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Hölder-continuity of solutions for some Schrödinger equations

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0. Introduction.

Recently the local regularity properties for solutions of Schrödinger equations of the form

$$(*) \qquad Lu \equiv -(a_{ij}u_{x_i})_{x_j} = Vu$$

have been studied by many authors (see e.g. [A-S], [D-M], [C-F-G], [C-F-Z]) allowing V to be a very singular potential, precisely $V \in S$, the Stummel-Kato class (see definition 1.1).

Under this assumption in [C-F-G] was established a Harnack inequality and proved a local continuity result for solutions of $(*)$.

It is easy to see that if Ω is an open bounded set in R^n then $L^p(\Omega) \subseteq S$ for $p > n/2$; hence the result in [C-F-G] generalizes the well known Hölder estimates by Stampacchia [ST], Ladizhenskaia [L-U] etc.

We stress that high integrability of V does not play an essential role.

In fact also the Morrey space $L^{1,\lambda}(\Omega)$ is contained in S for $\lambda > n-2$ and being in $L^{1,\lambda}(\Omega)$, for any $0 < \lambda < n$, does not imply any extra integrability (see e.g. the examples in [P2]).

In this paper we assume V in $L^{1,\lambda}(\Omega)$ $(\lambda > n-2)$ and prove local hölder-continuity for solutions of $(*)$ hence, in this special situation, we improve the continuity result in [C-F-G].

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Our technique is very close to the one in [C-F-G] heavily relying on the exploitation of well known estimates for the Green function of L.

There is however a technical difficulty.

It is impossible to use the usual C^∞-approximation for L and V (as in [C-F-G]) because functions in Morrey spaces are not close, in general, to bounded functions in $L^{1,\lambda}(\Omega)$ (see [P1] p. 22 for an example of an $L^{1,\lambda}(\Omega)$ function with distance from $L^\infty(\Omega)$ equal to 1). We overcame this difficulty by developping a representation formula for solutions of (*) that extends classical results on the Green function (see e.g. [ST]).

1. Some function spaces.

Let Ω be an open bounded set of R^n $(n > 3)$.

We will need some mild regularity assumption to be satisfies by $\partial\Omega$ e.g.

$$\exists A \in]0,1[: |\Omega_r(x)| \leqslant A|B_r(x)| \qquad \forall x \in \partial\Omega$$

where $r: 0 < r < \operatorname{diam}(\Omega)$ (1).

DEFINITION 1.1 (*Stummel-Kato* class). *We say that* $V: \Omega \to R$ *belongs to the Stummel-Kato class* S *iff there exists a non decreasing function* $\eta(r) > 0$ *with* $\lim_{r\to 0} \eta(r) = 0$ *such that*

$$\operatorname*{Sup}_{x\in\Omega} \int_{\Omega_r(x)} |V(y)||x-y|^{2-n}\, dy \leqslant \eta(r) \tag{1.1}$$

Obviously $S \subseteq L^1(\Omega)$.

DEFINITION 1.2 (*Morrey spaces*). $L^{1,\lambda}(\Omega)$ $(0 < \lambda < n)$ is the space of functions $f \in L^1(\Omega)$ such that

$$\|f\|_{L^{1,\lambda}(\Omega)} =: \operatorname*{Sup}_{\substack{x\in\Omega\\ r>0}} r^{-\lambda} \int_{\Omega_r(x)} |f(y)|\, dy < +\infty\,.$$

(1) $|E|$ denotes the Lebesgue measure of a measurable subset E of R^n:

$$B_r(x) =: \{y \in R^n: |x-y| < r\}\,; \qquad \Omega_r(x) =: \Omega \cap B_r(x)\,.$$

LEMMA 1.1. *If u belongs to $L^{1,\lambda}(\Omega)$ $(n-2<\lambda<n)$ then u belongs to the Stummel-Kato class and*

$$\int_{\Omega_r(x)} |u(y)|\,|x-y|^{2-n}\,dy \leqslant Cr^{\lambda-n+2}\|u\|_{L^{1,\lambda}(\Omega)}$$

where C depends only on λ and n.

