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On the Convolution in the Space $\mathcal{D}_{L^2}'^{(M_p)}$.

STEVAN PILIPOVIĆ (*)

SUMMARY. - We investigate the convolution and the Fourier transformation in the space of Beurling ultradistributions $\mathcal{D}_{L^2}'^{(M_p)}$. We give some simple conditions on a convolutor S for the solvability and the hypoelipticity of a convolution equation $S * U = V$ in $\mathcal{D}_{L^2}'^{(M_p)}$.

1. The space $\mathcal{D}_{L^2}'^{(M_p)}$ introduced in [3] is a subspace of the space of Beurling ultradistributions ([2]). This is a natural generalization of the Schwartz space $\mathcal{D}_{L^2}'$ and we investigated them in [3] in connection with the Hilbert transformation of ultradistributions. With suitable assumptions on the sequence M_p, we determined in [4] elements of $\mathcal{D}_{L^2}'^{(M_p)}$ as boundary values of certain holomorphic functions.

In [3] some questions on the convolution in $\mathcal{D}_{L^2}'^{(M_p)}$ occured. This was the motivation for our investigations of the convolution in $\mathcal{D}_{L^2}'^{(M_p)}$. The Fourier transformation maps $\mathcal{D}_{L^2}'^{(M_p)}$ into a subspace of L^2_{loc}. So by proving the exchange formula we obtain simple conditions for the solvability and the hypoelipticity of a convolution equation in $\mathcal{D}_{L^2}'^{(M_p)}$.

2. Our notation is the same as in [2]. Let M_p, $p \in \mathfrak{N}_0 = \mathfrak{N} \cup \{0\}$ be a sequence of positive numbers such that

(M.1) $$M_p^2 \leqslant M_{p-1} M_{p+1}\,, \qquad p \in \mathfrak{N}\,;$$

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(M.2) There are constants A and H such that

$$M_p \leqslant AH^p \min_{0 \leqslant q \leqslant p} \{M_q M_{p-q}\}, \qquad p \in \mathfrak{N}_0;$$

(M.3) There is a constant A such that

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \leqslant Ap M_p/M_{p+1}, \qquad p \in \mathfrak{N}.$$

We assumed in [3] instead of (M.2) and (M.3) the weaker conditions: (M.2)′ and (M.3)′. (All these conditions were analysed in [2].) The reason for that is the structural theorem for $\mathfrak{D}'^{(M_p)}_{L^2}$ which we need for the full characterization of the convolution in $\mathfrak{D}'^{(M_p)}_{L^2}$.

The associated function M is defined by

$$M(\varrho) = \sup_{p \in \mathfrak{N}_0} \left\{ \log \frac{M_0 \varrho^p}{M_p} \right\}, \qquad \varrho > 0.$$

From [2, Proposition 3.6] it follows

(1) $2M(\varrho) \leqslant M(H\varrho) + \log AM_0$ (A and H are from (M.2)).

The space of Beurling ultradistributions is defined ([2]) as the strong dual of the space

$$\mathfrak{D}^{(M_p)} = \operatorname{inj\,lim}_{m \in \mathfrak{N}} \operatorname{proj\,lim}_{n \in \mathfrak{N}} \mathfrak{D}^{(M_p)}_{m,n}$$

where

$$\mathfrak{D}^{(M_p)}_{m,n} = \{\varphi \in C^\infty;\ \operatorname{supp} \varphi \subset K(m),\ \|\varphi\|_{m,n} < \infty\},$$

$C^\infty = C^\infty(\mathfrak{R}^q)$is the space of smooth functions on $\mathfrak{R}^q$, $K(m)$ is the closed ball with the center at zero and with the radius $m > 0$,

$$\|\varphi\|_{m,n} = \sup_{\substack{\alpha \in \mathfrak{N}_0^q \\ x \in K(m)}} \left\{ \frac{n^{|\alpha|} |\varphi^{(\alpha)}(x)|}{M^{|\alpha|}} \right\} \qquad (|\alpha| = \alpha_1 + \ldots + \alpha_q).$$

