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Linear Groups with Large Cyclic Subgroups and Translation Planes.

U. DEMPWOLFF (*)

SUMMARY - Let V be a finite dimensional vectorspace over $GF(q)$ and R be a cyclic subgroup of prime order r in $GL(V)$, such that R has precisely one nontrivial irreducible submodule on V. Then we call R 1-irreducible. We consider irreducible groups $G \subseteq GL(V)$ generated by 1-irreducible subgroups of prime order. We generalize some results of Hering where this problem was treated under the additional assumption that V itself is the nontrivial irreducible submodule of R. Further we give an application of our results to collineation groups of translation planes.

1. Introduction.

Let q_0 be a prime and denote by q a fixed power of q_0. Let V be a finite dimensional vectorspace over $GF(p)$. For $X \subseteq GL(V)$ set

$$V_X = \{v \in V : vx = v \text{ for } x \in X\}$$

and

$$V^X = \langle v(x-1) : v \in V, x \in X \rangle .$$

We call a subgroup $X \subseteq GL(V)$ *1-irreducible* if $V = V_X \oplus V^X$ and V^X is an irreducible X-space. In the remainder of this article we consider 1-irreducible, cyclic subgroups $R \subseteq GL(V)$, where $|R| = r$ is a prime. So if dim $V_R = n$, we have that r is a p-primitive divisor of $q^n - 1$

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(or in the notation of Hering [7]$r|\varphi_n^*(q)$). Notice that by a theorem of Zsygmondy [22]$\varphi_n^*(q) \neq 1$ unless $n = 6$, $q = q_0 = 2$ or in some cases $n = 2$, $q = q_0$. Suppose $\mathcal{S}$ is a collection of 1-irreducible groups of prime order r and let $G = \langle \mathcal{S} \rangle$ be an irreducible subgroup of $GL(V)$. If $V = V^R$, then Hering [7,8] has determined these groups G. We consider first the general case and show that in the generic case $F^*(G)$ (generalized Fitting group of G) is a quasi-simple group. Then we make the restriction $2 \cdot \dim V^R \geqslant \dim V$ and consider in particular the case where the simple nonabelian composition factor of G is a Chevalley group. Modular representation theory of Chevalley groups will determine G together with the possible module V. This applies to the problem of determining subgroups $X \subseteq GL(V), X = \langle R_1, R_2 \rangle$, where R_1, R_2 are 1-irreducible cyclic subgroups of $GL(V)$ of prime order (see 2.1). In the final section we give a further application of these results to translation planes, which have a collineation group with a «large» orbit on l_∞.

2. The normal subgroup structure of irreducible subgroups of $GL(V)$ generated by 1-irreducible subgroups and preliminary results.

For the remainder of this section denote by V a finite dimensional vectorspace over $GF(q)$. Denote by r a prime dividing $\varphi_n^*(q)$ and by $\mathcal{S}$ a collection of 1-irreducible subgroups of order r in $GL(V)$. Let $G = \langle \mathcal{S} \rangle$ be an irreducible subgroup of $GL(V)$. Note that by [22] we have $r \equiv 1 \pmod{n}$. We also assume that $n \geqslant 2$ (i.e. R is not a group of dilatations).

2.1. PROPOSITION. *Let $R_1, R_2 \in \mathcal{S}$ such that $[R_1, R_2] \neq 1$. Set $X = \langle R_1, R_2 \rangle$. Then there is a decomposition $V = W \oplus U$ in X-invariant subspaces with $U \subseteq V_X$, $V^X \subseteq W$ and one of the following holds*:

(i) $\dim W = 2n$ *and* W *contains an irreducible X-subspace* W_1, $\dim W_1 = n$, *such that* $O_{q_0}(X)$ *stabilizes the chain* $0 \subset W_1 \subset W$ *and* $X/O_{q_0}(X) \simeq Z_r \times Z_r$.

(ii) $O_{q_0}(X)$ *stabilizes the chain* $0 \subseteq W_X \subset W^X \subseteq W$, W_X *and* W/W^X *are trivial X-spaces, and W^X/W_X is an irreducible X-space of dimension* $\leqslant 2$n.

PROOF. Let U be a complement of $V_X \cap V^X$ in V_X. Clearly U is X-invariant and $V^X \cap U = 0$. Pick W as a subspace of V with $V^X \subseteq W$

and $V = U \oplus W$. Again W is X-invariant and the first assertion holds.