Indeed,

$$\int_{\Omega}|u(y)||x-y|^{2-n}\,dy=\sum_{k=0}^{+\infty}\int_{\Omega\cap\{r/2^{k+1}\leqslant|x-y|<r/2^k\}}|u(y)||x-y|^{2-n}\,dy\leqslant$$

$$\leqslant\sum_{k=0}^{+\infty}(r2^{-k-1})^{2-n}\int_{\Omega_{r/2^k}(x)}|u(y)|\,dy\leqslant r^{\lambda-n+2}C\|u\|_{L^{1,\lambda}(\Omega)}\,.$$

REMARK 1.1:

$$L^{1,\lambda}(\Omega)\subseteq S\subseteq L^{1,\mu}(\Omega)\qquad \text{where } 0<\mu\leqslant n-2<\lambda<n\,.$$

Indeed the inclusion $L^{1,\lambda}(\Omega)\subseteq S$ is an immediate consequence of Lemma 1.1 and the other inclusion is obvious.

We now recall the definitions of the Sobolev spaces $H^{1,p}(\Omega)$, $H_0^{1,p}(\Omega)$ and $H^{-1,p}(\Omega)$.

DEFINITION 1.3. *We say that u belongs to $H^{1,p}(\Omega)[H^{1,p}_{\mathrm{loc}}(\Omega)](1<$ $<p<+\infty)$ iff u,*

$$\frac{\partial u}{\partial x_i}\in L^p(\Omega)[L^p_{\mathrm{loc}}(\Omega)]\qquad (i=1,2,\dots,n)$$

$H^{1,p}(\Omega)$ is a Banach space under the norm

$$\|u\|_{H^{1,p}(\Omega)}=\|u\|_{L^p(\Omega)}+\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}$$

$H_0^{1,p}(\Omega)$ is the closure of $\mathfrak{D}(\Omega)$ with respect to the $H^{1,p}(\Omega)$ norm; $H^{-1,p}(\Omega)$ is the dual space of $H_0^{1,q}(\Omega)$, where $1/p+1/q=1$. We have $T\in H^{-1,p}(\Omega)$ iff, $\exists f_i\in L^p(\Omega)$ $(i=1,2,\dots,n)$ such that $T=\sum_{i=1}^{n}\partial f_i/\partial x_i$.

2. Green's function and a representation formula.

In the following sections we will consider the operator $L - V$ where L is the divergence form elliptic operator

$$L = -\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)$$

satisfying

$$\left\{\begin{array}{l} a_{ij} \in L^\infty(\Omega)\,, \quad a_{ij} = a_{ji} \quad (i, j = 1, 2, \dots, n) \\ \exists \nu > 0: \nu|\xi|^2 < a_{ij}\xi_i\xi_j < \nu^{-1}|\xi|^2 \quad \forall \xi \in R^n \end{array}\right. \tag{2.1}$$

and V is a function

$$V \in L^{1,\lambda}(\Omega) \quad (\lambda > n-2)\,. \tag{2.2}$$

DEFINITION 2.1. *We say that $u \in H^{1,2}_{\mathrm{loc}}(\Omega)$ is a local weak solution of the equation*

$$Lu = Vu$$

iff

$$\int_\Omega a_{ij}(x)\, u_{x_i}(x)\, \psi_{x_j}(x)\, dx = \int_\Omega V(x)\, u(x)\, \psi(x)\, dx\,; \qquad \forall \psi \in \mathfrak{D}(\Omega)\,. \tag{2.3}$$

Definition 2.1 is meaningful by the inclusion $L^{1,\lambda}(\Omega) \subseteq S$ and [S] p. 138-140.

We recall that under the weaker hypothesis $V \in S$ the following regularity result for weak solutions was proven in [C-F-G].

