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## On Some Subgroups of Infinite Rank Butler Groups.

MANFRED DUGAS (\*)

### 1. Introduction.

All groups in this note are abelian and torsion-free unless stated otherwise. Undefined notations are standard as in [F]. Some twenty years ago M. C. R. Butler [Bu] studied pure subgroups of finite rank completely decomposable groups. These groups were later called Butler groups by L. Lady. Following the lead of Bican and Salce [B-S] we call a group  $B$  a Butler group if  $B\text{ext}(B, T) = 0$  for all torsion groups  $T$ . The functor  $B\text{ext}(\cdot, \cdot)$  is the subfunctor of  $\text{Ext}(\cdot, \cdot)$  of all balanced extensions as introduced by R. Hunter [Hu]. Butler groups of finite rank have been studied to some extent, e.g. [A1], [B]. Recently, Butler groups of infinite rank attracted some attention, see [A2], [B-S-S], [A-H] and [D-R]. D. Arnold raises in [A2] the question about Butler groups being closed with respect to pure countable subgroups. This question is of particular interest in the light of the results in [D-R]. All main results in [D-R] require the hypothesis « All pure finite rank subgroups of the Butler group  $B$  ( $|B| = \aleph_1$ ) are again Butler groups ». The purpose of this paper is to show that for Butler groups of rank  $\aleph_1$  this hypothesis is redundant:

**THEOREM 1.** Each pure, countable subgroup of a Butler group of rank  $\aleph_1$  is again a Butler group.

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Observe that a countable torsion-free group is a Butler group iff each pure finite rank subgroup is a Butler group, c.f. [B-S]. The main theorem in [D-R] now reads as follows:

**THEOREM.** Assume  $V = L$  holds. The following are equivalent for a torsion-free group  $B$  of cardinality  $\aleph_1$ :

- (a)  $\text{Bext}(B, S) = 0$  for all torsion groups  $S$ .
- (b)  $\text{Bext}(B, T) = 0$  for any countable,  $\Sigma$ -cyclic torsion group  $T$ .
- (c)  $B$  is a  $B_2$ -group, i.e.  $B$  has an  $\omega_1$ -filtration  $B = \bigcup_{\alpha < \omega_1} B_\alpha$  into pure, countable subgroups  $B_\alpha$  with  $B_0 = 0$  and  $B_{\alpha+1} = B_\alpha + C_\alpha$ ,  $C_\alpha$  a finite rank Butler group.
- (d)  $B$  has an  $\omega_1$ -filtration where each  $B_\alpha$  is a (pure and countable) decent subgroup of  $B$ , c.f. [A-H] or [D-R].
- (e)  $B$  has an  $\omega_1$ -filtration such that each  $B_\alpha$  has the T.E.P. in  $B$ . (We refer to [D-R] for the definition of the torsion extension property T.E.P.)

This theorem, as stated above, is undecidable in  $ZFC$ . But we would like to conjecture, that without (b) it's valid in  $ZFC$ .

A subgroup  $A$  of a group  $B$  is called separable in  $B$ , c.f. [H1], if for any  $b \in B - A$  there is a countable sequence  $\{a_n | n < \omega\} \subset A$ , such that for any  $a \in A$ , the height sequences satisfy the inequality  $|b + a| \leq |b + a_n|$  for some  $n < \omega$ . Our Theorem 1 is an immediate consequence of the more general

**THEOREM 2.** Let  $A$  be a pure and separable subgroup of the torsion-free group  $B$  with  $B/A$  countable. If  $K$  is a generalized regular subgroup of  $A$ , then there exists a generalized regular subgroup  $L$  of  $B$  with  $L \cap A = K$ .

