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GIORGIO PIVA

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On Endomorphism Algebras over Admissible Dedekind Domains.

GIORGIO PIVA (*)

0. Introduction.

In 1963 A. L. S. Corner proved the following noteworthy

THEOREM A (cf. [C], Theorem A, p. 688). *Every countable, reduced and torsion-free ring is isomorphic with the endomorphism ring of some countable, reduced and torsion-free group.*

Some years later (1969) A. Orsatti generalized this result to the class of locally countable, reduced and torsion-free rings; precisely he established

THEOREM B (cf. [O], Theorem A*, p. 143). *Let A be a locally countable, reduced and torsion-free ring; then A is isomorphic with the endomorphism ring of some locally countable, reduced and torsion-free group G, having the same cardinality as A.*

In proving this result the Author substantially used Corner's methods but he simplified them by the introduction of a local-global argument.

Now, by means of this technique, we aim to extend Theorem B to a class of algebras over particular Dedekind domains. We first give the following

DEFINITION. *Let R be a Dedekind domain (not a field) and let Ω be the set of its non-zero prime ideals; R will be called « admissible »*

(*) Indirizzo dell'A.: Via A. Minto 3 - 35028 Piove di Sacco (Padova), Italy.

if its P-adic completion $\hat{R}_P$ has uncountable transcendence degree over R_P for each $P \in \Omega$.

We also recall that $\hat{R} = \prod_{P \in \Omega} \hat{R}_P$ where $\hat{R}$ is the natural completion of R, $\hat{R}_P$ the P-adic completion of R or, equivalently, the natural completion of R_P for each $P \in \Omega$.

We are now in position to state our

THEOREM C. *Let R be an admissible Dedekind domain; then if A is any R-algebra locally of countable rank which is reduced and torsion-free as an R-module, there exists a reduced, torsion-free R-module M locally of countable rank and of the same (global) rank as A such that $A \cong End_R(M)$.*

Recently several authors obtained « Corner-type » results on endomorphism algebras; the sharpest one has been achieved in 1985 by Corner himself and R. Göbel using a combinatorial argument due to S. Shelah (see [CG]). We refer to this paper also for a full bibliography on above cited works.

Nevertheless our Theorem C cannot be derived from the Main Theorem of [CG]. Infact in § 6 of [CG] every R-algebra considered in the statement of Theorem C (apart from countable case which is not treated by Corner and Göbel) is realized as the endomorphism algebra of a suitable R-module M, but the rank of M ($=$ rk M) results not less than the cardinality of A, i.e. $|A| \leqslant \text{rk}\, M$, hence one gets $\text{rk}\, A < \text{rk}\, M$ whenever $\text{rk}\, A < |A|$ and the « global rank requirement » of Theorem C is not satisfied.

1. Preliminaries.

Throughout R denotes an admissible Dedekind domain, Ω the set of its maximal ideals and, for each $P \in \Omega$, R_P is the localization of R at P: R_P is a discrete valuation ring. All modules considered are torsion-free. Let M be an R-module; M is said to be divisible if $rM = M$ for every non-zero $r \in R$, M is said to be reduced when it admits no divisible submodules other than zero.

Now let L be a sub-R-module of M; we recall that L is pure in M when $rL = L \cap rM$ for every $r \in R$, and L is P-pure in M when $P^n L = L \cap P^n M$ for every $n \in \mathbb{N}$ ($P \in \Omega$).

Given a subset X of M we denote by $\langle X\rangle_*$ the intersection of all pure submodules of M containing X; $\langle X\rangle_* = \{m \in M : rm \in \langle X\rangle$ for some $r \in R$, $r \neq 0\}$, where $\langle X\rangle$ is the submodule generated by X.

For each $P \in \Omega$ we set $P^\omega M = \bigcap_{n\in \mathbf{N}} P^n M$ and form the quotient $M/P^\omega M$ which results a reduced, torsion-free R-module. Next we endow M with the natural topology defined by taking the family of submodules rM $(r \in R, r \neq 0)$ as a basis of neighbourhoods of 0: M is Hausdorff whenever $M_\omega = \bigcap_{P\in\Omega} P^\omega M = 0$. In particular a torsion-free module is Hausdorff in its natural topology if and only if it is reduced. All homomorphisms between modules endowed with their natural topologies are continuous.

