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Constant mean curvature surfaces in 4-space forms

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Constant Mean Curvature Surfaces in 4-Space Forms.

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1. Introduction: constant mean curvature in 3-space.

Consider the following classical theorems about constant mean curvature surfaces in euclidean 3-space:

THEOREM (*a*) (Ricci) [2, 12]. Let (M^2, ds^2) be a simply connected surface with Gaussian curvature K. Then there exists an isometric immersion $f\colon M \to \mathbb{R}^3$ with constant mean curvature H without umbilic points if and only if $H^2 > K$ and the metric $d\tilde{s}^2 := (H^2 - K)^{\frac{1}{2}} ds^2$ is flat.

THEOREM (*b*) (H. Hopf) [9]. Let M^2 be a closed surface with Euler characteristic χ and $f\colon M \to \mathbb{R}^3$ an immersion with constant mean curvature. Then either $f(M)$ is a round sphere or $2\chi = -N$ where N is the number of umbilic points, counted with multiplicities.

REMARK. Hopf does not state explicitly this theorem but only its corollary: A topological sphere of constant mean curvature is round ([9], Ch. VI, 2.1). However, the more general statement can be read off from VI, 2.3., p. 139: In the non-umbilic case the singularities of the principal line fields, i.e. the umbilic points are isolated and of index $j = -n/2$ where n is the order of the umbilic point (see below).

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This implies Theorem (*b*) by the Poincaré-Hopf index theorem which was discussed before ([9], Ch. III, 2.2).

At first sight, the two theorems seem to be unrelated since in (*a*) the umbilics are excluded, in (*b*) they are counted. But it turns out that a sharpened version of (*a*) in fact implies (*b*). To see this, we must extend the local characterization of minimal surfaces given in (*a*) to the umbilic points. First observe that the flatness of a metric $d\tilde{s}^2 = a\, ds^2$ is expressed by the differential equation

$$\Delta \log a = 2K$$

where Δ is the Laplacian with respect to the metric ds^2. Unfortunately, in general the left hand side is undefined at the zeros of the function a, and the zeros of $a = (H^2 - K)^{\frac{1}{2}}$ are precisely the umbilic points. However, for a certain type of zeros the expression $\Delta \log a$ makes still sense:

DEFINITION. A function $a: M \to [0, \infty)$ on a surface (M, ds^2) is called of *absolute value type* (AVT) if every point in M has a neighborhood U such that $a = a_0|h|$ on U where a_0 is smooth and positive and $h \circ z^{-1}$ is holomorphic for some conformal coordinate $z: U \to \mathbf{C}$.

Here, *conformal* means that $ds^2 = \lambda^2\, dz\, d\bar{z}$ for some positive function λ, called *conformal factor*. Note that then $\Delta = 4\lambda^{-2}\, \partial_z \partial_{\bar{z}}$.

If an absolute value type function a does not vanish entirely its zeros are isolated, and around each zero we have $\Delta \log a = \Delta \log a_0$ which is still defined and smooth at the zeros. The *order* $\mathrm{ord}_p(a)$ of a at *a* zero $p \in M$ is, by definition, the order of its holomorphic part h at p. If M is a compact surface, we define the number of zeros of a (with multiplicities) to be

$$N(a) := \sum_{a(p)=0} \mathrm{ord}_p(a)\,.$$

Then we get as an easy consequence of the divergence theorem (see [7], 4.1)

LEMMA 1. Let (M, ds^2) be a compact surface and $a \not\equiv 0$ an AVT function on M. Then

$$\int_M \Delta \log a = -\, 2\pi N(a)\,.$$

REMARK. One may generalize the notion of AVT functions by allowing the holomorphic part h to have poles. Then the lemma is still correct with $N(a) :=$ number of zeros $-$ number of poles.

Now we can give the sharpened version of Ricci's theorem (a) for any 3-space form of constant sectional curvature c, Q^3_c:

THEOREM 0. Let (M, ds^2) be a simply connected surface with Gaussian curvature K. Then there exists an isometric immersion $f\colon M \to Q^3_c$ with constant mean curvature H if and only if $K \leqq c + H^2$ and $a := (c + H^2 - K)^{\frac{1}{4}}$ is AVT satisfying

$$\Delta \log a = 2K \tag{1}$$

unless $a \equiv 0$ and f is totally umbilic. In fact, there is exactly a one-parameter family of such immersions, up to local congruence.

Of course, for the « only if » part we do not have to assume simple connectivity. By local congruence we mean that the lifts to the universal cover $\hat{Q}^3_c$ are congruent.

This theorem has been stated by Lawson [12], but avoiding umbilic points. However, for global applications the umbilic points are essential as we saw. In fact, from Theorem 0 and Lemma 1 we get immediately Theorem (b), using the Gauss-Bonnet theorem:

COROLLARY. Let M be a compact surface and $f\colon M \to Q^3_c$ an immersion with constant mean curvature H. Then either f is totally umbilic or

$$2\chi = -N(a) \tag{2}$$

where χ is the Euler number of M and $a = (c + H^2 - K)^{\frac{1}{4}}$. Note that the zeros of a are precisely the umbilic points, by Gauss equations.