The notion of a generalized regular subgroup was defined in [B] and used to characterize countable Butler groups: The countable group  $B$  is a Butler group iff all localizations  $B_p$ ,  $p$  a prime, are completely decomposable and for each generalized regular subgroup  $K$  of  $B$  and each (pure) finite rank subgroup  $H$  of  $B$ ,  $(H/(H \cap K))_p = 0$  for almost all  $p$ , c.f. [A1], [B-S]. In order to show that Theorem 1 follows from Theorem 2, let  $B$  be a Butler group of rank  $\aleph_1$  and  $H$  a pure finite rank subgroup of  $B$ . Consider an  $\aleph_1$ -filtration  $B = \bigcup_{\alpha < \omega_1} B_\alpha$  into pure countable subgroups  $B_\alpha$  with  $B_0 = H$ . Let  $K_0$  be a generalized

regular subgroup of  $H = B_0$ . Since  $B_1$  is countable,  $B_1/B_0$  is countable and Theorem 2 shows the existence of a generalized regular subgroup  $K_1$  of  $B_1$  with  $K_1 \cap B_0 = K_0$ . By transfinite induction there are generalized regular subgroups  $K_\alpha$  of  $B_\alpha$  for all  $\alpha < \omega_1$  with  $K_{\alpha+1} \cap B_\alpha = K_\alpha$ . The subgroup  $K = \bigcup_{\alpha < \omega_1} K_\alpha$  is a generalized regular subgroup of  $B$ . Since  $B$  is a Butler group,  $H/K_0 \cong (H + K)/K \subset B/K$  and  $(B/K)_p = 0$  for almost all primes  $p$ . Hence  $(H/K_0)_p = 0$  for almost all  $p$  and the characterization of countable Butler groups mentioned above shows that  $H$  is a Butler group.

We give an example showing that the separability condition in Theorem 2 is indispensable. Therefore, our approach doesn't yield a complete answer to D. Arnold's question. We only get the following:

**COROLLARY.** A pure and countable subgroup of a Butler group  $B$  is Butler provided  $B$  satisfies the third axiom of countability [H2] with respect to separable subgroups.

**2. Notations.**

For an element  $x$  of a group  $G$ ,  $|x| = (|x|_p)_{p \text{ prime}}$  is the height sequence of  $x$  and we add a superscript  $|x|^G$  to indicate in which group the height is computed.  $|x|_p$  denotes the  $p$ -height. We say  $|x| \leq |y|$  if  $|x|_p \leq |y|_p$  for all primes  $p$ . Recall that a subgroup  $K$  of a torsion-free group  $A$  is a (full) generalized regular subgroup if  $A/K$  is torsion and for each pure rank one subgroup  $R$  of  $A$  the  $p$ -primary part  $(R/(R \cap K))_p = 0$  for almost all primes  $p$ . We call the generalized regular subgroup  $K$  of  $A$   $\omega$ -regular, if for any pure finite rank subgroup  $H$  of  $A$ ,  $(H/(H \cap K))_p = 0$  for almost all primes  $p$ . In this paper we find it more useful to deal with an equivalent concept:

Let  $A$  be a torsion-free group and  $T$  torsion. We call a map  $\varphi \in \text{Hom}(A, T)$  *regular* if for any pure rank 1 subgroup  $R$  of  $A$ ,  $(\varphi(R))_p = 0$  for almost all primes  $p$ . Observe that  $\varphi$  is regular iff the kernel of  $\varphi$  is a generalized regular subgroup. We call  $\varphi$   $\omega$ -regular, if  $(\varphi(H))_p = 0$  for almost all primes  $p$  whenever  $H$  is a pure finite rank subgroup of  $A$ . Observe that if  $A$  is completely decomposable or if  $A$  is the sum of finitely many pure rank 1 subgroups then each regular map is obviously  $\omega$ -regular. If  $A$  is any group,  $\bar{A}$  denotes the divisible hull of  $A$ . Let  $\omega$  denote the first infinite ordinal, i.e.  $\omega = \{0, 1, 2, \dots\}$ .

**3. Lifting regular maps.**

If  $\varphi \in \text{Hom}(A, T)$  is regular, we say  $\varphi$  lifts to a regular  $\bar{\varphi} \in \text{Hom}(\bar{A}, \bar{T})$  if  $\bar{\varphi}$  is regular and  $\bar{\varphi}|_A = \varphi$ .

