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Product measurability and Scorza-Dragoni's type property

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Product Measurability and Scorza-Dragoni's Type Property.

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SUMMARY - A new proof is given that « f: $T \times X \to \mathbb{R}$ is normal integrand if and only if f verifies Scorza-Dragoni's property in the semicontinuous case ».

1. Introduction.

First we recall the notions of measurability and semicontinuity for a real function. So, if (S, Σ) is a measurable space then a function h: $S \to \mathbb{R}$, where $\mathbb{R}$ denotes a real line with natural topology, is called Σ-measurable if $h^{-1}(A) \in \Sigma$ for each open subset A of $\mathbb{R}$. It is well known that h: $S \to \mathbb{R}$ is Σ-measurable if and only if $h^{-1}(A) \in \Sigma$ for each closed subset A of $\mathbb{R}$ or $h^{-1}(A_a) \in \Sigma$, $a \in \mathbb{R}$, where A_a may have one of the following forms: (a, ∞), or $[a, \infty)$, or $(-\infty, a)$ or $(-\infty, a]$. A function g: $E \to \mathbb{R}$, where E is a topological space, is called lower semicontinuous « lsc » (resp. uppersemicontinuous « usc ») if the set $g^{-1}((-\infty, a])$ $\left(g^{-1}([a, \infty))\right)$ is closed for each $a \in \mathbb{R}$.

Now let $(T, \mathcal{A}, \mu)$ be a measure space, where T is a metric compact Hausdorff space and μ a Borel, σ-finite, regular and complete measure defined on a σ-field $\mathcal{A}$ of subsets of T. Let X be a Polish space (i.e. separable, complete, metric) and let $\mathcal{B}(X)$ be the σ-field of Borel sets in X. By $\mathcal{A} \times \mathcal{B}(X)$ we denote the σ-field on $T \times X$ generated by sets $A \times B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}(X)$. Consider a function

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$f: T \times X \to \mathbb{R}$. We say that f is C_* type (resp. C^* type) if $f(t, \cdot)$ is lcs (resp. usc) for each $t \in T$ and $f(\cdot, x)$ is $\mathcal{A}$-measurable for each $x \in X$.

Furthermore we call $f: T \times X \to \mathbb{R}$ Scorza-Dragonian if it satisfies the following condition: «for every $\varepsilon > 0$ there exists a closed subset T_ε of T, with $\mu(T \setminus T_\varepsilon) < \varepsilon$, such that the restriction of f to the set $T_\varepsilon \times X$ is jointly semicontinuous (this condition is analogous to that of Scorza-Dragoni which had been introduced in [SD] for Caratheodory's functions). More precisely we shall say that f has SD_* property (resp. SD^* property) if $f_{|T_\varepsilon \times X}$ is lsc (resp. usc.).

The properties of Scorza-Dragoniaus have been investigated in [BA] and [Z]. Specially worthy of notice is the paper [BA] in which the author examined in detail the relationship between $\mathcal{A} \times \mathcal{B}(X)$-measurability and SD_* property. There Bottaro Aruffo showed, among others, that for C_* type functions the $\mathcal{A} \times \mathcal{B}(X)$-measurability and SD_* property are equivalent. The $\mathcal{A} \times \mathcal{B}(X)$-measurable and C_*-type function is also called the normal integrand (cf. [BA, Definizione 3.0]).

In the present note, based on [Z], we give another proof of the same fact.

2. The result.

Before stating our main theorem let us introduce some definitions.

We say that the function $f: T \times X \to \mathbb{R}$ has property Cl if the multifunction $F_A: T \to \mathcal{P}(X)$ ($\mathcal{P}(X)$ is the family of all subsets of X including the empty set) defined by $F_A(t) = \{x \in X: f(t, x) \in A\}$ is weakly $\mathcal{A}$-measurable for each closed subset A of $\mathbb{R}$. We recall that a multifunction $F_A: T \to \mathcal{P}(X)$ is weakly $\mathcal{A}$-measurable if the set $F_A^-(B) = \{t \in T: F_A(t) \cap B \neq \emptyset\}$ belongs to $\mathcal{A}$ for each open subset B of X (cf. [H, p. 54]).

The following theorem has been proved in [Z].

THEOREM 1. Let $f: T \times X \to \mathbb{R}$ be C_* type (resp. C^* type). Then f has property SD_* (resp. property SD^*) if and only if f has property Cl.

Now we can establish our main result

THEOREM 2. Let $f: T \times X \to \mathbb{R}$ be C_* type (resp. C^* type). Then f has property SD_* (resp. property SD^*) if and only if f is $\mathcal{A} \times \mathcal{B}(X)$-measurable.

PROOF. Let us assume that f has property SD_* (resp. property SD^*) and choose an arbitrary $a \in \mathbb{R}$. Since the interval $(-\infty, a]$ (resp. $[a, \infty)$) is closed in $\mathbb{R}$, the multifunction $F_{(-\infty,a]}$ (resp. $F_{[a,\infty)}$), by Theorem 1, is weakly $\mathcal{A}$-measurable. Moreover in view of lowersemicontinuity (resp. uppersemicontinuity) of functions $f(t, \cdot)$, $t \in T$, the multifunction $F_{(-\infty,a]}$ (resp. $F_{[a,\infty)}$) has closed values. Thus by [H, Theorem 3.5] its graph $\operatorname{Gr} F_{(-\infty,a]}$ (resp. $\operatorname{Gr} F_{[a,\infty)}$) belongs to $\mathcal{A} \times \mathcal{B}(X)$.

But

$$f^{-1}((-\infty, a]) = \{(t, x) \in T \times X : f(t, x) \leqslant a\} =$$
$$= \{(t, x) \in T \times X : x \in F_{(-\infty,a]}(t)\} = \operatorname{Gr} F_{(-\infty,a]} \in \mathcal{A} \times \mathcal{B}(X)$$

(resp. we have $f^{-1}([a, \infty)) = \operatorname{Gr} F_{[a,\infty)}$ for f having property SD^*). Conversely, let $f\colon T \times X \to \mathbb{R}$ be $\mathcal{A} \times \mathcal{B}(X)$-measurable. Then $f^{-1}(A) \in \mathcal{A} \times \mathcal{B}(X)$ for each closed $A \subset \mathbb{R}$ and hence, in particular, $f^{-1}(A) \cap T \times B \in \mathcal{A} \times \mathcal{B}(X)$ for each open $B \subset X$. By projection theorem [CV, Theorem III.23] we have $\operatorname{pr}_T (f^{-1}(A) \cap T \times B) \in \mathcal{A}$, where pr_T denotes the projection map from the product $T \times X$ onto T. Now let us observe that

$$F_A^-(B) = \{t \in T : F_A(t) \cap B \neq \emptyset\} = \Big\{t \in T : \bigvee_{x \in X} x \in F_A(t) \cap B\Big\} =$$
$$= \Big\{t \in T : \bigvee_{x \in X} f(t, x) \in A \wedge (t, x) \in T \times B\Big\} =$$
$$= \Big\{t \in T : \bigvee_{x \in X} (t, x) \in f^{-1}(A) \cap T \times B\Big\} = \operatorname{pr}_T (f^{-1}(A) \cap T \times B) \in \mathcal{A}.$$

So we see that F_A is weakly $\mathcal{A}$-measurable and this f has property Cl. Therefore by Theorem 1 f has property SD_* (resp. SD^*) which completes the proof of Theorem 2.

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