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A Finiteness Condition on Automorphism Groups.

FEDERICO MENEGAZZO - DEREK J. S. ROBINSON (*)

1. Introduction.

If G is a group, its automorphism group $\text{Aut } G$ acts on G in a natural way as a permutation group. Should this action be restricted by the imposition of a finiteness condition, there will be repercussions on the structure of the group G . The simplest case is where $\text{Aut } G$ is required to be finite and there is a considerable literature dealing with the resulting structure of G (see [3]). Recently groups G for which the automorphism classes (i.e. $\text{Aut } G$ -orbits) are finite have been studied ([5]). In the present work we study what is in a sense the dual property, that fixed point subgroups of automorphisms have finite index.

An automorphism α of a group G is said to be *virtually trivial* if $|\mathcal{G}:C_G(\alpha)|$ is finite. (Automorphisms with this property have also been considered in [7] under the name of « bounded automorphisms ».) The set of all virtually trivial automorphisms of G is readily seen to be a normal subgroup of $\text{Aut } G$, denoted here by $\text{Aut}_{vt}(G)$.

Should it happen that $\text{Aut } G = \text{Aut}_{vt}(G)$, that is, every automorphism is virtually trivial, we shall say that G is a *VTA-group*.

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Obviously every finite group is VTA and every VTA-group is an FC -group. It is not difficult to see that every abelian VTA-group must be finite (by considering the automorphism $x \mapsto x^{-1}$). The simplest example of an infinite VTA-group is

$$G = \langle x, y : y^5 = 1, y^x = y^2 \rangle.$$

This group has no outer automorphisms and each conjugacy class has at most 5 elements.

2. Finiteness of the commutator subgroup.

The basic result in the theory of VTA-groups is

PROPOSITION 1. *If G is a VTA-group, then G' is finite.*

PROOF. Suppose to the contrary that G' is infinite. Choose any element x_1 and X_1 a finitely generated normal subgroup containing x_1 , for example $\langle x_1^G \rangle$. Set $C_1 = C_G(X_1)$; then $|G:C_1|$ is finite since G is an FC -group. Hence there is a finitely generated normal subgroup H such that $G = HC_1$. Then $G' = H'[H, C_1]C_1'$. By standard results on FC -groups (see [2]) G' is a torsion group and the elements of finite order in H form a finite subgroup. Consequently C_1' is infinite and by a well-known theorem of B. H. Neumann [1] there are conjugacy classes of C_1 with arbitrarily large finite orders. Hence one can find an element c of C_1 such that $|C_1:C_{C_1}(c)|$ exceeds $|X_1:C_{X_1}(x_1)|$. Put $x_2 = x_1c$ and observe that $C_{C_1}(x_2) = C_{C_1}(c)$. Therefore

$$|G:C_G(x_2)| \geq |C_1:C_{C_1}(c)| > |X_1:C_{X_1}(x_1)|.$$

Now choose a finitely generated normal subgroup X_2 such that $G = X_2C_G(x_2)$ and $X_1 \leq X_2$. Then

$$|X_2:C_{X_2}(x_2)| = |G:C_G(x_2)| > |X_1:C_{X_1}(x_1)|.$$

Notice that conjugation by x_1 and x_2 produce the same inner automorphism on X_1 .

By repetition of this procedure one can construct a sequence of elements x_1, x_2, \dots and a chain of finitely generated normal subgroups $X_1 < X_2 < \dots$ with $x_i \in X_i$ such that conjugation by x_i and x_{i+1} have

the same effect on X_i and

$$|X_{i+1}:C_{x_{i+1}}(x_{i+1})| > |X_i:C_{x_i}(x_i)| \quad \text{for all } i = 1, 2, \dots$$

Let $U = \bigcup_{i=1,2,\dots} X_i$ and let α be the locally inner automorphism of U whose restriction to X_i is the inner automorphism induced by conjugation by x_i . Now it is always true that a locally inner automorphism of a subgroup of an FC-group can be extended to a locally inner automorphism of the group (see [6], Lemma 2.3, but note that the periodicity hypothesis is not essential). It follows that α is a virtually trivial automorphism of U and this yields the contradiction that the $|X_i:C_{x_i}(x_i)|$ are bounded. The proof is now complete.