The space $\mathfrak{D}^{(M_p)}_{L^2}$ is defined ([3]) as follows:

$$\mathfrak{D}^{(M_p)}_{L^2,h} = \left\{\varphi \in C^\infty \cap L^2;\ \gamma_h(\varphi) = \sum_{\alpha \in \mathfrak{N}_0^q} \frac{h^{|\alpha|}\|\varphi^{(\alpha)}\|_2}{M_{|\alpha|}} < \infty\right\}, \qquad h > 0\,,$$

$$\mathfrak{D}^{(M_p)}_{L^2} = \operatorname*{proj\,lim}_{\alpha \in \mathfrak{N}} \mathfrak{D}^{(M_p)}_{L^2,n}\,.$$

The space $\mathfrak{D}^{(M_p)}_{L^2}$ is an FG-space (Gelfand space, see [1]) and

$$\mathfrak{D}^{(M_p)} \hookrightarrow \mathfrak{D}^{(M_p)}_{L^2} \hookrightarrow \mathfrak{D}_{L^2} \ \ ([3])$$

where « $A \hookrightarrow B$ » means that A is a dense subspace of B and the inclusion mapping is continuous. ($\mathfrak{D}_{L^2}$ is well-known Schwartz space.)

The Fourier transformation of an $f \in L^2$ is defined by

$$(\mathcal{F}_2 f)(\xi) = \operatorname*{l.i.m.}_{A\to\infty} \int_{-A}^{A}\!\!\dots\!\int_{-A}^{A} f(x)\exp\left(i\langle x, \xi\rangle\right) dx_1, \dots, dx_q\,,$$

$$\xi \in \mathfrak{R}^q(\langle x, \xi\rangle = x_1\xi_1 + \dots + x_q\xi_q)\,,$$

where l.i.m. means the square-mean limit. Since we shall use in the paper the Fourier transformations of tempered distributions and of $\mathfrak{D}'^{(M_p)}_{L^2}$-ultradistributions, we indicate these transformations by $\mathcal{F}_t$ and $\mathcal{F}_M$, respectively. It is well-known that for $f \in L^2 \subset S'$, $\mathcal{F}_2 f = \mathcal{F}_t f$.

Obviously, the sequence of norms γ_k, $k \in \mathfrak{N}$, on $\mathfrak{D}^{(M_p)}_{L^2}$ is equivalent to the following one:

$$\tilde{\gamma}_k(\varphi) = \sup_{\alpha \in \mathfrak{N}_0^q}\left\{\frac{k^{|\alpha|}\|\varphi^{(\alpha)}\|_2}{M_{(\alpha)}}\right\} = (2\pi)^{-q/2}\sup_{\alpha \in \mathfrak{N}_0^q}\left\{\frac{k^{|\alpha|}\|(\xi^\alpha(\mathcal{F}_2\varphi)(\xi))\|_2}{M_{|\alpha|}}\right\}.$$

$$k \in \mathfrak{N}_0(\xi^\alpha = \xi_1^{\alpha_1}, \dots, \xi_q^{\alpha_q})\,.$$

This implies that $\mathfrak{D}^{(M_p)}_{L^2}$ is isomorphic to the space $D^{(M_p)}_{L^2} = \mathcal{F}_2(\mathfrak{D}^{(M_p)}_{L^2})$ in which we transport the convergence structure from $\mathfrak{D}^{(M_p)}_{L^2}$.

The inverse Fourier transformation of an $f \in L^2$ is defined by

$$(\mathcal{F}_2^{-1} f)(\xi) = (2\pi)^{-q}(\mathcal{F}_2 f)(-\xi)\,, \qquad \xi \in \mathfrak{R}^q\,.$$

Clearly, $\mathcal{F}_2^{-1}$ is an isomorphism of $D^{(M_p)}_{L^2}$ onto $\mathfrak{D}^{(M_p)}_{L^2}$. This implies that the adjoint mappings

$$\mathcal{F}_M\colon D_{L^2}^{\prime(M_p)} \to \mathfrak{D}_{L^2}^{\prime(M_p)}, \qquad \mathcal{F}_M^{-1}\ \mathfrak{D}_{L^2}^{\prime(M_p)} \to D_{L^2}^{\prime(M_p)}$$

are isomorphisms. One can easily prove that for $f \in L^2$

$$\mathcal{F}_M f = \mathcal{F}_2 f \quad \text{and} \quad \mathcal{F}_M^{-1} f = \mathcal{F}_2 f .$$