From now on we suppose M to be reduced and torsion-free and consider its natural completion $\hat{M}$ (that is the completion of M provided with the natural topology). $\hat{M}$ is complete, Hausdorff, reduced, torsion-free R-module containing M as a dense topological submodule; moreover its topology coincides with the natural topology.

Following the nomenclature adopted for groups and rings in [O], we introduce for each $P \in \Omega$ the Hausdorff P-localization M_P^* of M setting

$$M_P^* = (M/P^\omega M) \otimes_R R_P \qquad \text{(tensor product of } R\text{-modules)};$$

M_P^* is in a natural way an R_P-module and it is Hausdorff in its natural topology.

We now consider the projection $\varrho_P \colon M \to M/P^\omega M$, the inclusions $j_P \colon M/P^\omega M \to M_P^*$, $i_P \colon M_P^* \to \hat{M}_P^*$ (i_P exists by definition of natural completion) and form the pair of homomorphisms $\varphi_P \colon M \to M_P^*$, $\psi_P \colon M \to \hat{M}_P^*$ putting

$$\varphi_P = j_P \circ \varrho_P, \qquad \psi_P = i_P \circ \varphi_P;$$

then by means of the diagonal-homomorphisms φ and ψ of the families $\{\varphi_P : P \in \Omega\}$, $\{\psi_P : P \in \Omega\}$ we make the following diagrams commutative:

$$(*) \qquad \begin{tikzcd} & \prod_{P\in\Omega} M_P^* \arrow[dr, "\delta_P"] & \\ M \arrow[ur, "\varphi"] \arrow[rr, "\varphi_P"'] & & M_P^* \end{tikzcd}$$

$$(**)\qquad \begin{tikzcd} & \prod_{P\in\Omega}\hat{M}^*_P \arrow[dr, "\pi_P"] & \\ M \arrow[ur, "\psi"] \arrow[rr, "\psi_P"'] & & \hat{M}^*_P \end{tikzcd}$$

(where δ_P and π_P are the P-projections and $(\delta_P\circ\varphi)(x) = \varphi(x)_P = \varphi_P(x)$, $(\pi_P\circ\psi)(x) = \psi(x)_P = \psi_P(x)$ for each $x \in M$).

For the sake of convenience we put $M^* = \prod_{P\in\Omega} M^*_P$ (also called natural pre-completion of M) and $H = \prod_{P\in\Omega}\hat{M}^*_P$. It may be proved that $\hat{M} = \hat{M}^* = H$ where both natural topologies of M^* and H coincide with the product-topologies of the natural topologies of the components. H is in a natural way an $\hat{R}$-module as well as each component $\hat{M}^*_P$ is an $\hat{R}_P$-module; moreover ψ extends uniquely to a topological $\hat{R}$-isomorphism $\hat{\psi}\colon \hat{M} \to H$.

We conclude this section listing some technical results useful in the following.

LEMMA 1 (cf. [O], Lemma 2, p. 145). *Let us put* $M(P) = \operatorname{Im}\varphi_P$ *for each* $P \in \Omega$; *then the following hold*:

i) $M(P) \cong M/P^\omega M \cong \operatorname{Im}\psi_P$;

ii) *the pure submodule of* M^*_P *generated by* $M(P)$ *coincides with* M^*_P, *i.e.* $\langle M(P)\rangle_* = M^*_P$.

LEMMA 2 (originally due to Corner, cf. [C], Lemma 2.1., p. 699, and adapted to our context by R. B. Warfield Jr., cf. [W], Lemma 6, pp. 298-299). *Let R be a discrete valuation ring, $\hat{R}$ its natural completion and suppose that the transcendence degree of $\hat{R}$ over R is uncountable. Given a reduced, torsion-free R-algebra A of countable rank there exists a sub-R-algebra L of $\hat{R}$ such that L has countable rank and if*

$$\sum_{j=1}^{n} g_j a_j = 0$$

where the g_j are elements of $\hat{R}$ linearly independent over L, $a_j \in A$, $n \in \mathbb{N}$ then the a_j all vanish.

LEMMA 3 (adapted from [O], Lemma 3, p. 146). *Let $P \in \Omega$ and suppose that L is a P-pure sub-R-module of the reduced, torsion-free R_P-*

module K; then $L\otimes R_P$ (as a R-module) *is canonically isomorphic with the pure sub-R-module of K generated by L* (in symbols $\langle L\rangle_* = L\otimes R_P$).