It is an easy observation that if G is a group with finite derived subgroup, then $G/Z(G)$ has finite exponent. Also a theorem of P. Hall (see [2]) shows that $G/Z_2(G)$ is finite. Hence

COROLLARY 1. *If G is a VTA-group, then $G/Z(G)$ is centre-by-finite and has finite exponent.*

As a result of Proposition 1 one can give an equivalent definition for VTA-groups.

LEMMA 1. *A group G is a VTA-group if and only if $[G, \alpha]$ is finite for every automorphism α of G .*

PROOF. It is clear that $|G:C_G(\alpha)|$ will be finite if $[G, \alpha]$ is finite. Conversely assume that G is a VTA-group; set $C = Z(G)$. Since $a \mapsto [a, \alpha]$ is an endomorphism of C and $C/C_G(\alpha) \cong [C, \alpha]$, the subgroup $[C, \alpha]$ is finite. By Corollary 1 G/C has finite exponent, say e , and $[G, \alpha]^e \leq [G^e, \alpha] G' \leq [C, \alpha] G'$. Hence $[G, \alpha]^e$ is finite and $[G, \alpha]$ is locally finite. Since $[G, \alpha]$ is certainly finitely generated, it is finite, as required.

REMARK. In general, $\alpha \in \text{Aut } G$ and $|G:C_G(\alpha)| < \infty$ do not imply that $[G, \alpha]$ is finite. For let $G = \langle x, y \mid x^2 = 1, y^x = y^{-1} \rangle$ be the infinite dihedral group and let $\alpha \in \text{Aut } G$ be defined by $x \mapsto xy, y \mapsto y$.

The following lemma is a sharper form of [7], Theorem 1:

LEMMA 2. *Let G be an arbitrary group and let X be a finitely generated subgroup of $\text{Aut}_{vt}(G)$. Then $X/Z(X) \cap \text{Inn } G$ is finite. In particular $\text{Aut}_{vt}(G)$ is locally FC and $\text{Aut}_{vt}(G)/\text{Aut}_{vt}(G) \cap \text{Inn } G$ is locally finite.*

PROOF. We know that $|G:C_G(X)|$ is finite. Let N be the core of $C_G(X)$ in G , so that $|G:N|$ is finite. Put $X_1 = C_X(G/N)$. Then X/X_1 is finite and there is an obvious injection

$$\mu: X_1 \rightarrow \text{Der}(G/N, Z(N)).$$

Of course X_1 is finitely generated. Now $H^1(G/N, Z(N))$ is a bounded abelian group, being annihilated by $|G:N|$. Hence X_1/X_2 is finite where X_2 is the inverse image of $\text{Inn}(G/N, Z(N))$ under μ . But X_2 consists of inner automorphisms induced by elements of $Z(N)$, and such automorphisms commute with X . Hence $X_2 \leq Z(X) \cap \text{Inn } G$.

COROLLARY 2. *If G is a VTA-group, then $\text{Aut } G$ is locally finite.*

3. Structure of the centre.

LEMMA 3. *If G is a VTA-group, then $Z(G)$ is reduced and its primary components are all finite.*

PROOF. Let $C = Z(G)$ and $Q = G/C$. By Corollary 1 it is possible to express Q as a direct product $Q_1 \times Q_2$ where Q_1 is finite and Q_2 is abelian. By the Universal Coefficients Theorem $H^2(Q, C)$ is bounded, say $l \cdot H^2(Q, C) = 0$ with $l > 0$.

Suppose that C is not reduced and so $C = D \times E$ where D is either a p^∞ -group or \mathbb{Q} . Then there is an automorphism of C in which $d \mapsto d^k$ and $e \mapsto e$ ($d \in D, e \in E$); here $k > 1$ and $k \equiv 1 \pmod{l \text{ or } pl}$ according as $D \simeq \mathbb{Q}$ or p^∞ . Since $l \cdot H^2(Q, C) = 0$ this α extends to an automorphism of G . Hence $|D:C_D(\alpha)| < \alpha$ which is of course impossible.

