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Oka-Analyticity of the Essential Spectrum.

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SUMMARY - Let X be a complex Banach space, G an open set in $\mathbb{C}$, and $\lambda \mapsto T_\lambda$ a holomorphic family of closed operators on X. We show here that $\lambda \mapsto \sigma_e(T_\lambda)$ is an analytic multifunction, where σ_e denotes the essential spectrum in *any* one of its several definitions.

1. Introduction.

Let X, Y be (nonzero) complex Banach spaces, and $\mathcal{B}(X, Y)$ the Banach space of bounded linear operators from X to Y. It is known (see [9, Corollary 3.3 p. 371]) that, if G is an open set of $\mathbb{C}$, and $\lambda \mapsto T_\lambda: G \to \mathcal{B}(X) = \mathcal{B}(X, X)$ is a holomorphic map, then the multifunction spectrum $\lambda \mapsto \sigma(T_\lambda): G \to \mathrm{cl}\,(\mathbb{C}) = \{\text{closed subsets of } \mathbb{C}\}$ is *Oka-analytic*, i.e. it is upper semicontinuous (u.s.c.; viz. $\{\lambda \in G: \sigma(T_\lambda) \subset A\}$ is open in G if A is open in $\mathbb{C}$ and $\mathbb{C} \setminus A$ is compact) and the open set $\Omega = \{(\lambda, z) \in G \times \mathbb{C}: z \notin \sigma(T_\lambda)\}$ is pseudoconvex: by abuse of terminology we say that σ is Oka-analytic on $\mathcal{B}(X)$. Denoting by $\mathcal{C}(X, Y)$ the set of closed linear operators from X to Y, the same result holds, more generally, (see [10, Theorem 1 p. 121]) if $G \ni \lambda \mapsto T_\lambda$ is a Kato-holomorphic family with values in $\mathcal{C}(X) = \mathcal{C}(X, X)$: that is (cf. [4, in Section VII.1.2 p. 366]) if, for every $\lambda_0 \in G$ there exist G_0 open neighborhood of λ_0 in G, a Banach space Y, and holomorphic families $\lambda \mapsto U_\lambda$, $V_\lambda: G_0 \to \mathcal{B}(Y, X)$ such that U_λ is one-to-one and $T_\lambda = V_\lambda U_\lambda^{-1}$ for every $\lambda \in G_0$: we say that σ is Oka-analytic on $\mathcal{C}(X)$.

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We shall prove the same results for other multifunctions of spectral type, each one usually referred to as essential spectrum. The standard consequences of Oka-analyticity (we refer to [1], [11], [12] for precise statements and proofs) thus extend to them: for instance, several functions of each essential spectrum (such as the radius, the inverse of the distance from a fixed point, the k-th diameter, for any $k \in \mathbb{N}$, the capacity, etc.) are plurisubharmonic on $\mathcal{B}(X)$ (or $\mathcal{C}(X)$). Also, we have the analyticity of spectral sets, the finite scarcity and countable scarcity theorems; and many others.

2. Essential spectrum.

A linear operator $T \in \mathcal{C}(X, Y)$ will be said to be *Fredholm* if its range $R(T)$ is closed, and if the dimensions of its kernel $N(T)$ and of its co-kernel $Y/R(T)$ are finite: such dimensions will be called *nullity* and *deficiency* of T, resp., and denoted by nul (T) and def (T); while the *index* of T will be ind $(T) =$ nul $(T) -$ def (T). If $\mathcal{C}(X, Y)$ is endowed with the «gap» (metrizable) topology (see [4, in Section IV.2.4 p. 201-202]), which induces the norm topology on the open set $\mathcal{B}(X, Y)$, then $\mathcal{F}(X, Y) = \{T \in \mathcal{C}(X, Y)$ that are Fredholm$\}$ is open in $\mathcal{C}(X, Y)$, and on each of its connected components the function ind is constant, while nul and def are, in general, just u.s.c. (cf. [4, Theorem IV.5.17 p. 235]). By $\mathcal{F}_0(X, Y)$ we shall denote the union of the components of $\mathcal{F}(X, Y)$ where ind $\equiv 0$; and by $\mathcal{K}(X, Y)$ the set of compact operators from X to Y.