As a consequence, a topological sphere of constant mean curvature in Q^3_c is totally umbilic. This is well known: See Almgren [1] and Calabi [4] for $Q^3_c = S^3$ and Chern [6] for general space forms.

Theorem 0 turns out to be a special case of a theorem in dimension 4 which we will discuss in the next two sections. But the theorems (a) and (b) stated at the beginning may serve as a model case as well: Local characterization in terms of curvatures leads to global results. In the present paper we want to carry out this program for minimal surfaces in 4-space of constant sectional curvature (see also [7] for the case of constant holomorphic curvature). We wish to mention that everything can be done for branched immersions as well which we will discuss in a subsequent paper (*).

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2. Constant mean curvature in 4-space.

In the following sections, let Q_c^4 be any 4-dimensional space of constant sectional curvature c and (M, ds^2) a surface with Gaussian curvature K which is always assumed to be simply connected if we claim existence of an immersion. One way of carrying over the notion of constant mean curvature surfaces to 4-space is the assumption of parallel mean curvature vector: If $f: M \to Q_c^4$ is an immersion with 2nd fundamental form $\alpha(x, y) = (D_x y)^\perp$, then $\eta := \frac{1}{2}$ trace α is called the mean curvature vector field, and this is called parallel if it is a parallel section of the normal bundle, i.e. $(D\eta)^\perp = 0$.

THEOREM 1. There exists an isometric immersion $f: M \to Q_c^4$ with parallel mean curvature vector of length $H > 0$ if and only if $K \leqq c + H^2$ and $a := (c + H^2 - K)^{\frac{1}{4}}$ is AVT satisfying

$$\Delta \log a = 2K . \tag{1}$$

In fact, up to local congruence, there is exactly a two-parameter family $f_{\sigma\tau}$, $-\pi \leqq \sigma, \tau \leqq \pi$ of such immersions.

REMARK. It is known by Yau [17] that a surface of nonzero parallel mean curvature vector always lies in a totally umbilic hypersurface $Q_{c'}^3 \subset Q_c^4$ with constant mean curvature $H' = (H^2 - (c' - c))^{\frac{1}{2}}$, for some $c' \geqq c$. Therefore, Theorem 0 and Theorem 1 are essentially the same. The two parameters σ and τ are easy to explain geometrically: One parameter τ is obtained by rotating the second fundamental form in the tangent plane, i.e. replacing α with $R_\tau^* \alpha$ where R_τ is the rotation of angle τ. The other parameter σ arises as the angle between the

(*) J.-H. ESCHENBURG and R. TRIBUZY, *Branch points of conformal mappings of surfaces*, Math. Ann., **279** (1988), pp. 621-633.

normal vector of $Q^3_{c'}$ in Q^4_c and the mean curvature vector η in the normal plane of M in Q^4_c. In other words, we have

$$H' = H \sin \sigma , \qquad (c' - c)^{\frac{1}{2}} = H \cos \sigma .$$

(Observe that $(c' - c)^{\frac{1}{2}}$ is the mean curvature of $Q^3_{c'}$ in Q^4_c). In fact, we see that for fixed $|\sigma|$, the immersions $f_{\sigma\tau}$ all belong to the same totally umbilic hypersurface $Q^3_{c'}$, up to local congruence, and that $f_{\sigma,\tau}$ is congruent to $f_{-\sigma,\tau}$ under a reflection in $Q^3_{c'}$.

There is yet another way of generalizing the notion of constant mean curvature to higher codimension, namely by only assuming that $\|\eta\| = H = \text{const}$. S. T. Yau has investigated this situation [17]. In particular he claimes: If M is diffeomorphic to a sphere, then any immersion $f\colon M \to Q^4_c$ with $\|\eta\| = \text{const} > 0$ is totally umbilic. However, his proof relies on the claim that for any surface $M \subset Q^4_c$ with $\|\eta\| = \text{const}$, like in the minimal case $\|\eta\| = 0$, the quartic form $\Phi(x) = \|\alpha'(x, x)\|^2 - \|\alpha'(x, Jx)\|^2$ where J denotes the $90°$-rotation on M is the real part of a holomorphic quartic form; here we let $\alpha'(x, y) := \alpha(x, y) - \langle x, y\rangle \eta$, the trace free part of the 2nd fundamental form α. We will show by a counterexample that in general this is wrong (see Appendix). So the question remains open whether there exists a non-trivial immersion of S^2 into Q^4_c with $\|\eta\| = \text{const} \neq 0$.