Next assume that the p -component C_p is infinite for some prime p . Then $C[p] = \{a \in C: a^p = 1\}$ is infinite since C is reduced. By constructing central automorphisms of G corresponding to elements of $\text{Hom}(Q_2, C[p])$ one can see that Q_2/Q_2^p must be finite. Since Q_2 has finite exponent, this implies that $(Q_2)_p$ is finite. There is nothing to be lost in supposing Q_2 to be a p' -group. Since C_p/C_p^p is infinite, so is C/C^p . Also G' is finite; thus G'/C^p has an infinite elementary abelian quotient. Therefore $C^p[p]$ is necessarily finite, which shows that $(C^p)_p$ is finite. It follows that C_p has finite exponent. Hence it is possible to write $C = C_p \times F$ for some subgroup F .

Since $Q_2 \equiv G_2/F$ is a p' -group, G_2/F splits over C/F and

$$G_2/F = X/F \times C/F,$$

say. Thus $G_2 = X \times C_p$. Because Q_1 is finite, there is a finitely generated subgroup Y such that $G = YG_2$ and $Y \cap G_2 \leq C$. But C_p has finite exponent, so one can write $C_p = C_{p,0} \times C_{p,1}$ where $C_{p,0}$ is finite and $Y \cap G_2 = X \cap C_{p,0}$. Hence

$$G = YG_2 = Y \times F C_{p,0} C_{p,1} = (YX C_{p,0}) C_{p,1}.$$

If $yx c_0 = c_1$ where $y \in Y$, $x \in X$, $c_i \in C_{p,i}$, then $y \in Y \cap G_2 \leq X \times C_{p,0}$. Hence one can assume that $y = 1$ and $x c_0 = c_1$. But now $x \in X \cap C_p = 1$ and $c_0 = c_1 = 1$. Consequently $G = (Y \times C_{p,0}) \times C_{p,1}$. Since G is a VTA-group and $C_{p,1}$ has finite exponent, $C_{p,1}$ is finite. This gives the contradiction that C_p is finite.

LEMMA 4. *If G is a torsion VTA-group, then $Z(G)$ is finite and G has finite exponent.*

PROOF. By Lemma 3 and Corollary 1, it is sufficient to prove that the set π of prime divisors of orders of elements of G is finite. Let L denote the second centre of G . Then $|G:L|$ is finite and there is a finite normal subgroup N such that $G = NL$; since G' is finite, one can assume that $G' \leq N$. If π_0 is the set of prime divisors of $|N|$, then $G = N(L_{\pi_0} \times L_{\pi_0}') = (NL_{\pi_0}) \times L_{\pi_0}'$. Now $(L_{\pi_0}')' \leq L_{\pi_0}' \cap G' = 1$, so L_{π_0}' is abelian; therefore it is finite by the VTA-property.

However in general the torsion subgroup of the centre of a VTA-group can be infinite, as will be shown in § 5.

4. Necessary and sufficient conditions.

Let Q be a group and C an abelian group regarded as a trivial Q -module. Then there are natural left and right actions of $\text{Aut } Q$ and $\text{Aut } C$ respectively on $H^2(Q, C)$. If $C \rightarrow G \rightarrow Q$ is a central extension with cohomology class Δ , then a necessary and sufficient condition for there to exist an automorphism of G inducing in Q and C automorphisms \varkappa and γ is that $\varkappa \Delta = \Delta \gamma$. If we make $H^2(Q, C)$ into a right $\text{Aut } Q \times \text{Aut } C$ -module by means of the rule $\Delta(\varkappa, \gamma) = \varkappa^{-1} \Delta \gamma$,

then the above condition is equivalent to

$$(\varkappa, \gamma) \in C_{\text{Aut } Q \times \text{Aut } C}(\Delta).$$

These observations, the background to which may be found in [4], may be used to give necessary and sufficient conditions for a group to be a VTA-group.

THEOREM 1. *Let G be a group with centre C and central quotient group Q . Let Δ be the cohomology class of the extension $C \hookrightarrow G \twoheadrightarrow Q$. Then G is a VTA-group if and only if the following hold:*

- (i) G' is finite;
- (ii) each primary component of C is finite;
- (iii) $C_{\text{Aut } Q \times \text{Aut } C}(\Delta)$ is contained in $\text{Aut}_{vt}(Q) \times \text{Aut}_{vt}(C)$.