THEOREM 2.1. *There exist two positive constants $C = C(\nu, n)$, $r_0 = r_0(\nu, n, \eta)$ (η from definition 1.1) and a non decreasing function $\omega(r)$: $\lim_{r\to 0} \omega(r) = 0$ such that, for any local weak solution of $Lu + Vu = 0$ in Ω and for every ball $B_r(x_0)$: $B_{4r}(x_0) \subseteq \Omega$ $(0 < r \leq r_0)$ we have:*

$$\operatorname*{osc}_{B_r(x_0)} u \leq C\omega(r) \operatorname*{Sup}_{B_{3r}(x_0)} |u|\,.$$

We now define a different class of solutions:

DEFINITION 2.2. *Let L be such that* (2.1) *holds, let μ be a bounded variation measure in Ω and $T = \sum_{i=1}^{n} \partial f_i/\partial x_i \in H^{-1,2}(\Omega)$.*

We say that $u \in L^1(\Omega)$ is a very weak solution of the equation

$$Lu = \mu + T$$

if and only if

$$\int_\Omega u(x)\, L\psi(x)\, dx = \int_\Omega \psi(x)\, d\mu - \sum_{i=1}^{n} \int_\Omega f_i(x) \frac{\partial \psi}{\partial x_i}\, dx \tag{2.4}$$

for every $\psi \in H_0^{1,2}(\Omega) \cap C^0(\bar{\Omega})$ such that $L\psi \in C^0(\bar{\Omega})$. In much the same way as in [ST] it is possible to show

LEMMA 2.1. *Assume μ is a bounded variation measure and $T = \sum_{i=1}^{n} \partial f_i/\partial x_i \in H^{-1,2}(\Omega)$. If $u \in H_0^{1,2}(\Omega)$ is a weak solution of the equation*

$$Lu = \mu + T$$

i.e.

$$\int_\Omega a_{ij}(x)\, u_{x_i}(x)\, \psi_{x_j}(x)\, dx = \int_\Omega \psi(x)\, d\mu - \sum_{i=1}^{n} \int_\Omega f_i(x) \frac{\partial \psi}{\partial x_i}\, dx; \quad \forall \psi \in H_0^{1,2}(\Omega) \tag{2.5}$$

then u is the very weak solution of the same equation.

The proof is an easy consequence of the definitions above. We now recall the definition of fundamental solution.

Let $y \in \Omega$ and δ_y the Dirac mass at y.

Consider the equation

$$Lu = \delta_y .$$

We call its (very weak) solution the Green's function relative to the operator L with pole at y and we denote it by $g(x, y)$.

By the definition above the solution $\varphi \in H_0^{1,2}(\Omega) \cap C^0(\bar{\Omega})$ of $L\varphi = \psi$, where $\psi \in C^0(\Omega)$ is given by the formula

$$\varphi(y) = \int_\Omega g(x, y)\,\psi(x)\,dx = \langle \psi(x), g(x, y)\rangle .$$

Consider:

$$Lu = \mu + T \tag{2.8}$$

where μ is a bounded variation measure, $T \in H^{-1,p}(\Omega)$ $(p > n)$. We have the following

THEOREM 2.2:

$$u(x) = \langle \mu(y), g(x, y)\rangle + \langle T(y), g(x, y)\rangle$$

is the very weak solution of (2.8).

PROOF. We consider only the case $\mu = 0$ (for the case $T = 0$ see [ST] Th. 8.3 p. 227).

We will show that

$$u(x) = \langle T(y), g(x, y)\rangle$$

satisfies:

$$\langle L\psi(x), \langle T(y), g(x, y)\rangle\rangle = \langle T(y), \psi(y)\rangle ; \qquad \forall \psi \in H_0^{1,2}(\Omega) \cap C^0(\bar{\Omega})$$

such that $L\psi \in C^0(\bar{\Omega})$.