If W^X/W_X is not irreducible, then as $R_1, R_2 \subseteq X$, we must have $W_X = 0$, $W^X = W$ and wlog $W_1 = W^{R_1}$ is an X-invariant subspace, and $W = W_1 \oplus W^{R_2}$. Now (i) follows. If W^X/W_X is irreducible then as $V^X = V^{R_1} + V^{R_2}$, we have $\dim W^X/W_X \leqslant 2n$ and all assertions of (ii) are obvious.

REMARK. In case (ii) of 2.1 we have $\dim W/W^X$, $\dim W_X \leqslant n - k$, where $\dim W^X/W_X = n + k$. We will determine $X/O_{q_0}(X)$ in sections 3-5.

NOTATION. Suppose X is a finite group with splitting field $GF(q_0^a)$. Let M be an absolutely irreducible $GF(q_0^a)$-module, which affords the character χ. Denote by $GF(q_0)(\chi)$ the field generated by adjoining the values of χ to $GF(q_0)$. It is well known, that M can be realized over $GF(q_0)(\chi)$—the *field of definition* for M—i.e. there is an irreducible $GF(q_0)(\chi)$-module M' of X with $M \simeq M' \otimes GF(q_0^a)$.

2.2. Suppose $q = q_0^b$. *Let $GF(q_0^a)$ be a splitting field for G with $GF(q_0^b) \subseteq GF(q_0^a)$. Suppose $V^* = V \otimes GF(q_0^a) = V_1 \oplus \dots \oplus V_s$ with absolutely irreducible $GF(q_0^a)$-modules $V_1, \dots, V_s$.*

(i) *Suppose $GF(q_0^c)$ is the field of definition for the $GF(q_0^a)$-module V_1. Then $GF(q_0^c)$ is the field of definition for the modules $V_2, \dots, V_s$ too. Further $s = c/(b, c)$ and* $\dim V_i = (\dim V)/s$ $(1 \leqslant i \leqslant s)$. *There are $GF(q_0^c)$-modules $\hat{V}_i (1 \leqslant i \leqslant s)$ with $V_i \simeq \hat{V}_i \otimes GF(q_0^a)$ and $|\hat{V}_i \otimes GF(q_0^{bs})| = |V|$.*

(ii) *Suppose R is a 1-irreducible subgroup of G of order r i.e. $r | \varphi_n^*(q_0^b)$. Then with the notation of* (i) *we have $r | \varphi_{n/s}^*(q_0^c)$ and R acts 1-irreducible on $\hat{V}_i$* $(1 \leqslant i \leqslant s)$.

PROOF. Follow the proof of [11; V, 13.3]. There is a subgroup Γ_1 of $\Gamma = \mathrm{Gal}\,(GF(q_0^a) : GF(q_0^b))$ with $\Gamma = \Gamma_1 \partial_1 \cup \dots \cup \Gamma_1 \partial_s$ and $V^* = V_0 \partial_1 \oplus \dots \oplus V_0 \partial_s$ for an irreducible $GF(q_0^a)$-module V_0. Set $V_i = V_0 \partial_i$. As Γ is abelian we have that $GF(q_0^c)$ is the field of definition for $(2 \leqslant i \leqslant s)$ too. Set $t = c/(b, c)$. Then $V \otimes GF(q_0^{bt}) = \hat{V}_1 \oplus \dots \oplus \hat{V}_s$, where $V_i = \hat{V}_i \otimes GF(q_0^a)$. Hence s divides t. Set $\hat{\Gamma} = \{\gamma \in \Gamma \colon V_0 \gamma = V_0\}$, then $|\Gamma : \hat{\Gamma}| = s$. Suppose $s < t$ and let $GF(q_0^{\bar{c}})$ be the fixed field of $\hat{\Gamma}$. Then $\bar{c} = bs$ and $GF(q_0^{\bar{c}}) \subsetneqq GF(q_0^{bt})$. Thus $GF(q_0^c)$ is not the field of definition V_0, a contradiction.

Thus $s = t$ and $\hat{\Gamma} = \mathrm{Gal}\,(GF(q_0^{bs}) : GF(q_0^b))$ acts transitively on $\{\hat{V}_1, \dots, \hat{V}_s\}$. In particular $|\hat{V}_i| = |V|$. (i) follows.