Each of the following sets is customarily referred to as essential spectrum of $T \in \mathcal{C}(X)$ (cf. [7, p. 365; 13, § 1 p. 142; 2, Definition 11 p. 107]):

a) the *Wolf spectrum*

$$\sigma_{ew}(T) = \{z \in \mathbb{C} \colon T - zI \notin \mathcal{F}(X) = \mathcal{F}(X, X)\};$$

b) the *Weyl spectrum* $\sigma_{em}(T) = \bigcap_{K \in \mathcal{K}(X) = \mathcal{K}(X, X)} \sigma(T + K)$;

c) the *Browder spectrum* $\sigma_{eb}(T) = \Big\{z \in \mathbb{C} \colon z$ is an accumulation point of $\sigma(T)$, or $R(T - zI)$ is not closed, or z is an eigenvalue of T of infinite algebraic multiplicity $\Big($i.e. $\dim \Big(\bigcup_{k=1,\dots,\infty} N(T - zI)^k\Big) = \infty\Big)\Big\}$.

Thus the Weyl spectrum is the largest subset of the spectrum which is invariant under compact perturbations: furthermore (see [8, Theorem VII.5.4. p. 180])

$$\sigma_{em}(T) = \{z \in \mathbf{C}\colon T - zI \notin \mathcal{F}_0(X) = \mathcal{F}_0(X, X)\}\,.$$

If Cal $(X) = \mathcal{B}(X)/\mathcal{K}(X)$ is the Calkin algebra of X, then (see [8]) the Wolf spectrum of $T \in \mathcal{B}(X)$ coincides with the (Banach algebra) spectrum $\sigma_{\mathrm{Cal}(X)}$ in Cal (X) of the coset $[T]_{\mathrm{Cal}(X)}$ of T.

Fixed $T \in \mathcal{C}(X)$, due to forementioned properties the function $z \mapsto \mathrm{ind}\,(T - zI)$ is constant on each component W of the complement of $\sigma_{ew}(T)$: but we also have that $z \mapsto \mathrm{nul}\,(T - zI)$, $\mathrm{def}\,(T - zI)$ are constant on W, except for a discrete subset of W (see [4, Theorem IV.5.31 p. 41]). This implies that $\sigma_{eb}(T)$ is the complement of the union of those W's which are not contained in $\sigma(T)$, viz. such that $W \ni z \mapsto \mathrm{nul}\,(T - zI) \equiv \mathrm{def}\,(T - zI) \equiv 0$ « a.e. » (see [2]).

Thanks to the various observations made so far, we have $\sigma_{ew}(T) \subset \subset \sigma_{em}(T) \subset \sigma_{eb}(T) \subset \sigma(T)$ (all closed subsets), and each inclusion may be strict, even if $T \in \mathcal{B}(X)$. (If X is finite-dimensional, all the essential spectra are obviously empty; while if it is Hilbert and T is self-adjoint, then they coincide.)

THEOREM 1. *The Wolf spectrum is Oka-analytic on* $\mathcal{C}(X)$.

PROOF. The upper semicontinuity of σ_{ew} easily follows from the openness of $\mathcal{F}(X)$ in $\mathcal{C}(X)$.

Let Cal $(X, Y) = \mathcal{B}(X, Y)/\mathcal{K}(X, Y)$ as a Banach space: the product of composition $\mathcal{B}(X, Y) \times \mathcal{B}(Y, X) \to \mathcal{B}(Y)$ induces a continuous bilinear product $\mathrm{Cal}\,(X, Y) \times \mathrm{Cal}\,(Y, X) \to \mathrm{Cal}\,(Y)$. As in the case $X = Y$, one shows that $T \in \mathcal{B}(X, Y)$ is Fredholm if and only if its coset $[T]_{\mathrm{Cal}(X, Y)}$ has a two-sided inverse in $\mathrm{Cal}\,(Y, X)$: if so, such inverse is unique by the associativity of the product, and (just like when $X = Y$) it depends continuously and holomorphically on $[T]_{\mathrm{Cal}(X, Y)}$, so on T.