LEMMA 4 (cf. [LD], Lemma 1, p. 218). *Let Q be the quotient field of R and suppose that Q is countably generated as an R-module; then, given a torsion-free R-module M, the following assertions are equivalent:*

i) *M is countably generated;*

ii) *M is of countable rank.*

2. The proof of Theorem C.

By convention, we put $\pi_P(a) = a_P$ for every $a\in \hat{A}$.

Now let X be a maximal linearly independent subset of A; evidently $|X| = \operatorname{rk} A$. Define $X_P = \{x_P\colon x\in X\}$; then X_P is a subset of $A(P)$ such that $A(P) = \langle X_P\rangle_*$ and since $A_P^* = \langle A(P)\rangle_*$ (via Lemma 1) we have also $A_P^* = \langle X_P\rangle_*$ ($\langle\ \rangle_*$ now considered in A_P^*).

A locally of countable rank means that $A/P^\omega A$ has countable rank as an R-module for each $P\in\Omega$. This implies that A_P^* has countable rank too (infact $A_P^* = (A/P^\omega A)\otimes_R R_P$ and R_P has rank 1). As a consequence it is possible to choose a countable subset C_P of A_P^* such that $\langle C_P\rangle_* = A_P^*$. Given $c\in C_P$ there exist $r(c)\in R-0$, $n(c)\in \mathbb{N}$ such that

$$r(c)\,c = \sum_{i=1}^{n(c)} r(c,i)\,a(c,i)$$

where $a(c,i)\in X_P$, $r(c,i)\in R$; then we define

$$B_P = \bigcup_{c\in C_P}\{a(c,1),\dots,a(c,n(c))\}.$$

B_P is a countable subset of X_P and it is easy to recognize that $\langle B_P\rangle_* = A_P^*$. Next, for each $b\in B_P$, we choose $x\in X$ such that $x_P = b$ and define A^P to be the set of all x so selected: clearly $|A^P| = |B_P|\leqslant\aleph_0$. Set $B = \bigcup_{P\in\Omega} A^P$. B is a subset of X and $|B|\leqslant|X| = \operatorname{rk} A$.

Notice that R_P fulfils all conditions of Lemma 2 so $\hat{R}_P$ contains a sub-R_P-algebra L_P with the properties there described. Since L_P has countable rank, the transcendence degree of $\hat{R}_P$ over L_P is uncountable, then it is possible to choose a subset $\{\alpha_P(a), \beta_P(a) : a \in A^P\}$ of $\hat{R}_P$ algebraically independent over L_P for each $P \in \Omega$ and construct in $\hat{R}$ a family $\{\alpha(b), \beta(b) : b \in B\}$ of elements defined component-wise as follows:

$$\alpha(b)_P = \alpha_P(b),\ \beta(b)_P = \beta_P(b) \qquad \text{if } b \in A^P;$$

$$\alpha(b)_P = \beta(b)_P = 0 \qquad \text{if } b \notin A^P.$$

In $\hat{A}$ (considered as an $\hat{R}$-module) we pick the elements $e(b) = \alpha(b)1 + \beta(b)b$, where $b \in B$; of course $e(b)_P = \alpha(b)_P 1_P + \beta(b)_P b_P$, so we have $e(b)_P = 0$ whenever $b \notin A^P$.

We are now in position to build the R-module M required in our theorem; infact in $\hat{A}$ we set

$$M = \langle A,\ Ae(b) : b \in B\rangle_* .$$

Note that $\operatorname{rk} M = \operatorname{rk} A$ because the $Ae(b)$ have all the same rank as A and $|B| \leqslant \operatorname{rk} A$. In order to show that M is locally of countable rank it is convenient to introduce the pure sub-R-module M^P of $\hat{A}^*_P$ $(P \in \Omega)$ by means of the position