Let

$$T = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \qquad \text{where } f_i \in L^p(\Omega) , \quad i = 1, 2, \ldots, n .$$

Then

$$\langle L\psi(x), \langle T(y), g(x, y)\rangle\rangle = \int_\Omega L\psi(x) \left(-\int_\Omega \frac{\partial g}{\partial y_i} f_i(y)\,dy\right) dx .$$

We observe that

$$|L\psi(x) \frac{\partial g}{\partial y_i} f_i(y)| \in L^1(\Omega \times \Omega) .$$

Indeed we have:

$$\int_\Omega\left(\int_\Omega |L\psi(x)|\left|\frac{\partial g}{\partial y_i}\right| |f_i(y)|\,dy\right)dx = \int_\Omega |L\psi(x)|\left(\int_\Omega\left|\frac{\partial g}{\partial y_i}\right| |f_i(y)|\,dy\right)dx \leqslant$$

$$\leqslant \int_\Omega |L\psi(x)|\left\|\frac{\partial g}{\partial y_i}\right\|_{L^{p'}(\Omega)} \|f_i\|_{L^p(\Omega)}\,dx \leqslant \max_{\bar\Omega} |L\psi(x)|\,\|f_i\|_{L^p(\Omega)}\int_\Omega\left\|\frac{\partial g}{\partial x_i}\right\|_{L^{p'}(\Omega)} dx\,.$$

Then (see [ST] p. 220 (8.6))

$$\int_\Omega\left(\int_\Omega |L\psi(x)|\left|\frac{\partial g}{\partial y_i}\right| |f_i(y)|\,dy\right) \leqslant C \max_{\bar\Omega} |L\psi(x)|\,\|f_i\|_{L^{p'}(\Omega)}\,.$$

By Tonelli and Fubini's theorems we have:

$$\int_\Omega L\psi(x)\left(-\int_\Omega \frac{\partial g}{\partial y_i} f_i(y)\,dy\right)dx = \int_\Omega f_i(y)\left(-\int_\Omega \frac{\partial g}{\partial y_i} L\psi(x)\,dx\right)dy =$$

$$= \int_\Omega f_i(y)\left(-\frac{\partial}{\partial y_i}\int_\Omega g(x,y)\,L\psi(x)\,dx\right)dy =$$

$$= \int_\Omega f_i(y)\left(-\frac{\partial}{\partial y_i}\langle g(x,y), L\psi(x)\rangle\right) = \left\langle\frac{\partial f_i}{\partial y_i}, \langle g(x,y), L\psi(x)\rangle\right\rangle =$$

$$= \langle T(x), \psi(x)\rangle\,.$$

Remark 2.1. In the proof above we may differentiate under the integral; i.e.

$$-\int_\Omega \frac{\partial g}{\partial y_i} L\psi(x)\,dx = -\frac{\partial}{\partial y_i}\int_\Omega g(x,y)\,L\psi(x)\,dx\,.$$

In fact, for every $\varphi\in\mathfrak{D}(\Omega)$ we have, using Fubini's theorem:

$$-\left\langle\frac{\partial}{\partial y_i}\int_\Omega g(x,y)\,L\psi(x)\,dx, \varphi(y)\right\rangle = \left\langle\int_\Omega g(x,y)\,L\psi(x)\,dx, \frac{\partial\varphi}{\partial y_i}\right\rangle =$$

$$= \int_\Omega \left(\int_\Omega g(x, y) L\psi(x)\, dx \right) \frac{\partial \varphi}{\partial y_i}\, dy = -\int_\Omega \left(\int_\Omega \frac{\partial g}{\partial y_i} \varphi(y)\, dy \right) L\psi(x)\, dx =$$

$$= -\left\langle \int_\Omega \frac{\partial g}{\partial y_i} L\psi(x)\, dx, \varphi(y) \right\rangle .$$