The notion of analytic multifunction being obviously local, we can assume the holomorphic families $\lambda \mapsto U_\lambda$, V_λ of operators in $\mathbf{B}(Y, X)$ relative to $\lambda \mapsto T_\lambda$ (see introduction) to be defined on the whole of G (Y being a suitable Banach space). If $(\lambda, z) \in G \times \mathbf{C}$, then $T_\lambda - zI = = (V_\lambda - zU_\lambda)\, U_\lambda^{-1}$, and the range, nullity and deficiency of $T_\lambda - zI$

are the same, resp., of $V_\lambda - zU_\lambda$; so

$$\Omega_{ew} = \{(\lambda, z) \in G \times \mathbf{C} \colon z \notin \sigma_{ew}(T_\lambda)\} =$$
$$= \{(\lambda, z) \in G \times \mathbf{C} \colon V_\lambda - zU_\lambda \in \mathcal{F}(Y, X)\} =$$
$$= \{(\lambda, z) \in G \times \mathbf{C} \colon [V_\lambda - zU_\lambda]_{\mathrm{Cal}(Y,X)} \text{ has a two-sided inverse in } \mathrm{Cal}(X, Y)\}.$$

In view of the remarks made above, the conclusion can be drawn exactly as in [3, Proof of the Theorem, p. 1] using the nonextendability to $(G \times \mathbf{C}) \cap \partial\Omega$ of any restriction of the holomorphic mapping

$$\lambda \mapsto \big([V_\lambda - zU_\lambda]_{\mathrm{Cal}(Y,X)}\big)^{-1} \colon \Omega \to \mathrm{Cal}(X, Y). \quad \square$$

(Examination of $V_\lambda - zU_\lambda$ instead of $(V_\lambda - zU_\lambda)U_\lambda^{-1}$, as made in the preceding proof, allows a quicker proof of the Oka-analyticity of the spectrum on $\mathcal{C}(X)$ than [10, Proof of Theorem 1 p. 123].)

THEOREM 2. *The Weyl spectrum is Oka-analytic on* $\mathcal{C}(X)$.

PROOF. Since Kato-holomorphic families are continuous (by [4, Theorem IV.2.29 p. 207]), if $\varphi \colon G \times \mathbf{C} \to \mathcal{B}(X)$ is given by $\varphi(\lambda, z) = T_\lambda - zI$, then the open set $\Omega_{em} = \{(\lambda, z) \in G \times \mathbf{C} \colon z \notin \sigma_{em}(T_\lambda)\} = \varphi^{-1}(\mathcal{F}_0(X))$ is a union of components of the open set $\Omega_{ew} = \varphi^{-1}(\mathcal{F}(X))$, which is pseudoconvex by Theorem 1. $\square$

As to the Browder spectrum, we cannot infer its Oka-analyticity directly from that of the Wolf or the Weyl spectrum as done for Theorem 2. In fact $\Omega_{eb} = \{(\lambda, z) \in G \times \mathbf{C} \colon z \notin \sigma_{eb}(T_\lambda)\}$ is not, in general, a union of components of Ω_{em}, because of the lack of lower semicontinuity of nul, def. Yet one could conjecture, in view of some of the properties recalled earlier, that the functions nul′, def′ are locally constant on $\mathcal{F}_0(X)$, where

$$\mathrm{nul}'(T) = \mathrm{nul}(T - zI), \qquad \mathrm{def}'(T) = \mathrm{def}(T - zI), \qquad \text{for } 0 < |z| \ll 1.$$

This conjecture is true only when X is finite-dimensional (in which case nul′, def′ vanish identically), as the following counter-example shows.

COUNTEREXAMPLE 3. Let: X be the Hilbert space $l^2 = L^2(\mathbb{Z}, \nu)$ (where ν is the counting measure); $P, A \in \mathcal{B}(X)$ the projection on the zeroth coordinate and the onestep shift to the right, resp.; $G = \mathbf{C}$; $\lambda \mapsto T_\lambda = A(I - \lambda P)\colon G \to \mathcal{B}(X)$. If $\{e_j\}_{j\in\mathbf{Z}}$ is the canonical basis of l^2, then $N(T_1) = [e_0]$, $R(T_1) = [e_1]^\perp$: so $T_1 \in \mathcal{F}_0(X)$, therefore $T_\lambda - zI \in \mathcal{F}_0(X)$ for $|\lambda - 1| \ll 1$, $|z| \ll 1$. But $N(T_1 - zI) = [\sum_{j=0,\dots,+\infty} z^j e_{-j}]$ for $|z| \ll 1$, so $\mathrm{nul}'(T_1) = \mathrm{def}'(T_1) = 1$; while T_λ is invertible for $0 < |\lambda - 1| \ll 1$, thus $\mathrm{nul}'(T_\lambda) = \mathrm{def}'(T_\lambda) = 0$. Hence $(1, 0) \in \Omega_{em} \setminus \Omega_{eb}$, whereas $(\lambda, 0) \in \Omega_{eb}$ if $0 < |\lambda - 1| \ll 1$.