3. Minimal surfaces in 4-space.

In this case, as a new ingredience we have the normal curvature K_N, which is defined as follows. Let $f\colon M \to Q^4_c$ be any isometric immersion. Let D^N be the induced connection on the normal bundle NM and R^N its curvature tensor. Then R^N is determined by a single function, called *normal curvature*,

$$K_N(p) = \langle R^N(e_1, e_2)\, e_4, e_3\rangle$$

where $p \in M$ and (e_1, e_2, e_3, e_4) is an oriented orthonormal basis of $T_{f(p)}Q^4_c$ with e_1, e_2 tangent and e_3, e_4 normal. (TM and NM are always considered as subbundles of $f^*TQ^4_c$.) Thus K_N can be defined only if the bundle $f^*TQ^4_c$ over M is orientable. If the mean curvature vector η is parallel and non-zero, the plane bundle NM has a parallel section and hence $K_N = 0$. But now we consider minimal surfaces, i.e. immersions with $\eta = 0$. Then it follows from Gauss and Ricci equation that

$$|K_N| \leqq c - K \tag{3}$$

with equality exactly at those points $p \in M$ where the so called *ellipse of curvature*

$$E_p := \{\alpha(x, x);\ x \in T_p M,\ |x| = 1\}$$

in $N_p M$ is a circle (see ch. 4); these points are called *circular* (or *semi-umbilic*). A minimal immersion is called *superminimal* if every point is circular.

THEOREM 2. Let $K_N\colon M \to \mathbf{R}$ be a smooth function satisfying (3). Then there exists an isometric minimal but not superminimal immersion $f\colon M \to Q_c^4$ with normal curvature K_N if and only if the functions $a_\pm := (c - K \pm K_N)^{\frac{1}{4}}$ are AVT with

$$\Delta \log a_\pm = 2K \mp K_N\,. \tag{4}$$

In fact, up to local congruence, there is exactly a one-parameter family f_τ, $-\pi \leqq \tau \leqq \pi$, of such immersions.

This theorem was obtained in [16] under the assumption $|K_N| < c - K$ and no circular points.

If M is closed and oriented with oriented normal bundle, the Gauss-Bonnet-Chern-Weil theorem gives the Euler number χ_N of the normal bundle:

$$\int_M K_N\, dv = 2\pi\chi_N\,.$$

Hence integrating (2) and using Lemma 1, we get immediately:

COROLLARY. Let M be a closed oriented surface and $f\colon M \to Q_c^4$ a minimal but not superminimal immersion with oriented normal bundle. Then we get

$$2\chi \mp \chi_N = -N(a_\pm) \tag{5}$$

and in particular

$$4\chi = -N(a_+ a_-) \tag{6}$$

where $a_\pm = (c - K \pm K_N)^{\frac{1}{4}}$.

Recall that $N(a_+ a_-)$ is the number of circular points (counted with multiplicities). One may interpret (6) as follows: At any $p \in M$ consider the longest principal axis of the ellipse of curvature E_p. Since the mapping $x \to \alpha(x, x): T_p^1 M \to E_p$ has degree two, there are exactly four vectors $\pm x_1 \perp \pm x_2$ with $\|\alpha(x_i, x_i)\|$ maximal. This defines a field of two perpendicular lines outside the circular points, and in the circular points the index of this two-line field is $-\operatorname{ord}_p(a_+ a_-)$ (see [8]). Thus we may get (6) from the Poincaré-Hopf index theorem. This argument is very similar to Hopf's proof of Theorem (*b*) (see ch. 1).

In particular it follows from (6) that a minimal sphere in Q_c^4 is always superminimal which was proved first by Calabi [4] and that a minimal torus is either superminimal or has no circular points at all.

Now consider superminimal immersions in Q_c^4. For these we have

$$(3') \qquad |K_N| = c - K$$

and the following intrinsic characterization:

THEOREM 3. There exists an isometric superminimal immersion $f: M \to Q_c^4$ which is not totally geodesic if and only if $K \leqq c$ and the function $a := (c - K)^{\frac{1}{4}}$ is *AVT* with

$$(7) \qquad \Delta \log a = 3K - c = 2K - |K_N|\,.$$

Equation (7) was known to Calabi [4]. The theorem was obtained by Tribuzy and Guadalupe [16] in the case $K < c$ (i.e. no points for which α vanishes). For $c = 0$, $Q_c^4 = \mathbb{R}^4$, the theorem shows that the superminimal immersions are exactly those which are holomorphic with respect to a suitable isometry between $\mathbb{R}^4$ and $\mathbb{C}^2$, and they are also totally real with respect to another such isometry (see 3.6, 3.8 and the concluding note in [7]). For $c = 1$, $Q_c^4 = S^4$, we see that there is only one superminimal immersion of constant curvature which is not totally geodesic, namely the Veronese embedding of $\mathbb{R}P^2$ with $K = \frac{1}{3}$. The superminimal surfaces in S^4 have been investigated by R. Bryant ([3], see also [10]).

As a global consequence of (3), (3′) and Theorem 3 we get immediately a theorem of Rodriguez and Guadalupe [13]:

COROLLARY (see [13]). Let M be a closed oriented surface and $f: M \to Q_c^4$ a minimal immersion with orientable normal bundle. Then

$$c \cdot \operatorname{area}(M) \geqq 2\pi(\chi + |\chi_N|)$$

with equality if and only if f is superminimal. In this case we have

$$|\chi_N| = 2\chi + N(a) \tag{8}$$

where $a = (c - K)^{\frac{1}{2}}$, unless f is totally geodesic.

Recall that $N(a)$ counts the number of totally geodesic points ($\alpha = 0$) with multiplicities.