Similarly, we define $\mathcal{F}_M\colon \mathfrak{D}_{L^2}^{\prime(M_p)} \to D_{L^2}^{\prime(M_p)}$ and $\mathcal{F}_M^{-1}\colon D_{L^2}^{\prime(M_p)} \to \mathfrak{D}_{L^2}^{\prime(M_p)}$. Let $f \in \mathfrak{D}_{L^2}^{\prime(M_p)}$ and $\varphi \in \mathfrak{D}_{L^2}^{(M_p)}$. We have:

$$(2) \qquad \langle \mathcal{F}_M^{-1} f, \mathcal{F}_2 \varphi \rangle = \langle f, \varphi \rangle = \langle \mathcal{F}_M f, \mathcal{F}_2^{-1} \varphi \rangle .$$

Let

$$D^\alpha = D_1^{\alpha_1} \ldots D_q^{\alpha_q} = \left(i \frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(i \frac{\partial}{\partial x_q}\right)^{\alpha_q}.$$

An operator of the form $P(D) = \sum_{\alpha \in \mathfrak{N}_0^q} a_\alpha D^\alpha$, a_α are complex numbers, is called the ultradifferential operator of class (M_p) if there are $L > 0$ and $C > 0$ such that

$$(3) \qquad |a_\alpha| \leqslant C L^{|\alpha|} / M_{|\alpha|}, \qquad \alpha \in \mathfrak{N}_0^q \ ([2]) .$$

It was proved in [4] that the mapping $\varphi \mapsto P(D)\varphi$ from $\mathfrak{D}_{L^2}^{(M_p)}$ into $\mathfrak{D}_{L^2}^{(M_p)}$ is continuous. We have

$$\mathcal{F}_2(P(D)\varphi)(\xi) = P(\xi)\, \mathcal{F}_2(\varphi)(\xi), \qquad \xi \in \mathfrak{R}^q .$$

We proved in [4] the following structural theorem: $f \in \mathfrak{D}_{L^2}^{(M_p)}$ iff there exist an ultradifferential operator of class (M_p), $P(D)$, and an $F \in L^2$ such that

$$(4) \qquad f = P(D) F .$$

By (2) and (4) we have

$$(5) \qquad (\mathcal{F}_M f)(\xi) = P(\xi)(\mathcal{F}_2 F)(\xi), \qquad \xi \in \mathfrak{R}^q .$$

Thus, we see that $\mathcal{F}_M f$ is a function from L^2_{loc}.

For the analytic function $P(\zeta) = \sum_{\alpha\in\mathfrak{N}_0^q} a_\alpha \zeta^\alpha$, $\zeta \in \mathfrak{C}^q$, (a_α satisfy (3)) we have

$$|P(\zeta)| \leqslant (C/M_0) \sup_{\alpha\in\mathfrak{N}_0^q} \left\{ \frac{L^{|\alpha|} M_0 |\zeta|^{|\alpha|}}{M_{|\alpha|}} \right\} \leqslant C_1 \exp\left(M(L_1|\zeta|\right), \qquad \zeta \in \mathfrak{C}^q, \tag{6}$$

where $C_1 = (C/M_0) \cdot \sum_{\alpha\in\mathfrak{N}_0^q} (L/L_1)^{|\alpha|}$ and $L_1 > L$.

3. Let $f \in \mathfrak{D}_{L^2}'^{(M_p)}$ and $\varphi \in \mathfrak{D}_{L^2}^{(M_p)}$; we define

$$(f * \varphi)(x) = \langle f(t), \varphi(x-t)\rangle, \qquad x \in \mathfrak{R}^q.$$

Let us put $\psi(x) = (f * \varphi)(x)$, $x \in \mathfrak{R}^q$, and assume that f is of the form (4). We have

PROPOSITION 1. *For any $\alpha \in \mathfrak{N}_0^q$, $\psi^{(\alpha)}$ is bounded and continuous. Moreover, for any $k > 0$*

$$\beta_k(\psi) = \sup_{\substack{x\in\mathfrak{R}^q \\ \alpha\in\mathfrak{N}_0^q}} \left\{ \frac{k^{|\alpha|} |\psi^{(\alpha)}(x)|}{M_{|\alpha|}} \right\} < \infty .$$

PROOF. For any $x \in \mathfrak{R}^q$ we have

$$\psi^{(\alpha)}(x) = \langle f(t), \varphi^{(\alpha)}(x-t)\rangle = \langle F(t), P(-D)\varphi^{(\alpha)}(x-t)\rangle = \\ = \int_{\mathfrak{R}^q} F(t) P(-D)\big(\varphi^{(\alpha)}(x-t)\big)\, dt .$$

Since $P(D)\,\varphi^{(\alpha)} \in L^2$, the first part of the assertion follows.