(ii) Use the notation of (i). Assume $\dim V = n + k$, i.e. $\dim V_R = k$. Then $\dim \tilde{V}_i = (n+k)/s$ and $\dim(\tilde{V}_i)_R = \dim(\tilde{V}_j)_R$ $(1 \leqslant i,j \leqslant s)$. Thus $\dim(\tilde{V}_i)_R = k/s$. Set $n' = n/s$, then r divides $\varphi^*_{n'}(q_0^{bs})\varphi^*_{n'-1}(q_0^{bs})\ldots$. However r is a q_0^b-primitive divisor of $\varphi^*_n(q_0^b)$ and therefore even a q_0^{bs}-primitive divisor with $r \mid \varphi^*_{n'}(q_0^{bs})$. Hence R is 1-irreducible on $\tilde{V}_i$.

2.3. *Let $R \in \mathcal{S}$ and suppose U is an R-invariant subspace with $U = U_1 \oplus \ldots \oplus U_r$ and R permutes $\{U_1, \ldots, U_r\}$ transitively. Then* $\dim U_i = 1$ $(1 \leqslant i \leqslant r)$, $r = n + 1$, *and V^R is a subspace of codimension* 1 *in U.*

Proof. As $\dim V^R = n$, the assertion is immediate.

2.4. *Suppose $N \subseteq G$ and $R \in \mathcal{S}$ normalizes but does not centralize N. Set $X = RN$ and suppose W is a faithful irreducible X-subspace, which is reducible as a N-space. Then one of the following assertions is true:*

(i) *N is abelian and all homogenous N-components on W have dimension* 1, $r = n + 1$ *and R permutes the homogenous N-components cyclic.*

(ii) $\dim W = 2n$, *N is isomorphic to an irreducible subgroup of* $GL(2, q^{n/2})$ *and* $r \mid |N|$.

(iii) $\dim W = 2n = 4$, $r = 3$, $N = Q * Z$, *where* $Q \simeq Q_8$ *and Z induces scalars on W.*

Proof. As we only work in X we assume $V = W$.

Case 1. N has more then one homogenous component. By Cliffords theorem and 2.3 we have (i).

Case 2. All irreducible N-composition factors are isomorphic. Suppose U is an irreducible N-submodule of W. For $R = \langle x \rangle$ define $U_1 = U$ and $U_i = U_{i-1}x$ $(i = 2, 3, \ldots)$. As $W = U_1 + U_2 + \ldots$ we have an s such that $W = U_1 \oplus \ldots \oplus U_s$. In particular $\dim W = m \cdot s$ for $\dim U = m$. By choosing a suitable basis of W adapted to the above decomposition we have a matrix representation of X such that

$$R \ni x \to \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_{s-1,s} \\ A_{s1} & A_{s2} & A_{s3} & \cdots & A_{ss} \end{pmatrix}$$

and $N \ni y \to \mathrm{diag}\,(D(y), \dots, D(y))$, where we have $(ms \times ms)$-matrices decomposed into blocks of seize $m \times m$. Now $A_{12}, A_{23}, \dots, A_{s-1,s}, A_{s1}$ are nonsingular and if $y \in N$ with $y^x = y'$ then $A_{i,i+1}^{-1} D(y) A_{i,\,i+1} = D(y')$. Thus each of these matrices induces by conjugation an automorphism of order r on the group $N_0 = \{D(y) : y \in N\}$ of $m \times m$-matrices. (By a basis transformation of the form $\mathrm{diag}(I, X_2, \dots, X_s)$, $X_i \in C_{\mathrm{End}(U)}(N_0)$ we may even assume that $A_{12}, A_{23}, \dots, A_{s-1,s}$ are matrices of order r). Thus $m \geqslant n$.

On the other hand $W_R \cap U_1 \oplus \dots \oplus U_{s-1} = 0$ and we have $\dim W_R \leqslant m$. Thus $m \geqslant n = \dim W^R = \dim W - \dim W_R \geqslant m(s-1)$. Hence $s = 2$ and $m = n$.

Set $A_{12} = A$, $A_{21} = B$, and $A_{22} = C$, and define $g = \mathrm{diag}\,(A, A) \in GL(W)$. Then xg^{-1} centralizes N and g induces by conjugation the same automorphismon N as x.