The upper semicontinuity of the Browder spectrum is interesting in its own, so we give it separately.

LEMMA 4. *The Browder spectrum is u.s.c. on* $\mathcal{C}(X)$.

PROOF. Let $T \in \mathcal{C}(X)$. Then $\sigma(T) \setminus \sigma_{eb}(T)$ consists only of isolated points of $\sigma(T)$: if A is a compact-complemented open subset of $\mathbf{C}$ containing $\sigma_{eb}(T)$, then $\sigma(T) \setminus A = \{z_1, \dots, z_k\}$. Since σ is u.s.c., for $T' \in \mathcal{C}(X)$ near T we have $\sigma(T') \subset A \cup \big(\bigcup_{j=1,\dots,k} B(z_j, \varepsilon)\big)$ (where $B(z_j, \varepsilon) = \{z \in \mathbf{C}\colon |z - z_j| < \varepsilon\}$), with $\varepsilon > 0$ such that the above union is disjoint. Using the Dunford integral calculus, for every such T' [4, Theorem IV.3.16 p. 212] provides a splitting of X into a $(k+1)$-ple direct sum $X = \bigoplus_{j=0,\dots,k} X_j(T')$ of the generalized eigenspaces associated to $\sigma(T') \cap A$, $\sigma(T') \cap B(z_j, \varepsilon)$, $j = 1, \dots, k$; furthermore $\dim X_j(T')$ is independent of T'. Fix $j = 1, \dots, k$. Since the restriction of $T - z_j I$ to $X_j(T)$ is a quasinilpotent operator, and its approximated nullity and deficiency (defined equal to the nullity and deficiency, resp., if the range is closed, or to $+\infty$ otherwise: cf. [4, Theorem IV.5.10 p. 233]) are finite (because $T - z_j I$ is Fredholm), [4, Theorem IV.5.30 p. 240] yields that $\dim X_j(T)$ is finite. Hence $\sigma(T') \cap B(z_j, \varepsilon)$ is a finite set $\{z_{j1}, \dots, z_{jk_j}\}$, and for $j' = 1, \dots, k_j$ the range of $T' - z_{jj'} I$ is closed and the algebraic multiplicity of $z_{jj'}$ in T' is finite; that is, $\sigma_{eb}(T') \cap B(z_j, \varepsilon)$ is empty. Therefore $\sigma_{eb}(T') \subset A$. □

Let $\lambda \mapsto \Sigma_\lambda\colon G \to \mathrm{cl}\,(\mathbf{C})$ be an Oka-analytic multifunction, and $\lambda \in G$. An isolated point z of Σ_λ is a *good isolated point*, or *g.i.p.*, (for Σ) at λ if there exists $\delta > 0$ such that $\sigma(T_{\lambda'}) \cap B(z, \delta)$ is finite for $|\lambda' - \lambda| < \delta$. The Oka-Nishino theorem (cf. [6, Corollary 5.5 p. 557]) asserts that the multifunction $\lambda \mapsto D\Sigma'_\lambda = \Sigma_\lambda \setminus \{\text{g.i.p.'s at } \lambda\}$ is itself Oka-analytic.

THEOREM 5. *The Browder spectrum is Oka-analytic on* $\mathcal{C}(X)$.