One consequence of this corollary is a theorem of E. Ruh [14]: A minimal 2-sphere (which has to be superminimal as we saw) with trivial normal bundle ($\chi_N = 0$) is totally geodesic. E.g. it is well known that $\chi_N = 0$ for an embedded oriented closed surface in S^4 ([11], Cor. 3.2); hence an embedded minimal sphere in S^4 is a great sphere.

4. Structure equations of surfaces in 4-space forms.

Let M be a surface and $f: M \to Q^4_c$ an immersion. Let (e_1, e_2, e_3, e_4) be a Darboux frame along f, i.e. an orthonormal frame with e_1, e_2 tangent and e_3, e_4 normal, and which is understood to be oriented if orientations of TM and NM are given. Then the 1-forms on M

$$\omega_a := \langle df, e_a \rangle, \qquad \omega_{ab} := \langle De_a, e_b \rangle,$$

$1 \leqq a, b \leqq 4$, satisfy $\omega_3 = \omega_4 = 0$, $\omega_{ab} = -\omega_{ba}$ and the Cartan structure equations

$$\begin{aligned} d\omega_a &= \sum_b \omega_{ab} \wedge \omega_b, \\ d\omega_{ab} &= \sum_d \omega_{ad} \wedge \omega_{db} - c\omega_a \wedge \omega_b, \end{aligned} \tag{C}$$

and the induced metric is $ds^2 = \omega_1^2 + \omega_2^2$. Vice versa, if 1-forms ω_a, ω_{ab} with these proporties are given on a simply connected surface M and if the quadratic form $ds^2 = \omega_1^2 + \omega_2^2$ is nondegenerated, there exists an immersion $f: M \to Q^4_c$ with Darboux frame $e_1, \ldots, e_4$ such that ω_a, ω_{ab} are the corresponding 1-forms, and this immersion is unique up to local congruence (e.g. see Spivak [15]). We will call two immersions $f, \tilde{f}: M \to Q^4_c$ *locally congruent* if their lifts to the universal cover $\tilde{Q}^4_c$ are congruent.

Let $h^{\alpha}_{ij} = \langle D_{e_i} e_j, e_\alpha \rangle = \langle \alpha(e_i, e_j), e_\alpha \rangle$ where we assume always $i, j \in \{1, 2\}$, $\alpha, \beta \in \{3, 4\}$.

Then we have

$$\omega_{i\alpha} = \sum h_{ij}^{\alpha} \omega_j .$$

Now we put

$$h_\alpha = \tfrac{1}{2}(h_{11}^{\alpha} + h_{22}^{\alpha}), \qquad k_\alpha = \tfrac{1}{2}(h_{11}^{\alpha} - h_{22}^{\alpha}) - i h_{12}^{\alpha},$$

$$h_{\pm} = h_3 \pm i h_4, \qquad k_{\pm} = k_3 \pm i k_4 .$$

Note that the mean curvature vector is $\eta = \sum h_\alpha e_\alpha$, so $\|\eta\| = h_{\pm}$. The functions $k_{\pm}$ are closely related to the ellipse of curvature which consists of the normal vectors

$$\alpha(ce_1 + se_2, ce_1 + se_2) = (c^2 - s^2)\alpha'(e_1, e_1) + 2cs\alpha(e_1, e_2) + \eta$$

where $c = \cos\tau$, $s = \sin\tau$ for arbitrary angles τ. This ellipse is a circle if and only if $\alpha'(e_1, e_1)$ and $\alpha(e_1, e_2)$ are perpendicular and of the same length, i.e. iff $k_+ = 0$ or $k_- = 0$ at the particular point.

Now we introduce the complex valued 1-forms

$$\varphi = \omega_1 + i\omega_2 ,$$

$$\psi_\alpha = \omega_{1\alpha} - i\omega_{2\alpha} = k_\alpha \varphi + h_\alpha \bar{\varphi}, \qquad \psi_{\pm} = \psi_3 \pm i\psi_4 = k_{\pm}\varphi + h_{\pm}\bar{\varphi} .$$

Then one easily checks that the Cartan equations (C) are equivalent to the following set of equations:

(9) $$d\varphi = -i\omega_{12} \wedge \varphi ,$$

(10) $$0 = \psi_+ \wedge \varphi + \overline{\psi_- \wedge \varphi} ,$$

(11) $$2i d\omega_{\mp} = \bar{\psi}_{\pm} \wedge \psi_{\pm} - c\bar{\varphi} \wedge \varphi ,$$

(12) $$d\psi_{\pm} = i\omega_{\mp} \wedge \psi_{\pm}$$

with $\omega_{\pm} := \omega_{12} \pm \omega_{34}$.

Remark. Under the usual isomorphism of the Lie algebra $SO(4)$ onto the Lie algebra $Sp(1) \times Sp(1)$, the connection form $\omega = (\omega_{ab})$ is mapped onto the pair

$$\left(-\tfrac{1}{2}(\omega_+ i + \bar{\psi}_- j);\ \tfrac{1}{2}(\omega_- i + \psi_+ j)\right)$$

where (i, j, ij) is the usual basis of $Sp(1)$, viewed as the space of imaginary quaternions. The equations (9)-(12) are just a translation of (C) into the Lie algebra $Sp(1) \times Sp(1)$ (compare [10]).