PROOF. Let G be a VTA-group; then (i) and (ii) hold by Proposition 1 and Lemma 3. If (\varkappa, γ) belongs to $C_{\text{Aut } Q \times \text{Aut } C}(\Delta)$, then \varkappa and γ are induced by an automorphism of G . Hence

$$\varkappa \in \text{Aut}_{vt}(Q) \quad \text{and} \quad \gamma \in \text{Aut}_{vt}(C).$$

Conversely, assume that the three conditions hold. Let α be an automorphism of G inducing \varkappa in Q and γ in C ; then $\varkappa \in \text{Aut}_{vt}(Q)$ and $\gamma \in \text{Aut}_{vt}(C)$ by (iii). Set $K/C = C_Q(\varkappa)$ and $L = C_C(\gamma)$. Then $|G:K|$ and $|C:L|$ are finite. The mapping $x \mapsto [x, \alpha]$ is a homomorphism from K onto $[K, \alpha]$ whose kernel contains L . Now condition (i) implies that G/C has finite exponent, whence so does G/L . Consequently $[K, \alpha]$ has finite exponent. Since $[K, \alpha] \leq C$, it follows that $[K, \alpha]$ is contained in T , the torsion subgroup of C . Since T has finite primary components by (ii), the subgroup $[K, \alpha]$ is finite. This shows that $|K:C_K(\alpha)|$ is finite, therefore $|G:C_G(\alpha)|$ is finite and G is a VTA-group, as claimed.

It is of course the third condition which is difficult to deal with. For torsion groups this can be simplified slightly.

THEOREM 2. *Let G be a torsion group with centre C and central quotient group Q . Let Δ be the cohomology class of the extension $C \hookrightarrow G \twoheadrightarrow Q$. Then G is a VTA-group if and only if the following con-*

ditions hold:

- (i) G' is finite;
- (ii) C is finite;

(iii) $St_{\text{Aut } Q}(\Delta^{\text{Aut } C})$, the stabilizer in $\text{Aut } Q$ of the $\text{Aut } C$ -orbit containing Δ , is contained in $\text{Aut}_{\text{vt}}(Q)$.

The point to note here is that κ in $\text{Aut } Q$ is induced by an automorphism of G if and only if there exists a γ in $\text{Aut } C$ such that $\kappa\Delta = \Delta\gamma$, that is, κ stabilizes $\Delta^{\text{Aut } C}$. Theorem 2 is now a consequence of Theorem 1 and Lemma 4. (An example of an infinite torsion VTA-group is given in § 5.)

Finitely generated VTA-group admit a precise characterization.

THEOREM 3. *A finitely generated group G is a VTA-group if and only if it is either finite or the split extension of a finite group F by an infinite cyclic group $\langle x \rangle$ such that \hat{x} , the image of x in $\text{Out } F$, is not conjugate to its inverse.*

PROOF. Let G be a finitely generated infinite VTA-group, and write $C = Z(G)$ and $Q = G/C$. By Corollary 1 the group Q is finite, so C is finitely generated. Suppose that C had a free abelian direct factor of rank 2, say $\langle a \rangle \times \langle b \rangle$. If $|Q| = m$, then it is easy to see that $a \mapsto ab^m, b \mapsto b$ extends to an automorphism of G ; but this cannot be virtually trivial. Consequently C has torsion-free rank 1. Since G' is finite, the elements of finite order form a finite subgroup F and G/F is infinitely cyclic. Write $G = \langle x \rangle F$ with $\langle x \rangle \cap F = 1$. If $\alpha \in \text{Aut } G$ induces the inversion automorphism in G/F , then $C_\alpha(\alpha) \leq F$ and $|G : C_\alpha(\alpha)|$ is infinite. Therefore every automorphism of G acts trivially on G/F .

Conversely let $G = \langle x \rangle \rtimes F$ have the property that all automorphisms of G act trivially on G/F . Let $\alpha \in \text{Aut } G$. Now G/C is finite where $C = Z(G)$, and $C = D \times E$ where D is finite of order d , say, and E is infinite cyclic. Hence there is an $m > 0$ such that $x^m \in C^d = E^d$. Now C^d is characteristic in G and infinite cyclic, so $\langle x^m \rangle$ is α -invariant. Since α clearly cannot map x^m to its inverse, it follows that $x^m \in C_\alpha(\alpha)$ and $|G : C_\alpha(\alpha)|$ is finite. Thus G is a VTA-group.

It remains to decide when the inversion automorphism of Q lifts to G . A simple argument shows that this occurs if and only if \hat{x} is conjugated to its inverse in $\text{Out } F$ by some $\varphi \in \text{Aut } F$.