$$M^P = \langle A(P),\ A(P)\, e(b)_P : b \in A^P\rangle_* .$$

We now refer to the situation displayed in diagrams (**) to note that $M^*_P \cong \pi_P(M) \otimes_R R_P$; moreover $\pi_P(M)$ is P-pure in $\hat{A}^*_P$ so we can apply Lemma 3 and get the isomorphism $M^*_P \cong \langle \pi_P(M)\rangle_*$ which will allow us to show that $M^*_P = M^P$. Infact on the one hand $M^*_P \supseteq \pi_P(M)$ and $\pi_P(M)$ contains $A(P)$, $A(P)\,e(b)_P$ where $b \in B$, hence $M^*_P \supseteq M^P$ by definition of M^P; on the other hand let $x \in M$ and $r \in R - 0$ be such that $rx = a + \sum_{i=1}^{n} a_i e(b_i)$ where $n \in \mathrm{N}$, a, $a_i \in A$, $b_i \in B$; by projection on the P-component one gets $rx_P = a_P + \sum_i a_{iP}\, e(b_i)_P$ that is $x_P \in M^P$ whence $\pi_P(M) \subseteq M^P$ and $\langle \pi_P(M)\rangle_* \subseteq M^P$ which provides the required inclusion $M^*_P \subseteq M^P$. Then $M^*_P = M^P$.

Next, as the inclusion $M^P \subseteq \langle A^*_P,\ A^*_P e(b)_P : b \in A^P\rangle_* \subseteq \hat{A}^*_P$ holds and since A^*_P and the $A^*_P e(b)_P$ are of countable rank while their «indexing-set» A^P is countable, we may infer that M^*_P is of countable rank.

We have therefore proved that M is locally of countable rank.

The proof of our theorem will be complete when we show that $A \cong \mathrm{End}_R(M)$. Since $AM = M$, the monomorphism ($1_A \in M$!) which associates to each $a \in A$ the left multiplication by a in M provides the inclusion $A \subseteq \mathrm{End}_R(M)$. In order to show the opposite inclusion we prove the following assertions:

(a) if $\eta \in \mathrm{End}_R(M)$ then η coincides with the left multiplication by $\eta(1)$;

(b) $\eta(1) \in A$.

(It is evident that (a) together with (b) imply $\mathrm{End}_R(M) \subseteq A$.)

To prove (a) it is enough to show that each $\eta \in \mathrm{End}_R(M^P)$ coincides with the left multiplication by $\eta(1_P)$ infact it can be proved that each $\eta \in \mathrm{End}_R(M)$ extends uniquely to $\eta * \in \mathrm{End}_R(M^*)$ (recall the situation displayed in diagrams $(*)$) and

$$\mathrm{End}_R(M^*) = \mathrm{End}_R \prod_{P\in\Omega} M^P = \prod_{P\in\Omega} \mathrm{End}_R(M^P)$$

because every M^P is fully invariant in M^*.

Then let $\eta \in \mathrm{End}_R(M^P)$; η extends in a unique way to $\hat{\eta} \in \mathrm{End}_{\hat{R}_P}(\hat{A}_P^*)$, so we have for each $b \in A^P$

$$\text{(i)} \quad \eta(e(b)_P) = \hat{\eta}(\alpha_P(b)1_P + \beta_P(b)\, b_P) = \alpha_P(b)\eta(1_P) + \beta_P(b)\eta(b_P)\,.$$

$\eta(1_P)$, $\eta(b_P)$, $\eta(e(b)_P) \in M^P$ so there exist $r \in R - 0$, $n \in \mathbb{N}$ such that

$$\text{(ii)} \qquad \begin{aligned} r\eta(e(b)_P) &= u + \sum_{i=1}^{n} u_i\, e(b_i)_P\,; \\ r\eta(1_P) &= v + \sum_{i=1}^{n} v_i\, e(b_i)_P\,; \\ r\eta(b_P) &= z + \sum_{i=1}^{n} z_i\, e(b_i)_P\,; \end{aligned}$$

where the b_i are pairwise distinct elements of A^P, $b_1 = b$ (for simplicity) and $u, v, z, u_i, v_i, z_i \in A(P)$.

$\hat{A}_P^*$ being torsion-free, substitution of (ii) in (i) gives

$$u + \sum_i u_i\big(\alpha_P(b_i)\,1_P + \beta_P(b_i)\,b_{iP}\big) =$$
$$= \alpha_P(b)\Big(v + \sum_i v_i\big(\alpha_P(b_i)\,1_P + \beta_P(b_i)\,b_{iP}\big)\Big) +$$
$$+ \beta_P(b)\Big(z + \sum_i z_i\big(\alpha_P(b_i)\,1_P + \beta_P(b_i)\,b_{iP}\big)\Big).$$

Since $\alpha_P(b_i)$, $\beta_P(b_i)$ are algebraically independent over L_P and u, v, z, u_i, v_i, $z_i \in A(P) \subseteq A_P^*$, Lemma 2 provides the equalities $u_1 = v$, $u_1 b_P = z$ while the remaining u, u_i, v_i, z_i all vanish.