PROOF. Let $\lambda \in G$. From the proof of Lemma 4 we get that all the points in $\sigma(T_\lambda)\setminus\sigma_{eb}(T_\lambda)$ are g.i.p.'s at λ, but in general not conversely: for instance, if $T_\lambda \equiv 0$ on G, then 0 is a g.i.p. at any $\lambda \in G$, but belongs to $\sigma_{eb}(T_\lambda)$ if X is infinite-dimensional. Thus we have $D\sigma(T_\lambda) \subset \sigma_{eb}(T_\lambda) \subset \sigma(T_\lambda)$ for each $\lambda \in G$, the first and third multifunction being Oka-analytic. In order to prove that Ω_{eb} is pseudoconvex it will suffice to show that for each $\lambda_0 \in G$ and $z_0 \in \sigma_{eb}(T_{\lambda_0})\setminus D\sigma(T_{\lambda_0})$ there exists a neighborhood U of (z_0, λ_0) in $D\Omega = \{(\lambda, z) \in G \times \mathbf{C}: z \notin D\sigma(T_\lambda)\}$ such that $U \cap \Omega_{eb}$ is pseudoconvex. Because $D\Omega$ is open, U can be taken to be a bidisk; also, we will assume $\lambda_0 = z_0 = 0$.

Since 0 is isolated in $\sigma(T_0)$ and Ω is open we can choose $\varepsilon > \delta > 0$ so that $\sigma(T_\lambda) \cap B(0, \varepsilon) = \sigma(T_\lambda) \cap B(0, \varepsilon - \delta)$ for $|\lambda| < \delta$: so $\lambda \mapsto \Sigma_\lambda = \sigma(T_\lambda) \cap B(0, \varepsilon): B(0, \delta) \to \mathrm{cl}(\mathbf{C})$ is still Oka-analytic. Because 0 is a g.i.p. at 0, by [1, Theorem 3.8] δ may be taken small enough that the cardinality of Σ_λ be finite and independent of $\lambda \in B(0, \delta)\setminus\{0\}$, say k. Furthermore for any such λ there exists $\delta_\lambda > 0$ and k holomorphic functions $h_1, \ldots, h_k: B(\lambda, \delta_\lambda) \to \mathbf{C}$ such that $\Sigma_{\lambda'} = \{h_1(\lambda'), \ldots, h_k(\lambda')\}$ for each $\lambda' \in B(\lambda, \delta_\lambda)$. As in the proof of Lemma 4 we have that $h_j(\lambda') \in \sigma_{eb}(T_{\lambda'})$ if and only if the dimension of the generalized eigenspace of $T_{\lambda'}$ associated to $h_j(\lambda')$ is finite; such dimension being stable, either $h_j(\lambda') \in \sigma_{eb}(T_{\lambda'})$ for all $\lambda' \in B(\lambda, \delta_\lambda)$, or for none of them. Therefore k_0 exists, with $1 \leqslant k_0 \leqslant k$, such that for each $\lambda \in B(0, \delta)\setminus\{0\}$ the function h_j can be rearranged in such a way that the former alternative holds for exactly $h_1, \ldots, h_{k_0}$. If $f: U = B(0, \delta) \times B(0, \varepsilon) \to \mathbf{C}$ is defined through $f(\lambda, z) = \prod_{j=1,\ldots,k_0} (z - h_j(\lambda))$ for $\lambda \neq 0$, and $f(0, z) = z^{k_0}$, such f is well-defined, holomorphic where $\lambda \neq 0$, and (by the upper semicontinuity of $\lambda \mapsto \Sigma_\lambda$, which is implied by that of $\lambda \mapsto \sigma(T_\lambda)$) continuous where $\lambda = 0$. By Radò's theorem the function f is holomorphic on U: since $\{f = 0\} = U\setminus\Omega_{eb}$, no restriction of $1/f: U \cap \Omega_{eb} \to \mathbf{C}$ can be extended to any point of $U\setminus\Omega_{eb}$. Thus $U \cap \Omega_{eb}$ is pseudoconvex, because U is. □

The following corollary of the three preceding theorems appeared in [5, Theorem 13 p. 320] for the Weyl case, while it can be easily proven directly in the Wolf case using the Oka-analyticity of the spectrum on a Banach algebra (Cal(X) here: see [1, Theorem 3.2 p. 46]).

COROLLARY 6. *The Wolf, Weyl, and Browder spectrum are all Oka-analytic on* $\mathfrak{B}(X)$. □

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