**PROPOSITION 1.** Let  $\varphi: A \rightarrow T$  be a regular map from the torsion-free group  $A$  into the torsion group  $T$ . If  $\varphi$  lifts to a regular  $\bar{\varphi}: \bar{A} \rightarrow \bar{T}$ , then  $\varphi$  is  $\omega$ -regular. The converse holds provided that  $A$  is countable.

**PROOF.** If  $\varphi$  lifts to a regular  $\bar{\varphi}$ , then  $\bar{\varphi}$  is  $\omega$ -regular since  $\bar{A}$  is divisible and so is  $\varphi$ , being the restriction of  $\bar{\varphi}$  to  $A$ . Now assume  $\varphi$  is  $\omega$ -regular. If  $A$  has finite rank, there is a finite set  $P$  of primes and  $(\varphi(A))_p = 0$  for  $p \notin P$ . Then  $\varphi$  extends to a map  $\bar{\varphi} \in \text{Hom}(\bar{A}, \bigoplus_{p \in P} \bar{T}_p)$ . Since  $P$  is finite,  $\bar{\varphi}$  is regular. Hence we may assume that  $A$  has infinite rank, i.e.  $A = \bigcup_{i=1}^{\infty} A_i$ ,  $A_i$  pure of finite rank,  $A_i \subset A_{i+1}$  for all  $i$  and  $\text{rk}(A_{i+1}/A_i) = 1$ . Then  $\bar{A} = \bigcup_{i=1}^{\infty} \bar{A}_i$  and  $\bar{A}_{i+1} = \bar{A}_i \oplus e_i Q$ ,  $e_i \in A_{i+1}$ . Let  $\varphi_n = \varphi|_{A_n}$  and assume we constructed an  $\omega$ -regular  $\bar{\varphi}_n: \bar{A}_n \rightarrow \bar{T}$  already with  $\bar{\varphi}_n|_{A_n} = \varphi_n$ . Let  $\pi_0: \bar{A}_{n+1} \rightarrow \bar{A}_n$  and  $\pi_1: \bar{A}_{n+1} \rightarrow e_n Q$  be the natural projections induced by the decomposition  $\bar{A}_{n+1} = \bar{A}_n \oplus e_n Q$ . We define a map  $\varrho: \pi_1(A_{n+1}) \rightarrow \bar{T}$  by setting  $\varrho(\pi_1(x)) = \bar{\varphi}_n(\pi_0(x)) - \varphi_{n+1}(x)$  for all  $x \in A_{n+1}$ . We have to show that  $\varrho$  is well defined. Let  $x \in A_{n+1}$  such that  $\pi_1(x) = 0$ , i.e.  $x \in A_{n+1} \cap \bar{A}_n = A_n$  since  $A_n$  is pure in  $A_{n+1}$ . This implies  $\bar{\varphi}_n(\pi_0(x)) = \bar{\varphi}_n(x) = \varphi_n(x) = \varphi_{n+1}(x)$ . This shows  $\varrho \in \text{Hom}(\pi_1(A_{n+1}), \bar{T})$ . Let  $R$  be the subgroup of  $Q$  with  $e_n R = \pi_1(A_{n+1})$ . Then  $\varrho(e_n R) = \varrho(\pi_1(A_{n+1})) \subset \bar{\varphi}_n(\bar{A}_n) + \varphi_{n+1}(A_{n+1})$  and since  $\bar{\varphi}_n$  and  $\varphi_{n+1}$  are both  $\omega$ -regular,  $(\varrho(e_n R))_p = 0$  for almost all  $p$ . Hence  $\varrho$  lifts to a regular  $\bar{\varrho}: e_n Q \rightarrow \bar{T}$ . Now consider the map  $\bar{\varphi}_n - \bar{\varrho}: \bar{A}_n \rightarrow \bar{T}$  and let  $x \in A_{n+1}$ . Then

$$\begin{aligned} (\bar{\varphi}_n - \bar{\varrho})(x) &= (\bar{\varphi}_n - \bar{\varrho})(\pi_0(x) + \pi_1(x)) = \bar{\varphi}_n(\pi_0(x)) - \bar{\varrho}(\pi_1(x)) = \\ &= \bar{\varphi}_n(\pi_1(x)) - \varrho(\pi_1(x)) = \varphi_{n+1}(x). \end{aligned}$$