3. Hölder-continuity of local solutions.

We now state the main result of this paper

THEOREM 3.1. *There exist positive numbers* $r_0 = r_0(\nu, \|V\|_{1,\lambda}, \lambda, n)$ $\alpha = \alpha(\nu, n)$, $C = C(\nu, n, \|V\|_{1,\lambda}, \lambda)$ *such that for any local solution* u *of* $Lu = Vu$ *in* Ω *and for any ball* $B_r(x_0)$, *with* $B_{4r}(x_0) \subseteq \Omega$, $0 < r \leqslant r_0$ *we have*

$$|u(x) - u(x_0)| \leqslant C \operatorname*{Sup}_{B_{3r}(x_0)} |u| r^{\lambda - n + 2} \cdot$$

$$\cdot \left(|x - x_0|^{\alpha/2} r^{-\alpha/2} + |x - x_0|^{(\lambda - n + 2)/2} r^{-(\lambda - n + 2)/2} + \left(\frac{|x - x_0|}{r} \right)^\alpha \right).$$

PROOF. Let $V \in L^{1,\lambda}(\Omega)$ and u a local weak solution of $Lu = Vu$ i.e. $u \in H^{1,2}_{\mathrm{loc}}(\Omega)$ such that:

$$\int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_j}\, dx = \int_\Omega V(x) \psi(x)\, dx \qquad \forall \psi \in \mathcal{D}(\Omega) . \tag{3.1}$$

Let $\varphi \in \mathcal{D}(\Omega)$. It is easy to see that $u\varphi$ is such that

$$\int_\Omega a_{ij}(x) \frac{\partial (u\varphi)}{\partial x_i} \frac{\partial \psi}{\partial x_j}\, dx = \int_\Omega V(x) u(x) \psi(x) \varphi(x)\, dx +$$

$$+ \int_\Omega a_{ij}(x) u(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j}\, dx - \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \psi(x)\, dx$$

holds.

Therefore, by Lemma 2.1, $u\varphi$ is a very weak solution of

$$L(u\varphi) = V(x) u(x) \varphi(x) - \frac{\partial}{\partial x_j} \left(a_{ij}(x) u(x) \frac{\partial \varphi}{\partial x_i} \right) - a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} .$$

By Theorem 2.2 we have

$$u(x)\varphi(x)=\int_\Omega V(y)u(y)\varphi(y)g(x,y)\,dy+\int_\Omega \frac{\partial g}{\partial y_i}a_{ij}(y)u(y)\frac{\partial\varphi}{\partial y_j}\,dy-$$
$$-\int_\Omega g(x,y)a_{ij}(y)\frac{\partial u}{\partial y_i}\frac{\partial\varphi}{\partial y_j}\,dy\,.$$

Now we choose $\varphi\in\mathfrak{D}(\Omega)$ such that $0\leqslant\varphi\leqslant 1$, $\varphi(x)=1$ in $B_{\frac{3}{2}r}(x_0)$, supp $(\varphi)\subseteq B_{2r}(x_0)$, $|\nabla\varphi|\leqslant C/r$ where $0<r\leqslant r_0$ and r_0 is determined by the local boundedness theorem 1.4 in [C-F-G].

Obviously, for every $x\in B_{2r}(x_0)$ we have:

$$u(x)-u(x_0)=\int_\Omega V(y)u(y)\varphi(y)\big(g(x,y)-g(x_0,y)\big)\,dy-$$
$$-\int_\Omega \big(g(x,y)-g(x_0,y)\big)a_{ij}(y)\frac{\partial u}{\partial y_i}\frac{\partial\varphi}{\partial y_j}\,dy+$$
$$+\int_\Omega\left(\left(\frac{\partial g}{\partial y_i}\right)_{(x,y)}-\left(\frac{\partial g}{\partial y_j}\right)_{(x_0,y)}\right)a_{ij}(y)u(y)\frac{\partial\varphi}{\partial y_j}\,dy=\mathrm{I}-\mathrm{II}+\mathrm{III}\,.$$

We begin estimating I.

$$\mathrm{I}=\int_{|x-y|>N|x-x_0|}\big(g(x,y)-g(x_0,y)\big)V(y)u(y)\varphi(y)\,dy+$$
$$+\int_{|x_0-y|\leqslant N|x-x_0|}\big(g(x,y)-g(x_0,y)\big)V(y)u(y)\varphi(y)dy=A+B\,.$$

Where N is a positive number to be fixed later.