Using (6), for $x \in \mathfrak{R}^q$, $\alpha \in \mathfrak{N}_0^b$, we have

$$\frac{k^{|\alpha|}}{M_{|\alpha|}} |\psi^{(\alpha)}(x)| \leqslant \|F\|_2 \frac{k^{|\alpha|}}{M_{|\alpha|}} \|\big(P(D)\varphi^{(\alpha)}\big)(\xi)\|_2 \leqslant \\ \leqslant C_1 \frac{k^{|\alpha|}}{M_{|\alpha|}} \|F\|_2 \left\|\exp\big(M(L_1|\xi|)\big)\xi^\alpha (\mathcal{F}_2\varphi)(\xi)\right\|_2 .$$

Thus by (1) we have

$$\sup_{\substack{x\in\mathfrak{R}^q \\ \alpha\in\mathfrak{N}_0^q}} \left\{ \frac{k^{|\alpha|}}{M_{|\alpha|}} |\psi^{(\alpha)}(x)| \right\} \leqslant C \left\|\exp\big(M(L_1|\xi|) + M(k|\xi|)\big)(\mathcal{F}_2\varphi)(\xi)\right\|_2 \leqslant \\ \leqslant C_0 \left\|\exp\big(M((L_1 + k)|\xi|)(\mathcal{F}_2\varphi)(\xi)\big)\right\|_2$$

where C and C_0 are suitable constants. Since the sequence of norms

$$\nu_k(\varphi) = \|\exp\big(M(k|\xi|)\big)(\mathcal{F}_2\varphi)(\xi)\|_2\,, \qquad k \in \mathfrak{N}\,,$$

is equivalent to the sequence γ_k, $k \in \mathfrak{N}$ ([4, Proposition 2.1]), the second assertion of the proposition follows.

If $\varphi, \psi \in \mathcal{D}_{L^2}^{(M_p)}$, then $\mathcal{F}_2(\varphi * \psi) = (\mathcal{F}_2\varphi)(\mathcal{F}_2\psi)$ ([5]). (In this case $*$ is the ordinary convolution.)

PROPOSITION 2. *Let* $f \in \mathcal{D}_{L^2}^{\prime(M_p)}$ *and* $\varphi \in \mathcal{D}_{L^2}^{(M_p)}$. *Then*

$$\mathcal{F}_t(f * \varphi)(\xi) = (\mathcal{F}_M f)(\xi)(\mathcal{F}_2\varphi)(\xi)\,, \qquad \xi \in \mathfrak{R}^q\,.$$

PROOF. We assume that f is of the form (4). We have

$$(f * \varphi)(x) = \langle P(D)F(t), \varphi(x-t)\rangle = \langle F(t), P(-D)\varphi(x-t)\rangle =$$
$$= \int_{\mathfrak{R}^q} F(t)P(-D)\varphi(x-t)\,dt = \big(F * P(D)\varphi\big)(x)\,.$$

since $F \in L^2$ and $P(D)\varphi \in L^2$, from the remark given before Proposition 2 and (5), the assertion follows.

4. Let $g \in \mathcal{D}_{L^2}^{\prime(M_p)}$ such that for every $\varphi \in \mathcal{D}_{L^2}^{(M_p)}$, $g * \varphi \in \mathcal{D}_{L^2}^{(M_p)}$. Then we call g the convolution-operator or convolutor. The space of all convolutors is denoted by $\mathcal{O}_C'(\mathcal{D}_{L^2}^{\prime(M_p)}, \mathcal{D}_{L^2}^{\prime(M_p)})$ or in short, by $\mathcal{O}_C^{\prime(M_p)}$. (Note that we do not assume the continuity of the mapping $\varphi \to g * \varphi$, in $\mathcal{D}_{L^2}^{(M_p)}$. This will be proved in Proposition 5.)