Thus  $\bar{\varphi}_n - \bar{\varrho}$  is the desired  $(\omega)$ -regular map extending  $\varphi_{n+1}$ : Induction completes the proof.  $\square$

**PROPOSITION 2.** Let  $A$  be a pure subgroup of  $B$ ,  $C$  a finite rank Butler subgroup such that  $B = A + C$ . Let  $\varphi: B \rightarrow T$  be regular

and assume that  $\varphi|A$  lifts to a regular  $\psi: \bar{A} \rightarrow \bar{T}$ . Then there exists  $\Psi: \bar{B} \rightarrow \bar{T}$  regular with  $\Psi|A = \psi$  and  $\Psi|B = \varphi$  if only  $\varphi|C$  is  $\omega$ -regular.

PROOF. We will use a similar argument as in the proof of the previous proposition. Let  $\bar{B} = \bar{A} \oplus X$  and  $\pi_0: \bar{B} \rightarrow \bar{A}$  and  $\pi_1: \bar{B} \rightarrow X$  be the natural projections. Then  $\pi_1(B) = \pi_1(C)$  is a Butler group of finite rank. Again, define  $\varrho: \pi_1(B) \rightarrow \bar{T}$  by  $\varrho(\pi_1(x)) = \psi(\pi_0(x)) - \varphi(x)$  for all  $x \in B$ . Then  $\varrho(\pi_1(B)) = \varrho(\pi_1(C)) \subset \psi(\pi_0(C)) + \varphi(C)$ . Since  $C$  has finite rank and  $\psi$  is regular,  $\psi(\pi_0(C))_p = 0$  for almost all primes  $p$  and since  $\varphi|C$  is a regular map from the finite rank Butler group  $C$ ,  $(\varphi(C))_p = 0$  for almost all primes  $p$  as well. Hence  $\varrho(\pi_1(B))_p = 0$  for almost all  $p$  and since  $\pi_1(B)$  is countable, we may use Proposition 1 to conclude that  $\varrho$  lifts to a regular  $\bar{\varrho}: \pi_1(\bar{B}) = X \rightarrow \bar{T}$ . Again  $\Psi = \psi - \bar{\varrho}: \bar{B} \rightarrow \bar{T}$  is a regular map extending  $\varphi$ .  $\square$

COROLLARY. (a) A countable torsionfree group  $B$  is a Butler group iff  $B$  is locally completely decomposable and each regular map from  $B$  into a torsion group lifts to a regular map from the divisible hull of  $B$  into the divisible hull of the torsion group. (b) If  $B$  is a  $B_2$ -group then each regular map  $\varphi: B \rightarrow T$  lifts to a regular  $\bar{\varphi}: \bar{B} \rightarrow \bar{T}$ .

PROOF. Part (a) is just a reformulation of a result in [B-S]. In order to show (b), let  $B$  be a  $B_2$ -group, i.e.  $B = \bigcup_{\alpha < \lambda} B_\alpha$  is a union of a smooth chain of pure subgroups  $B_\alpha$  with  $B_0 = 0$  and  $B_{\alpha+1} = B_\alpha + H_\alpha$  for a finite rank Butler subgroup of  $B$ . Let  $\varphi: B \rightarrow T$  be a regular map. By transfinite induction we find regular maps  $\psi_\alpha: \bar{B}_\alpha \rightarrow \bar{T}$  such that  $\psi_{\alpha+1}|B_\alpha = \psi_\alpha$  and  $\psi_\alpha|B_\alpha = \varphi|B_\alpha$ . Set  $\psi_0 = 0$  and suppose  $\psi_\alpha$  is defined already. The map  $\varphi|H_\alpha$  is regular and  $H_\alpha$  is a Butler group of finite rank. Thus  $\varphi|H_\alpha$  is  $\omega$ -regular and Proposition 2 provides a map  $\psi_{\alpha+1}$  with the desired properties.

We would like to raise the following

QUESTION. If  $B$  is any Butler group, is (b) true? (This would answer D. Arnold's question mentioned in the introduction.)