Moreover, from the structure equations of the bundles TM and NM with induced connections we have

$$2i\, d\omega_{12} = - K\ \bar{\varphi} \wedge \varphi\,, \tag{13}$$

$$2i\, d\omega_{34} = - K_N\, \bar{\varphi} \wedge \varphi\,. \tag{14}$$

Hence together with (11) and the definition of the $\psi_\pm$ we get

$$|k_\pm|^2 = c - K \pm K_N + H^2 \tag{15}$$

where $H := \|\eta\| = |h_\pm|$.

In particular, the immersion is minimal if and only if ψ_+ and ψ_- are (1, 0)-forms, i.e. multiples of φ. Note that for any conformal coordinate $z\colon M \to \mathbf{C}$ preserving the orientation given by (e_1, e_2), we have $\varphi = \mu\, dz$ for some complex valued function μ. Since $ds^2 = \varphi\bar{\varphi} = |\mu|^2 dz\, d\bar{z}$, so $\lambda := |\mu|$ is the conformal factor.

The mean curvature vector field is parallel if and only if

$$0 = \langle D\eta, e_\beta \rangle = \langle D\big(\textstyle\sum h_\alpha e_\alpha\big), e_\beta \rangle = dh_\beta + h_\alpha \omega_{\alpha\beta}\,,$$

hence

$$dh_\pm = \mp i h_\pm \omega_{34}\,. \tag{16}$$

So by (9), this is equivalent to $d(h_\pm \bar{\varphi}) = i\omega_\mp \wedge (h_\pm \bar{\varphi})$. Or, in other words, the (1, 0)-form

$$\psi'_\pm := \psi_\pm - h_\pm \bar{\varphi} = k_\pm \varphi\,,$$

the (1, 0)-part of $\psi_\pm$, satisfies

$$d\psi'_\pm = i\omega_\mp \wedge \psi'_\pm\,. \tag{12'}$$

It is easy to check that for an arbitrary immersion $f\colon M \to Q^4_c$ of an oriented surface M the quartic form

$$\Phi := \psi'_+ \psi'_- \varphi^2 = k_+ k_- \varphi^4$$

is independent of the choice of the Darboux frame and hence globally defined. In fact, its real part is $\Phi_R(x) = \|\alpha'(x, x)\|^2 - \|\alpha'(x, Jx)\|^2$ where J denotes the almost complex structure on M. If the immersion has parallel mean curvature vector of length $H \geqq 0$, this form is holomorphic: if $\varphi = \mu\, dz$ for some conformal coordinate z, putting $p_\pm := \mu k_\pm$ we get $\psi'_\pm = p_\pm dz$ and $\Phi = p_+ p_- \mu^2 dz^4$, and from (9) and (12′):

$$(d\mu + \mu i\omega_{12})\wedge dz = 0\,, \qquad (dp_\pm - p_\pm i\omega_\pm)\wedge dz = 0\,.$$

Therefore, $d(p_+p_-\mu^2)\wedge dz = 0$ which shows that $p_+p_-\mu^2$ is a holomorphic function. If $H \neq 0$, this quartic form splits as $\Phi = \Phi_+\Phi_-$ where $\Phi_\pm$ are holomorphic quadratic forms: We may choose in this case $e_3 = \eta/\|\eta\|$, then e_3, e_4 are parallel in the normal bundle. So ω_{34} vanishes and hence the 2-forms $\Phi_\pm := \psi_\pm \varphi$ are holomorphic quadratic forms, by a similar argument as above.

5. Proof of the theorems.

The link between the structure equations (9)-(12) and equations (1), (4), (7) is given by the following lemma which is essentially an easy special case of a theorem of Chern [5]. Let (M, ds^2) be a surface. A complex valued function p on M will be called of *holomorphic type* if locally $p = p_0 p_1$ with p_0 smooth and nowhere vanishing and $p_1 \circ z^{-1}$ holomorphic for a suitable conformal coordinate z on M. Clearly then $|p|$ is an AVT function.

LEMMA 2. Let p be a smooth complex valued function on M, $p \neq 0$, and ω a real valued 1-form on M. Let $\psi := p\, dz$ for some conformal coordinate z. Then the equality

$$(*) \qquad d\psi = i\omega\wedge\psi$$

is valid if and only if p is of holomorphic type and

$$(17) \qquad \omega = 2\,\mathrm{Im}\left(\partial_{\bar z}(\log p)\, d\bar z\right).$$

Moreover, then

$$(18) \qquad d\omega = -\frac{1}{2i}\,\Delta \log |p|\bar\varphi\wedge\varphi\,.$$

REMARK. Note that $\partial_{\bar z} \log p$ and $\Delta \log |p|$ are well defined even at the zeros of p, if p is of holomorphic type.

PROOF (see [7]). Since ω is real, we have $\omega = b\,dz + \bar b\,d\bar z$ for some complex function b. Thus $(*)$ is equivalent to the differential equation for p:

$$(**) \qquad \partial_{\bar z} p = i\bar b p\,.$$

Let u be a solution of the inhomogeneous Cauchy-Riemann equation $\partial_{\bar z} u = i\bar b$ and put $p_0 := e^u$. Then p_0 is a nowhere vanishing solution of $(**)$. If p is any other solution of $(**)$, then $p_1 := p/p_0$ is holomorphic. So p is of holomorphic type and $(**)$ is equivalent to $\bar b = (1/i)\partial_{\bar z}(\log p)$ and hence to (17). Differentiating (17), we get (18).