Proposition 2 implies.

PROPOSITION 3. *Let* $g \in \mathcal{O}_C^{\prime(M_p)}$ *and* $\varphi \in \mathcal{D}_{L^2}^{(M_p)}$. *Then*

$$\mathcal{F}(g * \varphi)(\xi) = (\mathcal{F}_M g)(\xi)(\mathcal{F}_2\varphi)(\xi)\,, \qquad \xi \in \mathfrak{R}^q\,.$$

PROPOSITION 4. *A* $g \in \mathcal{D}_{L^2}^{\prime(M_p)}$ *is from* $\mathcal{O}_C^{\prime(M_p)}$ *iff there exists* $k > 0$ *such that*

$$(\mathcal{F}_M g)\exp\big(-M(k|\cdot|)\big) \in L^\infty\,. \tag{7}$$

PROOF. From (7) it follows that for any $\varphi \in \mathfrak{D}^{(M_p)}_{L^2}$ and any $\alpha \in \mathfrak{N}_0^q$

$$\mathcal{F}_M(g * \varphi^{(\alpha)}) = (\mathcal{F}_M g)\,\mathcal{F}_2(\varphi^{(\alpha)}) \in L^2$$

and

$$\mathcal{F}_M(g * \varphi^{(\alpha)}) = \mathcal{F}_2(g * \varphi^{(\alpha)}) = (\mathcal{F}_M g)(-i\xi)^\alpha (\mathcal{F}_2 \varphi) .$$

Let $r > 0$. From (7) and (1) we obtain

$$\sup_{\alpha \in \mathfrak{N}_0^q} \left\{ \frac{r^{|\alpha|}}{M_{|\alpha|}} \|g * \varphi^{(\alpha)}\|_2 \right\} \leqslant (2\pi)^{-q/2} \left\| \sup_{\alpha} \left\{ \frac{r^{|\alpha|}}{M_{|\alpha|}} |(\mathcal{F}_M g)(\xi)|\,|\xi|^\alpha \,|(\mathcal{F}_2\varphi)(\xi)| \right\} \right\|_2 \leqslant$$
$$\leqslant C \|\exp\big(M(r|\xi|)\big) \exp\big(M(k|\xi|)\big)(\mathcal{F}_2\varphi)(\xi)\|_2$$
$$\leqslant \bar{C} \|\exp\big(M((r+k)|\xi|)\big)(\mathcal{F}_2\varphi)(\xi)\|_2 .$$

Thus, we have proved

$$\tilde{\gamma}_r(g * \varphi) \leqslant \bar{C} \nu_{r+k}(\varphi) , \qquad \varphi \in \mathfrak{D}^{(M_p)}_{L^2}. \tag{8}$$

Conversely, let $g \in \mathcal{O}'^{(M_p)}_C$ and φ be an arbitrary element from $\mathfrak{D}^{(M_p)}_{L^2}$. Since $\mathcal{F}_2(g * \varphi) = (\mathcal{F}_M g)(\mathcal{F}_2 \varphi)$ and $\nu_k(g * \varphi) < \infty$ for every $k > 0$, we have

$$\|(\mathcal{F}_M g) \exp\big(M(k|\cdot|)\big)(\mathcal{F}_2\varphi)\|_2 < \infty \qquad \text{for every } k > 0 . \tag{9}$$

It can be proved by (9) that $\mathcal{F}_M g$ is from L^∞_{loc}. Namely, for any open ball $\mathring{K}(m)$, $L^2(\mathring{K}(m))$ can be embedded into $D^{(M_p)}_{L^2}$ in a natural way:

$$L^2(\mathring{K}(m)) \in \psi \mapsto \tilde{\psi} = \begin{cases} \psi & \text{on } \mathring{K}(m) \\ 0 & \text{on } \mathfrak{R}^q \setminus \mathring{K}(m) \end{cases} \in D^{(M_p)}_{L^2} .$$

Since the condition

$$\|H(\xi)\psi(\xi)\|_{L^2(\mathring{K}(m))} < \infty \qquad \text{for every } \psi \in L^2(\mathring{K}(m)) ,$$

implies $H \in L^\infty(\mathring{K}(m))$, we obtain that $\mathcal{F}_M g \in L^\infty_{\text{loc}}$.

