $|F \cup G| + \aleph_0 < w(\alpha X)$. If kX is the compactification generated by $F \cup G$, one has $w(kX) < w(\alpha X)$. Then $kX < \alpha X$, against the assumption that F determines αX, because $F \subset C_k$.

COROLLARY 3.3. If $F \subset C^*(X)$ determines αX and $w(\alpha X) > w(X)$ then:

a) $d(F) = d(C_\alpha)$,

b) there exists $G \subset F$ which determines αX and such that $|G| + \aleph_0 = \delta(\alpha X)$.

PROOF. It is similar to the proof of 2.6 and 2.7.

In [1] it is proved that $\delta(\alpha X) \leqslant w(\alpha X - X)$ with the equality holding if X is locally compact (th. 4.2). It is clear that, from theorem 3.2, it follows that the assumption $w(\alpha X) > w(X)$ implies $\delta(\alpha X) = w(\alpha X - X)$. The conjecture made in [1], that $\delta(\alpha X) = \aleph_0$ should imply $w(\alpha X - X) \leqslant c$, gets a negative answer, as it is shown by the following example.

EXAMPLE 3.4. Let $\mathcal{A}$ be the collection of all the open subsets of $\mathbb{R}$ containing 0, and let $\mathcal{F} = \{A - F : A \in \mathcal{A} \text{ e } |F| < c\}$. The collection $\mathcal{F}$ has the finite intersection property, therefore there exists an ultrafilter $\mathcal{U} \supset \mathcal{F}$. For every $U \in \mathcal{U}$ one has $|U| = c$, otherwise $\mathbb{R} - U \in \mathcal{F} \subset \mathcal{U}$; moreover $\mathcal{U}$ is clearly a free ultrafilter. It is known that $\mathcal{U}$ can't have a filter base of cardinality $\leqslant c$ (see [7], 4G.2). Now put $Y = (\mathbb{R}, \tau)$, where τ is the following topology: every $x \neq 0$ is open, while the neighbourhoods of 0 are all the sets of the kind $U \cup \{0\}$ with $U \in \mathcal{U}$. τ is finer than the standard topology, and since every closed set not containing 0 is also open, Y is normal. Moreover $w(Y) > c$ because 0 cannot have a local base of cardinality $\leqslant c$ (otherwise $\mathcal{U}$ should have such a base). Since Y is a Tychonoff space, it is known that there exists a space X such that $\beta X - X$ is homeomorphic to Y and $\beta X - X$ is C^*-embedded in βX (see [5], Cor. 4.18). Let $g_1: \beta X - X \to Y$ be a homeomorphism, $j: Y \to \mathbf{R}$ the identity and $k: \mathbf{R} \to \mathbf{R}$ a bounded embedding. Then $g = k \circ j \circ g_1 \in C^*(\beta X - X)$ is injective. Therefore g has an extension, h, to βX. Put $f = h|_X$. Then $f^\beta = h$ has g as restriction to $\beta X - X$. Then f^β separates points of $\beta X - X$, which implies that $\{f\}$ determines βX (see [1], th 2.1). Then it turns out that $\delta(\beta X) = \aleph_0$, while $w(\beta X - X) = w(Y) > c$.

Note that, in the example above, $w(\beta X) = w(X)$ (see Th. 3.2).

4. Compactifications which do not increase the weight.

We set $K_w(X) = \{\alpha X \in K(X): w(\alpha X) = w(X)\}$. It is known that $K_w(X) \neq \emptyset$ (see also lemma 3.1); if $w(X) = \aleph_0$ then $K_w(X)$ is the family of all metric compactifications of X. In this section we study some lattice properties of $K_w(X)$.

We recall that $K(X)$ is a complete upper semi-lattice: if $\{\alpha_j X\} \subset K(X)$, then $\cup C_{\alpha_j}$ generates the least upper bound of $\{\alpha_j X\}$. It follows that $K(X)$ is a complete lattice if and only if $K(X)$ has a least element, that is, if and only if X is locally compact. In [4], [9], [10] some sufficient conditions are given for $K(X)$ to be a lattice.

Given a subset $\mathcal{F}$ of $K_w(X)$, we can consider the least upper bound (or the greatest lower bound) of $\mathcal{F}$ in $K(X)$ and the least upper bound (resp. the greatest lower bound) of $\mathcal{F}$ in $K_w(X)$. We denote by sup $\mathcal{F}$ and inf $\mathcal{F}$ respectively the least upper bound and the greatest lower bound of $\mathcal{F}$ in $K(X)$.

It is known that if $\alpha X \in K_w(X)$ and $\gamma X \leqslant \alpha X$, then $\gamma X \in K_w(X)$ (see, for example, [6], th. 3.1.22). This fact implies the next lemma.

LEMMA 4.1. Let $\mathcal{F} \subset K_w(X)$. $\mathcal{F}$ has a least upper bound in $K_w(X)$ if and only if sup $\mathcal{F} \in K_w(X)$.

PROPOSITION 4.2. If $\mathfrak{m} = w(X)$ then $K_w(X)$ is an $\mathfrak{m}$-complete upper semi-lattice.

PROOF. Let $\{\alpha_j X\} \subset K_w(X)$, with $|J| \leqslant \mathfrak{m}$. Applying lemma 2.5, for every $j \in J$ choose $F_j \subset C^*(X)$, of cardinality $\mathfrak{m}$, generating $a_j X$. Then $\alpha X = \sup \{\alpha_j X\} \in K_w(X)$, because $\cup F_j$ has cardinality $\mathfrak{m}$ and generates αX.

PROPOSITION 4.3. Sup $K_w(X) = \beta X$.