Suppose first $r \mid |N|$. By Sylows theorem x induces an inner automorphism on N. In particular there is a $y_0 \in N$ with $D(y_0) = A$ (i.e. $g = y_0 \in N$) and as xg^{-1} centralizes y_0 we have $ABA^{-1} = B$, $ACA^{-1} = C$, i.e. all matrices A, B, C commute. As $\dim W_R = n$, $\dim W_g = 0$, we have that $\langle g, x\rangle$ is abelian of type (r, r), i.e. $|xg^{-1}|$ is divisible by r. Now $F = C_{\mathrm{End}(U)}(N_0)$ is a field, say isomorphic to $GF(q^t)$. As x induces a proper inner automorphism, we have $t \leqslant n/2$. Then xg^{-1} can be considered as an element in $GL(2, q^t)$ and therefore $t = n/2$. Now as $F \simeq GF(q^{n/2})$, we have that N can be considered as an irreducible subgroup of $GL(2, q^{n/2})$ and we have assertion (ii).

Suppose next that r does not divide $|N|$. Then g and N both act irreducible on U. By [7; Th. A] and [7; 4.2] there is an extraspecial subgroup E in N such that $|E| = 2^{2a+1}$, $r = n + 1 = 2^a + 1$, and E is irreducible on U. As minimal faithful representations of E have degree 2^a we conclude that $C_{\mathrm{End}(U)}(N_0) \simeq GF(q)$ and BA^{-1}, CA^{-1} are scalars. Thus A, B, C all commute and again $|xg^{-1}|$ is divisible by r. Now xg^{-1} can be considered as an element of $GL(2, q)$ and thus r divides $q^2 - 1$, i.e. $n = 2$, $r = 3$. Suppose $B = \lambda A$, $C = \mu A$, then $x^3 = 1$ implies $\mu^{-1} = \lambda$, $\lambda + \mu^2 = 0$. Thus $\mu^3 = -1$ and as 3 does not divide $q - 1$ we have $\lambda = \mu = -1$.

Now as N is a solvable subgroup of $GL(2, q)$ normalized by an element of order 3, we conclude $[R, N] = Q$ and $N = Q * Z$, where $Q \simeq Q_8$ and Z is a subgroup of $Z(GL(W))$. (iii) follows.

2.5. *Suppose* $F^*(G) = N_1 * N_2$, $N_i \trianglelefteq G$ $(i = 1,2)$ *and* $[N_1, R] \neq 1$ *for some* $R \in \mathcal{S}$. *Then one of the following holds*:

(*a*) $N_2 \subseteq Z(G)$.

(*b*) *$F^*(G)$ is abelian, $r = n+1$, and G induces a nontrivial permutation group on the homogenous $F^*(G)$-components.*

(*c*) *N_1, N_2 are cyclic,* $\dim V = r = n+1$.

(*d*) $G = N_1 * N_2$, $\dim V = 2n$, $r \,\big|\, |N_i|$ $(i = 1,2)$ *and N_i is isomorphic to an irreducible, quasisimple subgroup of $SL(2, q^{n/2})$.*

(*e*) $\dim V = 4$, $r = n+1 = 3$, *and either $F^*(G) \simeq Q_8 * Q_8$ and $G/F^*(G) \simeq Z_3$, $Z_3 \times Z_3$ or $F^*(G) \simeq Q_8 * N$, where N is an irreducible, quasisimple subgroup of $SL(2, q)$, $(r \,\big|\, |N|)$, and $G = F^*(G)$ or $G/F^*(G) \simeq Z_3$.*

PROOF. Let W be an irreducible N_1-module in V_1.

Case 1. $V = W$. Then N_2 is cyclic. Suppose $N_2 \not\subseteq Z(G)$. Let $\tilde{R} \in \mathcal{S}$ such that $[N_2, \tilde{R}] \neq 1$. Extending if necessary the ground field, we see, that $\tilde{R}$ permutes the Eingenspaces of y, where $N_2 = \langle y \rangle$. Since $\dim V^{\tilde{R}} = n$, this implies by 2.3 $r = n+1$ and the Eingenspaces have dimension 1. Moreover $\dim V = r$. As N_1 centralizes N_2, N_1 is abelian too and therefore cyclic. In this case we have assertion (*c*). If however $N_2 \subseteq Z(G)$ we have assertion (*a*).

Case 2. $W \subsetneq V$. Suppose first that N_1 has more then one homogenous component on V. Then $G = \langle \mathcal{S} \rangle$ permutes these homogenous components and by 2.3 we have that all homogenous components are of dimension 1, i.e. N_1 is abelian and $\dim W = 1$. As N_2 must fix each homogenous component, $N_1 * N_2$ is abelian and thus assertion (*b*) is true.