Let us prove (7) by proving that g does not belong to $\mathcal{O}'^{(M_p)}_C$ if (7) does not hold.

If (7) does not hold there exists a sequence a_j such that $a_1 > 1$,

$a_{j+1} > a_j + 1$ and

$$|(\mathcal{F}_M g)(\xi)|\exp\big(-M(j|\xi|)\big) \geqslant j \quad \text{if } |\xi| \in A_j \subset (a_j, a_{j+1})$$

with mes $A_j = \varepsilon_j > 0$, $j \in \mathfrak{N}$.

Put

$$\psi_j(\xi) = \begin{cases} \varepsilon_j^{-1} \exp\big(-M(j|\xi|)\big), & |\xi| \in A_j,\ j \in \mathfrak{N}, \\ 0 & \text{elsewhere} \end{cases}$$

and $\psi = \sum_{j=1}^{\infty} \psi_j$. Since $\exp\big(-M(|\xi|)\big)$ decreases monotonically faster than any power of $1/|\xi|$ when, $|\xi| \to \infty$, one can prove that $\psi \in D_{L^1}^{(M_p)}$ and that $(\mathcal{F}_M g)\psi$ does not belong to $D_{L^1}^{(M_p)}$.

The proof is completed.

From (8) directly follows:

PROPOSITION 5. *If $g \in \mathcal{O}_C'^{(M_p)}$ then the mapping*

$$\varphi \mapsto g * \varphi \quad \textit{from } \mathcal{D}_{L^1}^{(M_p)} \textit{ into } \mathcal{D}_{L^1}^{(M_p)}$$

is continuous.

The last assertion enables us to define the convolution of an $f \in \mathcal{D}_{L^1}'^{(M_p)}$ and a $g \in \mathcal{O}_C'^{(M_p)}$ in a usual way:

$$\langle f \star g, \varphi \rangle = \langle f, \check{g} * \varphi \rangle \quad \text{where } \check{g}(x) = g(-x),\ x \in \mathfrak{R}.$$

PROPOSITION 6. *If $f \in \mathcal{D}_{L^1}'^{(M_p)}$ and $g \in \mathcal{O}_C'^{(M_p)}$ then,*

$$\mathcal{F}_M(f \star g) = (\mathcal{F}_M f)(\mathcal{F}_M g).$$

PROOF. For $\psi \in D_{L^1}^{(M_p)}$ we have

$$\begin{aligned}\langle \mathcal{F}_M(f \star g), \psi \rangle &= \langle f \star g, \mathcal{F}_2 \psi \rangle = \langle f, \check{g} * \mathcal{F}_2 \psi \rangle = \\ &= \langle \mathcal{F}_M f, \mathcal{F}_2^{-1}(\check{g} * \mathcal{F}_2 \psi) \rangle = \langle (\mathcal{F}_M f)(\xi), (2\pi)^{-q} \mathcal{F}_2(\check{g} * \mathcal{F}_2 \psi)(-\xi) \rangle = \\ &= \langle (\mathcal{F}_M f)(\xi), (2\pi)^{-q} (\mathcal{F}_M g)(\xi) \big(\mathcal{F}_2(\mathcal{F}_2 \psi)\big)(-\xi) \rangle = \\ &= \langle (\mathcal{F}_M f)(\xi), (\mathcal{F}_M g)(\xi) \psi(\xi) \rangle .\end{aligned}$$

This implies the assertion.

If g and f are from $\mathcal{O}_C'^{(M_p)}$ then $f \star g \in \mathcal{O}_C'^{(M_p)}$.

This follows from the definition of $\mathcal{O}_C'^{(M_p)}$ and the fact that

$$(f \star g) * \varphi = f \star (g * \varphi) = f * (g * \varphi)\,, \qquad \varphi \in \mathcal{D}_{L^2}^{(M_p)}\,.$$

Properties of the convolution are given in the next

PROPOSITION 7. *Let $g, h \in \mathcal{O}_C'^{(M_p)}$ and $f \in \mathcal{D}_{L^2}'^{(M_p)}$.*

(i) $g \star h = h \star g$;

(ii) $(f \star g) \star h = f \star (g \star h)\,.$

PROOF. (i) We have

$$\langle g \star h, \varphi\rangle = \langle g, \check{h} * \varphi\rangle = \big(g * (\check{h} \check{*} \varphi)\big)(0)\,;$$
$$\langle h \star g, \varphi\rangle = \big(h * (\check{g} \check{*} \varphi)\big)(0)\,.$$

Proposition 3 implies that $\big(h * (\check{g} \check{*} \varphi)\big)(x) = \big(g * (\check{h} \check{*} \varphi)\big)(x)$, $x \in \mathfrak{R}^q$.