We are now going to prove our main

THEOREM 2'. Let  $A$  be a pure separable subgroup of  $B$  with  $B/A$  countable. Then each regular  $\varphi: A \rightarrow T$  lifts to a regular  $f: B \rightarrow \bar{T}$ .

PROOF. Since  $A$  is separable in  $B$  and a countable extension of a separable subgroup is separable, there is a chain  $A = A_0 \subset A_1 \subset \dots \subset A_i \subset A_{i+1} \subset \dots \subset B$  with  $\text{rk}(A_{i+1}/A_i) = 1$ ,  $B = \cup A_i$  and each  $A_i$  is separable in  $A_{i+1}$ . Therefore we may assume that  $B/A$  has rank 1. Then  $\bar{B} = \bar{A} \oplus xQ$  with  $x \in B - A$ . If  $S$  is a subset of  $B$ , let  $\langle S \rangle^*$  be the pure subgroup of  $B$  generated by  $S$  and let  $R$  be the subgroup of  $Q$  with  $xR = B \cap xQ$ . For any  $q \in R$ , we have a sequence  $\{a_{q,n} : n < \omega\}$  such that for any  $a \in A$  there is some  $n$  such that  $|a + qx|^B \leq |a_{q,n} + qx|^B$ . Choose an enumeration of  $R$  and  $R \times \omega$  such that

$$\{a_{q,n} + qx : q \in R, n < \omega\} = \{a_i + q_i x : i < \omega\}.$$

Observe that the map ( $i \rightarrow q_i$ ) is not one to one. Let  $q_i = r_i/s_i$  with  $r_i, s_i$  relatively prime. We may assume  $r_i \neq 0$  and set  $y_i = a_i + q_i x$ . Define  $P_i = \{p \text{ prime} : p \text{ doesn't divide neither } r_i \text{ nor } s_i, |y_i|_p^B > \min\{|a_i|_p^A, |x|_p^B\}\}$ . Note that for  $p \in P_i$ ,  $|x|_p^B = |q_i x|_p^B$ . For  $xS$  the image of the projection of  $B$  into  $xQ$ , we define a map  $g : xS \rightarrow \bar{T}$ . First we define  $g(xR) = 0$ . In order to continue, we'll define  $g(p^{-j}x)$  for  $j < \omega$  with  $p^{-j} \notin R$ . In order to define a map  $g : xS \rightarrow \bar{T}$ , we only have to make sure that  $p(g(p^{-(j+1)}x)) = g(p^{-j}x)$  and avoid contradictory assignments. We'll define  $g$  in such a way that for all  $i < \omega$ ,  $g(p^{-n}x) = s_i r_i^{-1}(\bar{\varphi}(p^{-n}a_i))_p$  for almost all primes  $p$ . Here  $\bar{\varphi} : \bar{A} \rightarrow \bar{T}$  is a fixed map extending  $\varphi$ . For  $p \in P_1$ , set  $l_{1,p} = |y_1|_p^B$  and observe  $l_{1,p} > |x|_p^B = |a_1|_p^B = |a_1|_p^A$ . Now define  $g(p^{-1,1}x) = -(s_1 r_1^{-1})\bar{\varphi}(a_1)_p$ . Assume we defined  $g(p^{-i,1}x)$  for all  $i < n$ ,  $l_{i,p} = |y_i|_p^B$  and all  $p \in P_i$  such that  $g(p^{-i,1}x) = g(p^{-i,1}x)_p = -(s_i r_i^{-1})\bar{\varphi}(a_i)_p$  for almost all  $p \in \bigcup_{i=1}^{n-1} P_i$ . Let  $E = \{i | 1 \leq i < n, P_n \cap P_i \text{ is infinite}\}$ ,  $i \in E$  and  $p \in P_n \cap P_i$ . This implies  $l_{i,p} > \min\{|a_i|_p, |x|_p\}$  and  $l_{n,p} > \min\{|a_n|_p, |x|_p\}$  and therefore  $l_{n,p} > |a_n|_p = |x|_p = |a_i|_p$ . Let  $l_p = \min\{l_{i,p}, l_{n,p}\}$ . Then  $y_n, y_i, r_i s_n y_n$  and  $r_n s_i y_i$  are all in  $p^{l_p} B$ . Hence

$$r_i s_n y_n - r_n s_i y_i = r_i s_n a_n - r_n s_i a_i \in p^{l_p} B \cap A = p^{l_p} A.$$