(*a*) *The local equations* (1), (4), (7). Now let (M, ds^2) be a surface and $f\colon M \to Q_c^4$ an isometric immersion with parallel mean curvature vector field of length $H \geqq 0$. Let $e_1, \ldots, e_4$ be a Darboux frame along f. Then we may apply Lemma 2 to the forms $\psi_\pm = k_\pm \varphi = k_\pm \mu\,dz$, replacing $(*)$ with (12′). So the function

$$a_\pm := |k_\pm| = (c + H^2 - K \pm K_N)^{\frac{1}{2}}$$

is AVT $\big($since $|\mu| = \lambda > 0\big)$, and if $a_\pm \not\equiv 0$, we get from (18)

$$-\,2i\,d\omega_\mp = (\Delta \log a_\pm - K)\bar\varphi \wedge \varphi$$

since $\Delta \log \lambda = -\,K$ $\big($which also follows from (18), using (9) and (13)$\big)$. On the other hand, we get from (13) and (14)

$$-\,2i\,d\omega_\mp = (K \mp K_N)\bar\varphi \wedge \varphi$$

and therefore,

$$(19) \qquad \Delta \log a_\pm = 2K \mp K_N\,.$$

If $H \neq 0$, the form ω_{34} is exact by (16) and therefore $K_N = 0$, by (14); so we get (1). If $H = 0$, we get (4). If $a_+ a_- \equiv 0$ then either $a_+ \equiv 0$ or $a_- \equiv 0$ since a_+, a_- are both AVT. If $a_+ = a_- = 0$, the immersion is totally umbilic (i.e. $\alpha' = 0$), hence totally geodesic if $H = 0$. If $a_- = 0$, $a_+ \neq 0$, we have $H = 0$ (since otherwise $a_+ = a_-$) and therefore $K_N = c - K$ and $a_+ = (2(c-K))^{\frac{1}{2}}$, and similarly if $a_+ = 0$, $a_- \neq 0$. This proves (7).

(*b*) *Uniqueness.* Let M be oriented, $\hat{Q}_c^4$ simply connected and $f, \tilde{f}\colon M \to \hat{Q}_c^4$ two isometric immersions, both with the same normal curvature K_N and with parallel mean curvature vector field of length $H \geqslant 0$. Assume that $f, \tilde{f}$ induce the same holomorphic quartic form Φ on M. If $H \neq 0$, assume further that also Φ_+ and Φ_- are the same for both immersions. We claim that then $\tilde{f} = g \circ f$ for some oriented isometry g of $\hat{Q}_c^4$.

In fact, in case $H \neq 0$, the 1-forms ψ_+, ψ_- are the same for both immersions, so $\omega_\pm$ are the same by (12′) and hence the immersions are congruent. In case $H = 0$, let e_1, e_2, e_3, e_4 and $e_1, e_2, \tilde{e}_3, \tilde{e}_4$ be local oriented Darboux frames along f and $\tilde{f}$ resp. (We consider TM as a subbundle of $f^*T\hat{Q}$ and $\tilde{f}^*T\hat{Q}$.) Then we have $|k_\pm| = |\tilde{k}_\pm|$, $k_+k_- = \tilde{k}_+\tilde{k}_-$, and therefore

$$\tilde{\psi}_+ = e^{i\tau}\psi_+ , \qquad \tilde{\psi}_- = e^{-j\tau}\psi_-$$

for some real function τ. Rotating the normal frame of $\tilde{f}$ by this angle τ, we get $\tilde{\psi}_\pm = \psi_\pm$ for this new frame. As before, we get $\tilde{\omega}_\pm = \omega_\pm$ and so f and $\tilde{f}$ are congruent. In both cases, the isometry g of $\hat{Q}_c^4$ taking f to $\tilde{f}$ preserves the orientation of both the tangent and the normal bundle, so it is oriented.

REMARK. In fact, it is not necessary to assume that $\tilde{K}_N = K_N$. We have $k_+k_- = (c-K)^2 - K_N^2$, and since $\Phi = \tilde{\Phi}$, we have $\tilde{K}_N = \pm K_N$. In case $\tilde{K}_N = -K_N$, the immersion f and $\tilde{f}$ differ by an orientation reversing isometry. Similar, in the case $H \neq 0$, it is only necessary to assume that $\tilde{\Phi} = \Phi$ and $\tilde{\psi}_3 = \psi_3$ where e_3, $\tilde{e}_3$ are pointing in the direction of the mean curvature vector.