REMARK. In order to construct all finitely generated VTA-groups one must select a finite group F with the property that not every element of $\text{Out } F$ is conjugate to its inverse (or, equivalently, such that some irreducible \mathbb{C} -character of F is not real). If $\hat{\phi}$ is such an element, lift $\hat{\phi}$ in any way to an automorphism φ of F and form the semidirect product $\langle x \rangle \rtimes F$ of F with an infinite cyclic group $\langle x \rangle$ where x induces φ in F . The smallest possible F is the cyclic group of order 5; this gives rise to the VTA-group mentioned in § 1.

Hence the isomorphism classes of finitely generated VTA-groups are in one-one correspondence with pairs $(F, \hat{\phi})$ where F is a finite group as above and $\hat{\phi}$ is a conjugacy class of $\text{Out } F$ not equal to its inverse.

5. Examples.

We shall now construct the examples of infinite VTA-group promised in earlier sections.

(A) *There is an infinite VTA-group which is a p -group with nilpotency class 2.*

For $p = 2$ the group constructed in [5], Prop. 3 will do. Let $p > 2$ and let G be the group with generators a, b, x_1, x_2, \dots and relations

$$\begin{aligned} [x_{2i-1}, x_{2i}] &= a, & [x_{2i}, x_{2i+1}] &= b, & [x_i, x_j] &= 1 & \text{ if } i < j - 1, \\ [a, x_i] &= 1 = [b, x_i] = [a, b] = a^p = b^p, \\ x_1^p &= ab, & x_2^p &= a, & x_i^p &= 1 & \text{ if } i > 2. \end{aligned}$$

One quickly sees that $G' = Z(G) = \langle a \rangle \times \langle b \rangle$ and $Q \equiv G/Z(G)$ is an infinite elementary abelian p -group. For $i = 1, 2, \dots$ put $M_i = \langle x_i \rangle G'$ and $C_i = \langle x_{i+2}, x_{i+3}, \dots \rangle G'$. It is easily seen that M_1 is the set of all $y \in G$ such that $|G : C_G(y)| = p$. Thus M_1 is characteristic in G , as is $C_1 = \Omega_1(C_G(M_1))$. The elements y of G for which $|M_1 : C_{M_1}(y)| = p = |C_1 : C_{C_1}(y)|$ are those of the form $x_1^r x_2^s z$ where $s \not\equiv 0 \pmod{p}$ and $z \in G'$. From this it follows that: $\langle x_1, x_2 \rangle G'$ is characteristic in G . Among these y those that generate a cyclic normal subgroup of $\langle x_1, x_2 \rangle G'$ are of the form $x_2^s z$; hence M_2 is characteristic in G .

Now suppose that M_1, \dots, M_i are characteristic in G (where $i \geq 2$); then also $C_{i-1} = \Omega_1(C_G(M_1 \dots M_{i-1}))$ is characteristic. A little calculation shows that the elements y of C_{i-1} such that $||y, C_{i-1}|| = p$ are those of the form $x_{i+1}^r z$ ($r \not\equiv 0 \pmod p, z \in G'$). It follows that M_{i+1} is characteristic in G .

Next let $\alpha \in \text{Aut } G$. We have just shown that $x_i^\alpha \equiv x_i^{n_i} \pmod{G'}$ for all i and some integers n_i . Our relations imply $n_1 \equiv n_2 \equiv n_{2i-1} \cdot n_{2i} \equiv n_{2i} n_{2i+1}$, ($i = 1, 2, \dots$), i.e. $n_j \equiv 1 \pmod p$ for all j . Thus every automorphism of G acts trivially on $Z(G)$ and on Q . Hence $|G : C_G(\alpha)| \leq p^2$ for every automorphism α . Notice also that $\text{Aut } G \cong \text{Hom}(Q, Z(G))$, an elementary abelian p -group of cardinality 2^{8n} .

(B) *There exists a VTA-group with finite torsion-free rank whose centre has infinite torsion subgroup.*

The construction falls into two parts; first we assign the central quotient, then we construct the centre.

Let Q be a finite group satisfying the conditions

- (i) $Q_{ab} \equiv Q/Q'$ is not an elementary abelian 2-group;
- (ii) no automorphism of Q induces an automorphism of order 2 in Q_{ab} ;
- (iii) $Z(Q) = 1$.