(ii) Since

$$\langle f \star (g \star h), \varphi\rangle = \langle f, (g \check{\star} h) * \varphi\rangle = \langle f(x), (g \star h)(t), \varphi(x+t)\rangle =$$
$$= \Big\langle f(x), \langle g(t), \langle \check{h}(u), \varphi(x+t-\alpha)\rangle\rangle\Big\rangle =$$
$$= \big\langle f(x), \langle g(t), (\check{h} * \varphi)(x+t)\rangle\big\rangle = \langle f(x), \check{g} * (\check{h} * \varphi)(x)\rangle\,,$$

and

$$\langle (f \star g) \star h, \varphi\rangle = \langle (f \star g), (\check{h} * \varphi)\rangle = \langle f, \check{g} * (\check{h} * \varphi)\rangle\,,$$

the assertion is proved.

5. Observe the convolution equation in $\mathcal{D}_{L^2}'^{(M_p)}$:

$$S \star U = V\,, \tag{10}$$

where $S \in \mathcal{O}_C'^{(M_p)}$ and $V \in \mathcal{D}_{L^2}'^{(M_p)}$ are known ultradistributions and $U \in \mathcal{D}_{L^2}'^{(M_p)}$ is the unknown one.

Denote by $\mathcal{O}_{C,\mathcal{A}}'^{(M_p)}$ the space of all convolutors $S \in \mathcal{O}_C'^{(M_p)}$ for which $\mathcal{F}_M S$ is a smooth function which has the analytic continuation onto the whole $\mathfrak{C}^q$.

PROPOSITION 8. *Let* $S \in \mathcal{O}'^{(M_p)}_{C,\mathcal{A}}$. *The sufficient condition that the equation* (10) *is solvable in* $\mathcal{D}'^{(M_p)}_{L^2}$ *for any* $V \in \mathcal{D}'^{(M_p)}_{L^2}$ *is the following one*:

There exist $C > 0$ *and* $k > 0$ *such that*

$$|(\mathcal{F}_M S)(\zeta)| \geqslant \frac{C}{\exp\big(M(k|\zeta|)\big)}, \qquad \zeta \in \mathfrak{C}^q. \tag{11}$$

PROOF. Assume that (11) holds. Put

$$u(\xi) = \frac{(\mathcal{F}_M V)(\xi)}{(\mathcal{F}_M S)(\xi)}, \qquad \xi \in \mathfrak{R}^q.$$

From (5) it follows that for some ultradifferential operator P of class (M_p) and some $v \in L^2$

$$u(\xi) = \frac{P(\xi)}{(\mathcal{F}_M S)(\xi)}\, v(\xi), \qquad \xi \in \mathfrak{R}^q.$$

Let $P_1(\xi) = P(\xi)/(\mathcal{F}_M S)(\xi)$, $\xi \in \mathfrak{R}^q$. From [2, Proposizion 4.5] it follows that $P_1(D)$ is an ultradifferential operator of class (M_p). Thus, by (5) again, we obtain that the solution of (10) is

$$U = P_1(D)\, \mathcal{F}_2^{-1} v.$$

PROPOSITION 9. *Let* $S \in \mathcal{O}'^{(M_p)}_{C}$. *Then the necessary condition for the solvability of* (10) *for any* $V \in \mathcal{D}'^{(M_p)}_{L^2}$ *is the following one*:

There exist $C > 0$, $D > 0$ *and* $k > 0$ *such that*

$$|(\mathcal{F}_M S)(\xi)| \geqslant C \exp\big(-M(k|\xi|)\big), \qquad |\xi| \geqslant D. \tag{12}$$