To estimate A we use the inequality (see [G-T] p. 200 Th. 8.22 and Harnack's Theorem)

$$|g(x,y)-g(x_0,y)|\leqslant C(\nu,n)\left(\frac{|x-x_0|}{r}\right)^\alpha g(x_0,y)\leqslant$$
$$\leqslant\frac{C(\nu,n)}{N^\alpha}g(x_0,y)\leqslant\frac{C(\nu,n)}{N^\alpha|x_0-y|^{n-2}}$$

hence

$$A \leqslant \frac{C(\nu, n)}{N^\alpha} \int\limits_{B_{2r}(x_0)} \frac{|V(y)|}{|x_0 - y|^{n-2}}\, dy \operatorname*{Sup}_{B_{2r}(x_0)} |u|$$

and by Lemma 1.1

$$A \leqslant \frac{C(\|V\|_{L^{1,\lambda}(\Omega)}, \nu, n, \lambda)}{N^\alpha} r^{\lambda - n + 2} \operatorname*{Sup}_{B_{2r}(x_0)} |u| .$$

To estimate B we use Lemma 1.1 and the following bound

$$g(x, y) \leqslant \frac{C(\nu, n)}{|x - y|^{n-2}}$$

proven in [L-S-W].

We obtain:

$$|g(x, y) - g(x_0, y)| \leqslant \frac{C(\nu, n)}{|x - y|^{n-2}} + \frac{C(\nu, n)}{|x_0 - y|^{n-2}}$$

and therefore

$$\begin{aligned} B &\leqslant C(\nu, n) \int\limits_{|x_0 - y| \leqslant N|x - x_0|} \frac{|V(y)|}{|x - y|^{n-2}}\, dy \operatorname*{Sup}_{B_{2r}(x_0)} |u| \leqslant \\ &\leqslant C(\nu, n) \int\limits_{|x_0 - y| \leqslant (N+1)|x - x_0|} \frac{|V(y)|}{|x - y|^{n-2}}\, dy \leqslant \\ &\leqslant C(\nu, n \|V\|_{L^{1,\lambda}(\Omega)}, \lambda) \operatorname*{Sup}_{B_{2r}(x_0)} |u| \big((N+1)|x - x_0|\big)^{\lambda - n + 2} . \end{aligned}$$

Now, if we choose $N = (r/|x - x_0|)^{\frac{1}{2}} > 1$ we obtain

$$\begin{aligned} |\mathrm{I}| &\leqslant C(\|V\|_{L^{1,\lambda}(\Omega)}, \lambda, \nu, n) \operatorname*{Sup}_{B_{2r}(x_0)} |u|\, |x - x_0|^{\alpha/2} r^{\lambda - n + 2 - \alpha/2} + \\ &\quad + C(\|V\|_{L^{1,\lambda}(\Omega)}, \lambda, \nu, n) \operatorname*{Sup}_{B_{2r}(x_0)} |u| |x - x_0|^{(\lambda - n + 2)/2} r^{(\lambda - n + 2)/2} . \end{aligned}$$

Estimating II and III as in [C-F-G] we obtain

$$|\mathrm{II}| \leqslant C(\nu, n) \left(\frac{|x - x_0|}{r}\right)^\alpha \left(\fint_{B_{3r}(x_0)} u(y)^2\, dy\right)^{\frac{1}{2}}$$

and

$$|III| \leqslant C(\nu, n)\left(\frac{|x-x_0|}{r}\right)^{\alpha}\left(\ \fint_{B_{2r}(x_0)} u(y)^2\, dy\right)^{\frac{1}{2}}.$$

The theorem now follows.

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