So assume N_1 has only one homogenous component, i.e. W is faithful as an N_1-module. We may assume $W \subsetneq U = \langle Wy : y \in R \rangle$. Set $G_0 = N_1 R$. Assume U is not G_0-irreducible and U_1 is a proper G_0-space. Since N_1 acts faithful on U_1 and U/U_1, clearly R acts faithful on both factors, a contradiction. The same argument shows $V = U$. We now can apply 2.4. Thus $\dim V = 2n$.

Exclude first the case $r = n+1 = 3$. Then by 2.4 (ii) we have, that N_1 is isomorphic to an irreducible, nonabelian subgroup of $GL(2, q^{n/2})$ and $r \,\big|\, |N_1|$. As G is generated by 1-irreducible subgroups of order r we have further that $N_1 \simeq SL(2, q_0^f)$, f suitable or $N_1/Z(N_1) \simeq A_5$.

Suppose that assertion (*a*) does not hold. Then there is an $R^* \in \mathcal{S}$, with $[N_2, R^*] \neq 1$. We can not be in case 1 with the pair (N_2, R^*), as otherwise N_1 would be abelian. Also conclusion (*b*) does not hold, so that we have that N_2 is also an irreducible, quasisimple subgroup of $SL(2, q^{n/2})$ and r divides $|N_2|$. Assertion (d) follows.

Finally assume $\dim V = 4$, $r = n + 1 = 3$. If $N_2 \not\subseteq Z(G)$, then depending an weather 3 divides $|N_i|$ or not we have by 2.4 that either N_i is isomorphic to a nonabelian, subgroup of $SL(2, q^{n/2})$, (3 divides $|N_i|$), or that $N_i \simeq Q_8$. Now either (d) or (e) must hold.

2.6. *Suppose* $E(G) \neq 1$. *Then* $E(G)$ *is quasisimple or we have assertion* 2.5 (d).

Proof. Suppose $E = E(G) = E_1 * \ldots * E_t$ and $t \geqslant 2$. If some $E_i \lhd G$, then by 2.5 we are in case (d) of 2.5. So we assume $E_i \ntriangleleft G$ for $1 \leqslant i \leqslant t$. Thus some $R \in \mathcal{S}$ induces a non trivial permutation on $\{E_1, \ldots, E_t\}$ (i.e. $t \geqslant r$). Suppose $E_i^x = E_{i+1}$ $(1 \leqslant i < r)$, $E_r^x = E_1$ for $R = \langle x \rangle$.

Let p be a prime, $p \neq r$, q_0, and p divides $|E_1|$. Let $F \subseteq E_1 * \ldots * E.$ be an R-invariant, noncyclic abelian p-subgroup with $[F, R] = F$ and $F \cap E_i * E_j \not\subseteq Z(E_i * E_j)$ for $i \neq j$. By 2.3 $r = n + 1$ and the homogenous F-components on V^F have dimension 1. By the choice of F we have $\dim V^f \leqslant n + 1$ for $f \in F$.

Let W be an irreducible $E_1 * E_2$-module. As $E_1 * E_2$ is perfect, we have $\dim W^f \geqslant 2$, for $1 \neq f \in F \cap E_1 * E_2$. Let $V = W_1 \oplus \ldots \oplus W_k$ be a decomposition into irreducible $E_1 * E_2$-modules. By 2.4 E is irreducible on V and so $W \simeq W_i$ for $1 \leqslant i \leqslant k$. Moreover $E_3 * \ldots * E_t$ is isomorphic to a subgroup of $GL(k, q^s)$, where $C_{\mathrm{End}(W,W)}(E_1 * E_2) \simeq$ $\simeq GF(q^s)$. A faithful irreducible $E_3 * \ldots * E_t$-module has dimension at least 2^{t-2} and thus $k \geqslant 2^{t-2} \geqslant 2^{r-2} = 2^{n-1}$.

Hence $n + 1 \geqslant \dim V^f \geqslant 2^n$ for $1 \neq f \in F \cap E_1 * E_2$. Thus $n \leqslant 1$, a contradiction.

2.7. *Let* p *be a prime and* W *be a symplectic, nondegenerate finite dimensional* $GF(p)$*-space. Suppose* $\bar{p}$ *is an odd prime* $\bar{p} \neq p$, *and* $Q \subseteq Sp(W)$ *is a cyclic subgroup of order* $\bar{p}$. *Then there is a* $0 \neq w \in W$ *such that* $\langle wx \colon x \in Q \rangle$ *is a nondegenerate subspace of* W.

Proof. The verification of the assertion is straightfoward.

2.8. *Suppose* $R \in \mathcal{S}$ *and* $[F(G), R] \neq 1$. *Then one of the following assertions holds*:

(i) $\dim V = n + 1 = r$ and $F(G)$ *is cyclic*.