PROOF. Assume that (10) is solvable for any $V \in \mathcal{D}'^{(M_p)}_{L^2}$ but (12) does not hold. This implies that there exists a sequence of sets A_j, $j \in \mathfrak{N}$, such that mes $A_j = \varepsilon_j > 0$, $A_j \subset (a_j, a_{j+1})$ $a_1 > 1$, $a_{j+1} > a_j + 1$, $j \in \mathfrak{N}$ and

$$|(\mathcal{F}_M S)(\xi)| \leqslant j^{-1} \exp\big(-M(j^2|\xi|)\big), \qquad |\xi| \in A_j,\ j \in \mathfrak{N}.$$

As in the proof of Proposition 4, put

$$v_j = \begin{cases} \varepsilon_j^{-1} \exp\big(-M(j|\xi|)\big), & |\xi| \in A_j,\ j \in \mathfrak{N}, \\ 0 & \text{elsewhere} \end{cases}$$

and

$$v = \sum_{j=1}^{\infty} v_j .$$

Clearly $v \in D'^{(M_p)}_{L^2}$ but

$$v/(\mathcal{F}_M S) \notin D'^{(M_p)}_{L^2} .$$

Thus, for such $V = \mathcal{F}_M^{-1}(v)$ and S the solution of (10) does not exist in $\mathcal{D}'^{(M_p)}_{L^2}$.

6. We say that equation (10) satisfies the hypoelipticity condition if the existence of the solution $U \in \mathcal{D}'^{(M_p)}_{L^2}$ of (10) and $V \in \mathcal{D}^{(M_p)}_{L^2}$ imply that $U \in \mathcal{D}^{(M_p)}_{L^2}$. In this case we say that U is the hypoeliptic solution of (10).

PROPOSITION 10. *Let* $S \in \mathcal{O}'^{(M_p)}_C$. *Equation* (10) *is hypoeliptic iff* (12) *holds.*

PROOF. Let $\tilde{\varphi}(\xi) = 1$ for $|\xi| \leqslant D + 1$ and $\tilde{\varphi}(\xi) = 0$ for $|\xi| > D + 1$. This is an element of $D^{(M_p)}_{L^2}$.

Put

$$\tilde{P}(\xi) = \frac{1 - \tilde{\varphi}(\xi)}{(\mathcal{F}_M S)(\xi)} .$$

We have

$$|\tilde{P}(\xi)| \leqslant C \exp M(k|\xi|) , \qquad \xi \in \mathfrak{R}^q .$$

Obviously, $P = \mathcal{F}_M^{-1} \tilde{P} \in \mathcal{O}'^{(M_p)}_C$ and

$$S \mathbin{\text{☆}} P = \delta - \psi , \qquad \text{where } \psi = \mathcal{F}^{-1}(\tilde{\varphi}) .$$

Since

$$U = U * \delta = V \mathbin{\text{☆}} P + U \mathbin{\text{☆}} \psi$$

one can easily prove that $U \in \mathcal{D}^{(M_p)}_L$.

By the same arguments as in the second part of the proof of Proposition 9 one can prove that if (10) is hypoeliptic then (12) holds for S.

REFERENCES

[1] K. Floret - J. Wloka, *Einführung in die Theorie der lokalkonvexen Räume*, Lecture Notes in Math. 56, Springer, Berlin, Heidelberg, New York, 1968.

[2] H. Komatsu, *Ultradistributions*, I: *Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **20** (1973), pp. 25-105.

[3] S. Pilipović, *Hilbert transformation of ultradistributions*, Rend. Math. Univ. Padova, **77** (1987), pp. 1-13.

[4] S. Pilipović, *Boundary value representations for a class of Beurling ultradistributions*, Port. Math. (to appear).

[5] V. S. Vladimirov, *Generalized Functions in Mathematical Physics*, Mir, Moscow, 1979.

Added in proof. Proposition 4 implies that $(\mathcal{F}_M S)(\xi) = s(\xi)\exp[M(k|\xi|)]$, $\xi \in \mathfrak{R}^q$, for some $k > 0$ and some $s \in L^\infty$, and from (4) we have $(\mathcal{F}_M V)(\xi) = P(\xi)v(\xi)$, $\xi \in \mathfrak{R}^2$, where P is an ultradifferential operator of class (M_p) and $v \in L^2$. So, by using the Fourier transformation we have:

« (10) is solvable in $\mathcal{D}_{L^2}^{\prime(M_p)}$ if the equation $su = v$ has a solution $u \in L^2$ ».

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