Since  $\varphi$  is regular, there is a subset  $W_{i,n}$  of  $P_i \cap P_n$  with  $(P_i \cap P_n) - W_{i,n}$  finite and  $\varphi(p^{-l_p}(r_i s_n a_n - r_n s_i a_i))_p = 0$  for all  $p \in W_{i,n}$ . Thus  $(s_n r_n^{-1})\bar{\varphi}(p^{-l_p} a_n) = (s_i r_i^{-1})\bar{\varphi}(p^{-l_p} a_i)$  for all  $p \in W_{i,n}$ . We now consider two cases for a prime  $p \in W_n$ , where  $W_n = \bigcup_{i \in E} W_{i,n}$ .

*Case 1.* For all  $i \in E$  with  $p \in W_{i,n}$  we have  $l_{i,p} \leq l_{n,p}$ . Here we define  $g(p^{-l_{n,p}}x)$  to be an element in  $\bar{T}_p$  such that  $p^{l_{n,p}-l_{i,p}}g(p^{-l_{n,p}}x) = -(s_j r_j^{-1})\bar{\varphi}(p^{-l_{i,p}}a_j)$  where  $l_{j,p} = \max \{l_{i,p} \mid i \in E\}$ .

*Case 2.* There is  $i \in E$  such that  $p \in W_{i,n}$  and  $l_{n,p} < l_{i,p}$ . Again, let  $l_{j,p} = \max \{l_{i,p} \mid i \in E\}$  and observe that

$$g(p^{-l_{n,p}}x) = -(s_j r_j^{-1})\bar{\varphi}(p^{-l_{i,p}}a_j)p^{l_{i,p}-l_{n,p}}$$

was already defined. If  $p \in P_n - \bigcup_{i=1}^{n-1} P_i$ , set

$$g(p^{-l_{n,p}}x) = -q_n^{-1}\bar{\varphi}(p^{-l_{n,p}}a_n).$$

Our observations above show that these assignments define a map  $g'$  on a subgroup  $xS'$  of  $xS$ . Let  $g$  be any lifting of  $g'$  to  $xS$ . Our construction shows that our map  $g$  has the property that for any  $i < \omega$ ,  $(\bar{\varphi} \oplus g)(\langle y_i \rangle^*)_p = 0$  for almost all primes  $p$ . Let  $f = (\bar{\varphi} \oplus g)|_B$  and  $y \in B - A$ . Then there is a natural number  $k \neq 0$  such that  $ky = a + qx$  with  $a \in A$ ,  $q \in R$ . For some  $i < \omega$ , we have  $q = q_i$  and  $|a + qx|^p \leq |a_i + qx|^p$ . This implies  $|a - a_i|^p \geq |a + qx|^p$  and for  $e \leq |a + qx|_p$  we get

$$\begin{aligned} f(p^{-e}(a + qx))_p &= \bar{\varphi}(p^{-e}a)_p + g(p^{-e}qx)_p = \bar{\varphi}(p^{-e}a_i)_p + \\ &+ g(p^{-e}qx)_p = f(p^{-e}(a_i + qx))_p = 0 \end{aligned}$$

for almost all  $p$ , since  $p^{-e}(a - a_i) \in A$  and  $\varphi$  regular. This shows that  $f(\langle y \rangle^*)_p = 0$  for almost all primes  $p$ , i.e.  $f$  is the desired map.  $\square$

We conclude this paper with the promised

**EXAMPLE.** There is a torsion-free group  $B$  and a pure subgroup  $A$  of  $B$  with  $rk(B/A) = 1$  and a regular map  $\varphi: A \rightarrow T = \bigoplus_{p \text{ prime}} Z(p^\infty)$  such that  $\varphi$  can not be lifted to a regular  $\psi: B \rightarrow T$ .