(ii) $F(G)$ *contains an abelian, noncyclic normal subgroup* N *of* G, *all homogenous* N*-components have dimension* 1, *and* $r = n + 1$.

(iii) $F(G) = O_2(G)\cdot Z$, *Z is a cyclic $2'$-group,* $Z \subseteq Z(G)$, *and* $r = n+1$. *Moreover* $r = 2^c+1$ *respectively* 2^c-1 *and* $\dim V = n$ *respectively* $\dim V = n+2$. $O_2(G) = E * Z_0$, *where E is extraspecial of order* 2^{2c+1} *and Z_0 is cyclic,* $Z_0 \subseteq Z(G)$.

(iv) $r = n+1 = 3$ *G occurs in* 2.5(*e*) *or* $O_2(G) \simeq Q_8$ (*quaternion group*) *and* $G/O_2(G) \simeq Z_3$ *or* $Z_3 \times Z_3$

PROOF. First assume $r \neq 3$. By 2.5 $E(G) = 1$. Suppose p_1, p_2 are two primes with $[O_{p_1}(G), R_1] \neq 1 \neq [O_{p_2}(G), R_2]$ for $R_1, R_2 \in \mathcal{S}^*$. Again by 2.5 $F^*(G) = F(G)$ is abelian and (i) or (ii) holds. Thus we may assume that $F(G) = O_p(G)\cdot Z$, $[O_p(G), R] \neq 1$ for $R \in \mathcal{S}$ and $Z \subseteq Z(G)$. If $O_p(G)$ is cyclic again (i) is true.

If $O_p(G)$ contains a characteristic, abelian, noncyclic subgroup N, apply 2.3 and act with R on the homogeneous N-components. Now (ii) holds.

We now assume $O_p(G)$ is of symplectic type. Then by [11; III, 13.10] $O_p(G) = E * Z_0$, where E is 1 or extraspecial. If $p > 2$, then E has exponent p and Z_0 is cyclic. If $p = 2$, then Z_0 is cyclic, or $|Z_0| > 8$ and Z_0 is dihedral, generalized quaternion, or quasidihedral. As $r \geqslant 3$, $F = [O_p(G), R]$ is extraspecial, $F = [F, R]$, and R acts fixed-point-free on $F/Z(F)$. By 2.7 there is an extraspecial subgroup $F_1 \subseteq F$, such that $X = \langle R, R^*\rangle = F_1 R$, where $R^* \in \mathcal{S}$. R is fixed-point-free on $F_1/Z(F_1)$. Thus r divides p^c-1 or p^c+1 for $|F_1| = p^{2c+1}$. Further $V = V^X \oplus V_X$ with $\dim V^X \leqslant 2n$. Then $2n \geqslant p^c \geqslant r-1$, respectively $2n+1 \geqslant r$. So $r = 2n+1$ or $n+1$.

Assume first $rm = p^c+1$, then $2n \geqslant p^c \geqslant rm-1 \geqslant m(n+1)-1$. Thus $m = 1$, and $r = p^c+1$, which implies $p = 2$. Suppose $r = 2n+1 = 2^c+1$ i.e. $n = 2^{c-1}$. Since $\dim V^X \leqslant 2^c$ and faithful absolutely irreducible representations of F_1 have degree 2^c, we have $\dim V^X = 2^c = 2n$. Thus X affords an absolutely irreducible, faithful representation on V^X. We inspect the character table of X. Let D be a faithful, irreducible representation of degree 2^c. $D(y)$ has at most one eigenvalue 1 for $1 \neq y \in R$. As $\dim (V^X)_R = n = 2^{c-1}$, we conclude $c = 1$ and R is a group of dilatations, a contradiction. Hence $r = n+1 = 2^c+1$, and R is irreducible on $F_1/Z(F_1)$.

Suppose next $rm = p^c-1$, then $2n \geqslant p^c \geqslant rm+1 \geqslant m(n+1)+1$. Hence $m = 1$, $r = n+1$, and $p = 2$.

As $\tilde{Z} = \Omega_1(Z(F_1)) = \Omega_1\big(Z(O_2(G))\big)$, we have $V^X = V = V^{\tilde{Z}}$. Clearly V^X is an irreducible X-module. Because of 2.4 and $r > 3$, V^X is even an absolutely irreducible F_1-module. Thus $\dim V = 2^c = n$ or

$n+2$, where $r = 2^c + 1$ or $2^c - 1$ respectively and $F_1 = F = E$. $C_{F(G)}(E)$ is cyclic and therefore Z_0 is cyclic. Now (ii) follows.