Of course such groups abound; the simplest example is the holomorph of a cyclic group of order 5. Let σ be the set of all primes which do not divide the order of Q .

The centre of our group has to be chosen with some care. Let $\langle t_p \rangle$ be a cyclic group of order p , written additively, and define

$$T = \text{Dr}_{p \in \sigma} \langle t_p \rangle \quad \text{and} \quad T^* = \text{Cr}_{p \in \sigma} \langle t_p \rangle,$$

the direct and cartesian sums. We shall construct a group C such that

- (i)' $T \leq C \leq T^*$;
- (ii)' $\bar{C} = C/T$ has automorphism group of order 2;
- (iii)' $\text{Aut } C = \text{Aut}_{\sigma t}(C) \times \langle -1 \rangle$.

Here of course $\langle -1 \rangle$ refers to the automorphism $c \mapsto -c$ of C .

Assuming this C to be constructed, one regards it as a trivial Q -module. Then by the Universal Coefficients Theorem

$$H^2(Q, C) \cong \text{Ext}(Q_{ab}, \bar{C}).$$

This group has an element Δ with order > 2 by properties (i) and (ii)'. Let

$$C \twoheadrightarrow G \twoheadrightarrow Q$$

be a central extension with cohomology class Δ . Then $Z(G) = C$ since $Z(Q) = 1$. Let α in $\text{Aut } G$ induce automorphisms κ and γ in Q and C respectively; then $\kappa\Delta = \Delta\gamma$. If $\gamma \notin \text{Aut}_{vt}(C)$, then by (iii)' γ must induce -1 on \bar{C} ; in this case $\kappa\Delta = -\Delta$ by the above (natural) isomorphism. But κ cannot induce an automorphism of even order in Q_{ab} , so we reach a contradiction. It follows that $\gamma \in \text{Aut}_{vt}(C)$ and $\alpha \in \text{Aut}_{vt}(G)$ since Q is finite.

Construction of C . It remains to find an abelian group C satisfying (i)', (ii)', (iii)'. First we specify \bar{C} . Let

$$\sigma = \pi \cup \varrho$$

be a partition of σ into two infinite sets of primes. Define $x^*, y^* \in T^*$ by

$$\begin{aligned} (x^*)_p &= t_p, & (y^*)_p &= 0 & \text{if } p \in \pi \\ (x^*)_p &= 0, & (y^*)_p &= t_p & \text{if } p \in \varrho. \end{aligned}$$

Also let q be a prime not in σ . Define A_1, A_2, A_3 to be the subrings of \mathbb{Q} generated by $\pi^{-1}, \varrho^{-1}, q^{-1}$ respectively.

Now $u = x^* + T, v = y^* + T$ are independent vectors in the \mathbb{Q} -vector space $\bar{T} = T^*/T$, and we may define our group C by requiring $T < C < T^*$ and

$$\bar{C} \equiv C/T = A_1u + A_2v + A_3(u + v).$$

It is straightforward to check that \bar{C} has automorphism group of order 2. It remains to check property (iii)'. Let $\gamma \in \text{Aut } C$; replacing γ by $-\gamma$ if necessary, we can assume that γ acts trivially on \bar{C} . The task is now to show that $\gamma \in \text{Aut}_{vt}(C)$. Let $p \in \pi$. There is a unique

coset $b_p + T \in \bar{C}$ such that $x^* - pb_p \in T$; notice that, since pb_p obviously has trivial p -component, the p -component of $x^* - pb_p$ is t_p , and we may as well assume that $x^* - pb_p = t_p$. Now $b_p(\gamma - 1) = a_p$ (say) is in T , and one obtains

$$t_p(\gamma - 1) = x^*(\gamma - 1) - pa_p.$$

Looking at p -components one gets

$$t_p(\gamma - 1) \in \langle x^*(\gamma - 1) \rangle \quad \text{for all } p \in \pi.$$

This means that γ fixes almost all t_p with p in π ; a similar conclusion holds for ρ . Modifying γ by a virtually trivial automorphism of C , one can assume that γ operates trivially on T . However it is easily checked that $\text{Hom}(\bar{C}, T)$ is periodic, and this implies that $C(\gamma - 1)$ is finite and $|C: C_C(\gamma)|$ is finite. Therefore $\gamma \in \text{Aut}_{vt}(C)$.

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