(*c*) *Existence.* Let (M, ds^2) be a simply connected surface, $H \geqslant 0$ some number and K_N a function which is zero if $H > 0$, such that

$$a_\pm := (c + H^2 - K \pm K_N)^{\frac{1}{2}}$$

is AVT satisfying

$$\Delta \log a_\pm = 2K \mp K_N \tag{17}$$

unless $a_\pm \equiv 0$. Let $U \subset M$ be an open disk and $z\colon U \to \mathbb{C}$ a conformal coordinate. Choose an oriented orthonormal basis e_1, e_2 on U. Put $\omega_i = \langle\ ,e_i\rangle$ and $\varphi = \omega_1 + i\omega_2$. Then $\varphi = \mu\, dz$ with $|\mu| = \lambda$ the conformal factor. By (17) we have

$$\Delta \log \lambda^4 a_+ a_- = 0$$

(recall that $\Delta \log \lambda = -K$). Since $b := \lambda^4 a_+ a_-$ is AVT, there exists a holomorphic function g with $b = |g|$ (see [7], 3.12). In fact we have $b = b_0|h|$ with h holomorphic and $b_0 > 0$ and $\Delta \log b_0 = 0$. So $\log b_0$ is the real part of a holomorphic function l and thus $g = e^l \cdot h$ is holomorphic with $|g| = b$. In fact, g is uniquely determined up to a constant factor $e^{i\tau}$. Choose any such g. Let k_+, k_- be functions of holomorphic type satisfying $|k_\pm| = a_\pm$ and $k_+ k_- \mu^4 = g$ and put $\psi_\pm = k_\pm \varphi + H\bar{\varphi}$. Then (10) is automatically satisfied. To satisfy (12′) and hence (12), according to Lemma 2 we have to put

$$\omega_\mp = 2\,\mathrm{Im}\left(\partial_{\bar z}(\log p_\pm)\, d\bar z\right)$$

whenever $p_\pm := \mu k_\pm$ is not the zero function. Then we have

$$2i\, d\omega_\mp = -\,(\Delta \log a_\pm + \Delta \log \lambda)\bar\varphi \wedge \varphi = -\,(K \mp K_N)\bar\varphi \wedge \varphi\,.$$

On the other hand,

$$\bar\psi_\pm \wedge \psi_\pm = (a_\pm^2 - H^2)\bar\varphi \wedge \varphi = (c - K \pm K_N)\bar\varphi \wedge \varphi\,;$$

this proves Equation (11). Finitely, put

$$\tilde\omega_{12} := \tfrac{1}{2}(\omega_+ + \omega_-) = \mathrm{Im}\left(\partial_{\bar z}(\log p_+ p_-)\, d\bar z\right) = -\,2\,\mathrm{Im}\left(\partial_{\bar z}(\log \mu)\right) d\bar z\,.$$

Since $\varphi = \mu\, dz$, we get by Lemma 2

$$d\varphi = i\tilde\omega_{12} \wedge \varphi$$

and hence $\tilde\omega_{12} = \omega_{12}$. Thus (9) is satisfied.

It remains to consider the case $a_+ a_- \equiv 0$ which implies $H = 0$. If $a_- \equiv 0$, that means $K_N + K = c$, we put ψ_+, ω_- as above and $\psi_- = 0$, $\omega_+ = 2\omega_{12} - \omega_-$. Then ω_- still satisfies (11) and hence

$$2i\, d\omega_- = -\,(K - K_N)\bar\varphi \wedge \varphi$$

as before. Therefore,

$$2i\,d\omega_+ = 2i(2\,d\omega_{12} - d\omega_-) = -(K + K_N)\bar{\varphi}\wedge\varphi = -c\bar{\varphi}\wedge\varphi\,,$$

and hence ω_+ satisfies (11). Equation (12) is valid for ψ_+, ω_-, as before, and it is trivial for ψ_-, ω_+. (9) is satisfied by definition of ω_{12} (as an intrinsic quantity).

If instead $a_+ \equiv 0$, we argue similarly replacing $+$ with $-$. If both a_+ and a_- are the zero functions, i.e. $K = c$, $K_N = 0$, we put $\psi_+ = \psi_- = 0$, $\omega_+ = \omega_- = \omega_{12}$, and Equations (9)-(12) are satisfied.

Thus, by the general existence theorem, we get an isometric immersion $f\colon U \to \hat{Q}^4_c$ with parallel mean curvature vector of length H (since (12′) is satisfied), together with a Darboux frame $e_1, \ldots, e_4$ along f. In fact, we recover any quartic form Φ and in the case $H \neq 0$ any two quadratic forms Φ_+, Φ_-, holomorphic with given absolute values, by choosing the functions g, k_+, k_- suitably.

Now cover M by open coordinate balls U_t. Then we get immersions $f_t\colon U_t \to \hat{Q}^4_c$. Choose these so that the corresponding quartic or quadratic forms agree in the intersections $U_t \cap U_s$. By simple connectivity of M, this can be achieved. Then $f_t | U_t \cap U_s = g \circ f_s | U_t \cap U_s$ for some proper motion g of $\hat{Q}^4_c$, by the uniqueness part (b). Thus f_t and $g \circ f_s$ together define an immersion of $U_t \cup U_s$. Continuing this process, we get an immersion $f\colon M \to \hat{Q}^4_c$ with the desired properties.