PROOF. Let $\gamma X = \sup K_w(X)$ and let $F \subset C^*(X)$ be a family of cardinality $w(X)$, which separates points from closed subsets of X. For every $g \in C^*(X)$, $F \cup \{g\}$ generates a compactification $\alpha_g X \in K_w(X)$ and one has $g \in C_{\alpha_g} \subset C_\gamma$. Since g is arbitrary in $C^*(X)$, it follows $C_\gamma = C^*(X)$, that is, $\gamma X = \beta X$.

COROLLARY 4.4. $K_w(X)$ is a complete upper semi lattice if and only if $K_w(X) = K(X)$.

For instance, the condition of corollary 4.4 is satisfied if X is the space of ordinals $[0, \omega_1[$. Another example, with X not locally compact, is given in § 3, Ex. 3.4. However, if $w(X) = \aleph_0$ and X is not compact, then $K_w(X)$ is not a complete upper semi lattice, because βX is not metrizable. More generally, one has the following

PROPOSITION 4.5. If X contains a C^*-embedded discrete subset with cardinality $\mathfrak{m} = w(X)$, then $w(\beta X) = 2^{\mathfrak{m}}$. Therefore $K_w(X)$ is not a complete upper semi lattice.

PROOF. Easy consequence of [6], th. 3.6.11.

Now we give some conditions insuring that a subfamily $\mathcal{F}$ of $K_w(X)$ has a least upper bound in $K_w(X)$.

THEOREM 4.6. Let $\mathcal{F} = \{\alpha_j X\} \subset K_w(X)$. The following are equivalent:

a) $\sup \mathcal{F} \in K_w(X)$,

b) $d(\cup C_{\alpha_j}) = w(X)$,

c) there exists $\mathcal{G} \subset \mathcal{F}$ such that $|\mathcal{G}| \leqslant w(X)$ and $\sup \mathcal{G} = \sup \mathcal{F}$.

PROOF. *a*) $\Leftrightarrow$ *b*) This is a consequence of 2.5 and 2.6, since $\cup C_{\alpha_j}$ generates $\sup \mathcal{F}$.

b) $\Rightarrow$ *c*) Let F be dense in $\cup C_{\alpha_j}$ with $|F| = w(X)$. For every $f \in F$, let $\alpha_{j_f} X \in \mathcal{F}$ be such that $f \in C_{\alpha_{j_f}}$. By 2.6, F generates $\sup \mathcal{F}$, then $\cup C_{\alpha_{j_f}}$ generates $\sup \mathcal{F}$ as well.

c) $\Rightarrow$ *a*) Follows from 4.2.

Now, in order to establish the conditions under which $K_w(X)$ is a lower semi-lattice (hence a lattice), we start with the following:

LEMMA 4.7. Let $\mathcal{F} \subset K_w(X)$. $\mathcal{F}$ has a greatest lower bound in $K_w(X)$ if and only if $\mathcal{F}$ is bounded below in $K(X)$.

PROOF. Immediate consequence of the fact that $K(X)$ is a complete upper semi lattice (see also remark preceding lemma 4.1).

THEOREM 4.8. (*a*) $K_w(X)$ is a lattice if and only if $K(X)$ is.

(*b*) $K_w(X)$ is a complete lower semi lattice if and only if X is locally compact.

PROOF. (*a*) Let $K_w(X)$ be a lattice and let $\alpha_1 X$, $\alpha_2 X \in K(X)$. From 3.1 there exist $\gamma_1 X$, $\gamma_2 X \in K_w(X)$ such that $\gamma_i X \leqslant \alpha_i X$, $i = 1, 2$. Then $\inf\{\gamma_1 X, \gamma_2 X\}$ is a lower bound for $\{\alpha_1 X, \alpha_2 X\}$ which, therefore, has a greatest lower bound in $K(X)$.

The converse follows from 4.7 and 4.2.

(*b*) If $K_w(X)$ is a complete lower semilattice, then $\min K_w(X)$ is also $\min K(X)$, and this implies that X is locally compact. The converse follows from 4.7.

LEMMA 4.9 [Shirota]. Let X be first countable. Then $K(X)$ is a lattice if and only if X is locally compact.

COROLLARY 4.10. If X is first countable, then $K_w(X)$ is a lattice if and only if X is locally compact.

REFERENCES

[1] B. J. BALL - S. YOKURA, *Compactifications determined by subsets of $C^*(X)$*, Topology Appl., **13** (1982), pp. 1-14 (Erratum: Topology Appl., **14** (1982), p. 227).

[2] B. J. BALL - S. YOKURA, *Compactifications determined by subsets of $C^*(X)$, II*, Topology Appl., **15** (1983), pp. 1-6.

[3] J. L. BLASCO, *Hausdorff compactifications and Lebesgue sets*, Topology Appl., **15** (1983), pp. 111-117.

[4] A. CATERINO, *Special lattices of compactifications*, Comment. Math. Univ. Carolinae, **26**, 3 (1985), pp. 515-523.

[5] R. E. CHANDLER, *Hausdorff compactifications*, Marcel Dekker, New York, 1976.

[6] R. ENGELKING, *General Topology*, PWN-Polish Scientific Publishers, Warsaw, 1976.

[7] L. GILLMAN - M. JERISON, *Rings of continuous functions*, Van Nostrand, Princeton, 1960.

[8] T. SHIROTA, *On systems of structures of a completely regular space*, Osaka Math. J., **41** (1950), pp. 131-143.

[9] F. C. TZUNG, *Sufficient conditions for the set of Hausdorff compactifications to be a lattice*, Pacific J. Math., (1978), pp. 565-573.

[10] Y. UNLU, *Lattices of compactifications of Tychonoff spaces*, Topology Appl., **9** (1978), pp. 41-57.

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