**PROOF.** Let  $\{p_{n,i} \mid n, i < \omega\}$  be a list of all primes and  $F = \bigoplus_{i < \omega} e_i Z$  a free group of countable rank. Call two maps  $f, g: \omega \rightarrow \omega$  almost disjoint, if  $\{i < \omega \mid f(i) = g(i)\}$  is finite. Zorn's Lemma implies that there is a maximal family  $\Sigma$  of almost disjoint maps from  $\omega$  into  $\omega$ .



We introduce independent elements  $a_f$ ,  $f \in \Sigma$  and define  $A = \langle F, a_f, p_{n,f(n)}^{-1}(e_n - a_f) : f \in \Sigma, n < \omega \rangle$ . It's not hard to see that  $A$  is homogeneous of type 0 and  $p^{-1}(a_f - a_g) \in A$  iff  $p_{n,\sigma(n)} = p = p_{n,f(n)}$  for some  $n < \omega$ , i.e.  $f(n) = g(n)$ . Define  $\varphi: A \rightarrow T$  by setting  $\varphi(e_n) = \varphi(a_f) = 0$  for all  $n < \omega$ ,  $f \in \Sigma$  and  $\varphi(p_{n,f(n)}^{-1}(e_n - a_f)) = t_{n,f(n)}$ , an element of exponent 1 in  $Z(p_{n,f(n)}^\infty)$ . Since  $A$  is homogeneous of type 0,  $\varphi$  is regular.

We now define a rank 1 extension  $B$  of  $A$  by introducing a new independent element  $x$  and setting

$$B = \langle A, x, p_{n,f(n)}^{-1}(x + a_f), p_{n,i}^{-1}(x + e_n) \rangle : f \in \Sigma, n, i < \omega.$$

Routine verification shows that  $A$  is pure in  $B$  and  $B/A$  is rank 1 of type  $(1, 1, 1, \dots)$ .

By way of contradiction assume that  $\varphi: A \rightarrow T$  lifts to a regular map  $\psi: B \rightarrow T$  and let  $\Psi: \bar{B} \rightarrow T$  be any map with  $\Psi|_B = \psi$ . If  $n < \omega$ , there is  $i_0 = i_0(n)$  such that  $\psi(p_{n,i}^{-1}(x + e_n)) = 0$  for all  $i \geq i_0(n)$ , i.e.  $-\Psi(p_{n,i}^{-1}x) = \Psi(p_{n,i}^{-1}e_n)$ . On the other hand, if  $f \in \Sigma$ , there is  $n_1(f) < \omega$  such that  $\psi(p_{n,f(n)}^{-1}(x + a_f)) = 0$  for all  $n \geq n_1(f)$ , now let  $\sigma: \omega \rightarrow \omega$  be the map defined by  $\sigma(n) = i_0(n)$ . Then  $-\Psi(p_{n,\sigma(n)}^{-1}x) = \Psi(p_{n,\sigma(n)}^{-1}e_n)$  for all  $n$ . Since  $\Sigma$  is a maximal almost disjoint family of maps, there is some  $f \in \Sigma$  such that  $W = \{n < \omega : n_1(f) < n, f(n) = \sigma(n)\}$  is not empty. Let  $n \in W$ . Then

$$\Psi(p_{n,\sigma(n)}^{-1}e_n) = -\Psi(p_{n,\sigma(n)}^{-1}x) = -\Psi(p_{n,f(n)}^{-1}x) = \Psi(p_{n,f(n)}^{-1}a_f)$$

and  $\sigma(n) = f(n)$ . This implies

$$0 = \Psi(p_{n,f(n)}^{-1}(e_n - a_f)) = \psi(p_{n,f(n)}^{-1}(e_n - a_f)) = t_{n,f(n)},$$

a contradiction.  $\square$

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*Added in proof.* In a forthcoming paper (with P. Hill and K. M. Rangaswamy) we will show in *ZFC* that conditions (a) and (c) in our theorem are equivalent for any torsion-free group  $B$  of cardinality  $< \aleph_1$ . The same holds if  $|B| < \aleph_\omega$  and  $2^{\aleph_0} = \aleph_1$ .

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