Finally assume $r = n + 1 = 3$. If G is not as in 2.5(e), we have $F(G) = O_p(G) \times Z$, where Z is a cyclic p'-group in $Z(G)$. If $O_p(G)$ is abelian we have (i) or (ii). If $O_p(G)$ is of symplectic type and irreducible on V, then we get as above assertion (iii). If $O_p(G)$ is of symplectic type but not irreducible we apply 2.4 (iii). This gives us assertion (iv).

2.9. THEOREM. *Suppose* $\mathcal{S}$ *is a collection of* 1-*irreducible subgroups of order* r *(*r *a prime) of* $GL(V)$, $G = \langle \mathcal{S} \rangle$ *is irreducible, and* r *divides* $\varphi_n^*(q)$. *Then one of the following is true*:

(a) $F(G)$ *is cyclic*, G *is metacyclic, and* $\dim V = r = n + 1$.

(b) $F(G)$ *contains an abelian, noncyclic normal subgroup* N, $r = n + 1$, *and homogenous* N-*components have dimension* 1. G/N *induces a transitive permutation group on these homogeneous components.*

(c) $F(G) = O_2(G) \times Z$, Z *is a cyclic* $2'$-*group of* $Z(G)$ *and* $r = n + 1$. *Moreover* $r = 2^c + 1$ *or* $2^c - 1$ *and* $\dim V = n$ *or* $n + 2$ *respectively*, $O_2(G) = E * Z_0$, *where* E *is extraspecial of order* 2^{2c+1} *and* Z_0 *is cyclic*, $Z_0 \subseteq Z(G)$. $G/F(G)$ *acts irreducible on* $F(G)/Z(G)$ *if* $r = 2^c + 1$.

(d) $\dim V = 2n$, *and* G *is described in* 2.8 (iv) *or* 2.5 (d).

(e) $F^*(G) = E(G) * Z$, Z *is cyclic and contained in* $Z(G)$. $E(G)$ *is quasisimple and irreducible.*

PROOF. Suppose first $E(G) = 1$. By 2.8 assertion (a), (b), (c), or (d) follows. Suppose next $E(G) \neq 1$. By 2.6 either assertion (d) holds or $E(G)$ is quasisimple. If the conclusion of (d) does not hold, we have by 2.5 $F(G) = Z(G)$, i.e. $F(G) = Z$ is cyclic and (e) follows.

2.10. (a) *Suppose* $\dim V = 4$, q *is odd, and* 3 *does not divide* $q - 1$. *Let* $X \subseteq GL(V)$, *such that* $X_0 \trianglelefteq X$, $C_X(X_0) = Z(X)$ *and* $X/X_0 \simeq Z_3$. *Then* $X_0/Z(X)$ *is not isomorphic to* $L_3(4)$ *or* $U_4(2)$.

(b) *Let* X *be a perfect central extension of* $L_3(q)$ *or* $U_3(q)$. *Then* X *has no irreducible, projective module of dimension* 4 *over a field of characteristic* q_0.

PROOF. (a) Suppose the assertion is false and $X_0/Z(X) \simeq U_4(2)$ or $L_3(4)$. By [6; p. 302, Tab. 4.1] and the assumptions $Z(X)$ is a 2-group. By [14; 4.6, 5.2, 5.7] $Z(X) \neq 1$. Pick $M \subseteq X$, such that $Z(X) \subseteq M_0 = X_0 \cap M$, $M_0/Z(X) \simeq E_{16} \cdot A_5$, and $|M : M_0| = 3$. Then $F = F(M) =$

$= O_2(M)$ and the 2-rank of $M \leqslant 4$. If F is abelian, then F is of type $(2^a, 2, 2, 2)$ for $a \leqslant 2$, which is impossible.

Suppose F is not of symplectic type. Then there is an elementary abelian group E char F with $|E| \geqslant 4$, and $\Omega_1(Z(X)) \subseteq E$. Hence $|E| = 16$, as $E \nsubseteq Z(M)$. However $\Omega_1(Z(M))$ is fix under M, a contradiction.