Therefore $r\eta(1_P) = v$, $r\eta(b_P) = vb_P$ and finally

$$\eta(b_P) = \eta(1_P)\,b_P\,.$$

Moreover $\eta(1_P) \in A_P^*$ because A_P^* is pure in M^P and M^P is torsion-free.

So far we have seen that η coincides over B_P with the left multiplication by $\eta(1_P)$; but $\langle B_P\rangle_* = A_P^*$ and A_P^* is dense in M^P so we may conclude that η coincides over M^P with the left multiplication by $\eta(1_P)$.

This completes the proof of (a).

We now remark that for each $\eta \in \mathrm{End}_R(M)$ and each $P \in \Omega$ the P-component of $\eta(1)$ lies in A_P^* so $\eta(1) \in A^* = \prod_{P\in\Omega} A_P^*$. Of course $\eta(1) \in M$ too, thus in order to prove (b) it suffices to verify that $A^* \cap M = A$. Clearly $A^* \cap M \supseteq A$. Conversely let $g \in A^* \cap M$, then there exist $r \in R - 0$, $n \in \mathbf{N}$ such that $\mathrm{rg} = a + \sum_{i=1}^{n} a_i e(b_i)$ where a, $a_i \in A$, $b_i \in B$; in particular $\sum_i a_i e(b_i) = \mathrm{rg} - a = c \in A^*$. Now by projection on the P-component ($P \in \Omega$) we get $c_P = 0$: this is obvious whenever $\{b_1, \dots, b_n\} \cap A^P = \emptyset$ otherwise it follows by a further application of Lemma 2 to the equality $\sum_i a_{iP}\big(\alpha_P(b_i)1_P + \beta_P(b_i)\,bi_P\big) = c_P$ since the $\alpha_P(b_i)$, $\beta_P(b_i)$ are algebraically independent over L_P and a_{iP}, b_{iP}, $c_P \in A_P^*$. This holds for every $P \in \Omega$ therefore $c = 0$ that is $\mathrm{rg} = a \in A$ and finally $g \in A$ because A is pure in M. Hence $A^* \cap M = A$ and the theorem is proved.

3. Applications.

In this section we show that some previous results of Orsatti, A. Le Donne and Warfield are an easy consequence of Theorem C.

First of all we prove Orsatti's Theorem B (see Introduction).

Proof of Theorem B.

Put $R = \mathbb{Z}$; then Theorem C provides a group ($= \mathbb{Z}$-module) M satisfying all requirements of the assertion. Note infact that A locally countable means $|A/P^\omega A| \leqslant \aleph_0$ that is $|A(P)| \leqslant \aleph_0$, for every $P \in \Omega$; this implies that M^P is countable because $|A^P| \leqslant \aleph_0$ (see definitions of M^P and A^P in § 2).

Being $M^P = M_P^*$, this is equivalent to say that M is locally countable and the conclusion is reached.

From Theorem C can also be derived the following

Corollary 1 (improved form of [LD], Corollario, p. 224). *Let R be an admissible Dedekind domain such that Ω is countable; if A is a reduced, torsion-free R-algebra locally of countable rank then there exists a locally countably generated, reduced, torsion-free R-module M such that $A \cong \mathrm{End}_R(M)$. Moreover A and M have the same* (global) *rank.*

Proof. Ω being countable, the quotient field Q of R is countably generated as an R-module (cf. [S], Proposizione 4, p. 60). This enables us to apply Lemma 4 and obtain that each torsion-free R-module is locally of countable rank if and only if it is locally countably generated. Then the conclusion follows easily by Theorem C.

A slightly modified version of Corollary 1 is the

Corollary 2 (cf. [W], Theorem, p. 296). *Let R be a discrete valuation ring such that $\hat{R}$ has uncountable transcendence degree over R; then if A is any countably generated, reduced and torsion-free R-algebra there exists a countably generated, reduced and torsion-free R-module M such that $A \cong \mathrm{End}_R(M)$.*

Proof. R is clearly an admissible Dedekind domain and Ω is countable (infact $|\Omega| = 1$). Now A is of countable rank and « a fortiori » locally of countable rank so by Corollary 1 there exists a reduced, torsion-free R-module M of countable rank such that $A \cong \mathrm{End}_R(M)$; but by Lemma 4 M is also countably generated and the conclusion is reached.

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