It remains to verify the remark following Theorem 1. If $f\colon M \to Q^4_c$ is an immersion with parallel mean curvature vector of length $H \neq 0$, the holomorphic quadratic forms Φ_+ and Φ_- have the same absolute values $|k_+| = |k_-| = (c + H^2 - K)^{\frac{1}{2}}$. So they differ by a constant factor of unit length: Say $\varphi_+ = e^{2\delta i}\varphi_-$. Hence $\psi'_+ = e^{2\delta i}\psi'_-$, and putting $\psi'_\alpha := \psi_\alpha - h_\alpha\bar{\varphi}$ (see ch. 4), we get

$$\psi'_3 \sin\delta = \varphi'_4 \cos\delta\,.$$

In other words, the parallel vector $\xi := -(\sin\delta)\,e_3 + (\cos\delta)\,e_4$ is umbilic, i.e. $\langle\alpha', \xi\rangle = 0$ (compare [17]). It is easy to see ([17]), that $f(M)$ lies in a totally umbilic hypersurface Q^3_c in Q^4_c with normal vector ξ. Thus $\sigma := \pi/2 + \delta$ is the angle between the mean curvature vector $\eta = He_3$ and ξ.

On the other hand, if $f, \tilde{f}\colon M \to Q^4_c$ are isometric immersions with parallel mean curvature vector of length $H \geqq 0$ with $|\tilde{\Phi}| = |\Phi|$ and

$\tilde{\psi}_+/\tilde{\psi}_- = \psi_+/\psi_-$, then $\tilde{\Phi} = e^{4\tau i}\Phi$ for some constant angle τ. Hence $\tilde{\psi}_\pm = e^{2\tau i}\psi_\pm$ and so $\tilde{\alpha} = R_\tau^* \alpha$ where R_τ denotes the rotation by τ in the tangent plane.

Appendix: Mean curvature vector of constant length.

We want to give an example of a surface in 4-space whose mean curvature vector has constant length but whose quartic form Φ (see ch. 4) is not holomorphic (compare ch. 2).

Put $e: \mathbb{R} \to \mathbb{C}$, $e(t) = e^{it}$. Consider the immersion $f: I \times J \to \mathbb{C}^2 = \mathbb{R}^4$, $f(s, t) = \big(e(s), g(s)\, e(t)\big)$ where I, J are open intervals and $g: I \to (0, \infty)$ some smooth function. Let f_s, f_t denote the partial derivatives. We have $f_s \perp f_t$ and $\|f_s\| = (1 + g'^2)^{\frac{1}{2}} =: h$, $\|f_t\| = g$. Let us put

$$e_1 = \frac{1}{h} f_s = \frac{1}{h}(e', g'e)\,, \qquad e_2 = \frac{1}{g} f_t = (0, e')\,,$$

$$e_3 = (e, 0)\,, \qquad e_4 = \frac{1}{h}(-g'e', e);$$

this is a Darboux frame for the immersion f. For the 2nd fundamental form α we get in terms of this frame:

$$\alpha(e_1, e_1) = -h^{-2}e_3 + h^{-3}g''e_4\,,$$

$$\alpha(e_2, e_2) = -\frac{1}{gh}e_4\,,$$

$$\alpha(e_1, e_2) = 0\,.$$

Therefore, the mean curvature vector $\eta = \frac{1}{2}\big(\alpha(e_1, e_1) + \alpha(e_2, e_2)\big)$ has constant length if and only if

$$\text{(A1)} \qquad h^{-4} + \left(h^{-3}g'' - \frac{1}{gh}\right)^2 = c_1 = \text{const}\,.$$

On the other hand, the corresponding quartic form is

$$\Phi = \psi_+'\psi_-'\varphi^2 = (k_3^2 + k_4^2)\varphi^4$$

with

$$k_3 = -\frac{1}{2}h^{-2}, \qquad k_4 = \frac{1}{2}\left(h^{-3}g'' + \frac{1}{gh}\right)$$

and $\varphi = \omega_1 + i\omega_2$ where $\omega_i = \langle df, e_i \rangle$. Now putting $du = (\sigma/g)\,ds$ we have $\varphi = g(du + i\,dt)$, and so (u, t) is a conformal coordinate system for the metric on $I \times J$ induced by f, with conformal factor g. Since its coefficient function is real, $\Phi = (k_3^2 + k_4^2)g^4(du + i\,dt)^4$ is holomorphic if and only if $(k_3^2 + k_4^2)g^4 = \text{const}$, hence iff

$$\text{(A2)} \qquad \left(h^{-4} + \left(h^{-3}g'' + \frac{1}{gh}\right)^2\right)g^4 = c_2 = \text{const}\,.$$

Since $h^2 = 1 + g'^2$, Equations (A1) and (A2) are 2nd order differential equations for g. Since they are independent, one finds a function g which solves (A1) but not (A2).

To be specific, let $c_1 = 5$ and consider the solution g of

$$\text{(A1}'\text{)} \qquad g'' = g^{-1}(1 + g'^2) + \left(5(1 + g'^2)^3 - (1 + g'^2)\right)^{\frac{1}{2}}$$

with initial values

$$g(0) = 1\,, \qquad g'(0) = 0\,.$$

Then g solves also (A1), and it follows from (A1′) that g is even with $g''(0) = 3$, $g^{(4)}(0) = 78$. Instead, from (A2) we would get $g^{(4)}(0) = 72$, so g does not solve (A2).

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