Finally we assume that F is of symplectic type, say $F = E * Q$, where E is extraspecial of order 2^5 and Q is cyclic or $|Q| > 8$ and Q is dihedral, generalized quaternion, or quasidihedral. If Q is cyclic, then M/F is isomorphic to a subgroup of $Sp(4, 2)$, which is impossible. If Q is nonabelian, we have $F_0 = C_F(F') = C_F(Q')$ char F, and $|F:F_0| = 2$, $F_0 = E * C$, where C is cyclic and again M/F is isomorphic to a subgroup of $Sp(4, 2)$, a contradiction.

(*b*) This follows as $A_2(q)$ has no projective absolutely irreducible representation of degree 4 over characteristic q_0 (see [15]).

2.11. *Let $H = \mathfrak{X}(p^s)$ be a Chevalley group (twisted or not) over $GF(p^s)$ and σ a field automorphism of H with $GF(p^m)$ being the fixed field belonging to σ, and let s/m be a prime. Then there is a p-element $x \in C_H(\sigma)$ and a $y \in x^H$ such that $H = \langle y, C_H(\sigma)\rangle$.*

PROOF. Take $x = x_r(a)$, $a \in GF(p^m)$, where r corresponds to a long root of $\prod$, a simple root system belonging to $\mathfrak{X}$. By choosing $y = x_r(b)$, $b \in GF(p^s)$ suitable, the assertion becomes clear.

2.12. THEOREM. *Suppose* $\dim V \leqslant 2n$, $\mathcal{S}$ *is a collection of* 1-*irreducible subgroups of order r (r a prime) of $GL(V)$, and $G = \langle \mathcal{S} \rangle$ is irreducible, and r divides $\varphi_n^*(q)$. Assume further $E(G) \neq 1$. Then one of the following holds:*

(*a*) $\dim V = 2n$, $G = E(G) * Z(G)$, *$E(G)$ is an irreducible, quasisimple subgroup of $SL(2, q^n)$ or $SL(2, q^{n/2})$, and $Z(G)/(Z(G) \cap E(G))$ has order r or G is as in* 2.5 (*d*).

(*b*) $\dim V = 4 = r + 1$, $F^*(G) \simeq Q_8 * N$, *N is an irreducible, quasisimple subgroup of $SL(2, q)$, $G = F^*(G)$ or $G/F^*(G) \simeq Z_3$.*

(*c*) *$G = E(G)$ is quasisimple.*

PROOF. Apply 2.9. If we are in case (*d*) of 2.9. we conclude by 2.5 (*d*) and 2.8 (iv), that assertion (*a*) and (*b*) are true.

Suppose now that we are in case (*e*) of 2.9. Set $Z = Z(G)$.

(1) G/Z is isomorphic to a subgroup of Aut $(F^*(G)/Z)$ containing $F^*(G)/Z$.

Let $g \in G$ and assume $[F^*(G), g] \subseteq Z$. By the 3-subgroup lemma we have $[E(G), g] = 1$ and thus $g \in C_G(F^*(G)) = Z$.

(2) If r divides $|Z|$ then assertion (a) is true.

In this case G can be considered as an irreducible subgroup of $GL(2, q^n)$, as now $\dim V = 2n$. The subgroup structure of $GL(2, q^n)$ implies that assertion (a) holds.

From now on we can assume $F^*(G)$ is quasisimple. Suppose $E(G) \subsetneqq G$, i.e. r divides $|G/E(G)|$.

(3) $F^*(G)/Z \simeq \mathfrak{X}(q_0^t)$, where $\mathfrak{X}(q_0^t)$ denotes a Chevalley group (twisted or not) over $GF(q_0^t)$.

Since $r \geqslant 3$, $F^*(G)/Z$ is not sporadic or alternating. Assume $F^*(G)/Z \simeq \mathfrak{X}(p^s)$, where $\mathfrak{X}(p^s)$ denotes a Chevalley group (twisted or not) over $GF(p^s)$, $(p, q_0) = 1$. Denote by l the Lie rank of $\mathfrak{X}$. By a theorem of Steinberg [19] we have $n + 1 \leqslant r \leqslant M = \max\{(l+1,\ p^s - 1),\ (l+1,\ p^s + 1),\ s,\ 3\}$. Let $m = m(\mathfrak{X}(p^s))$ denote the minimal degree of an irreducible, nontrivial projective representation of $\mathfrak{X}(p^s)$ in characteristic q_0. By our assumption we have

$$(+)\qquad\qquad m \leqslant 2n < 2M - 1\,.$$

We now use the result of Landazuri Seitz [14]. This implies that $M \neq s$. Suppose $M = (l+1, p^s - 1) > s$. Then $\mathfrak{X} = A_l$, $l \geqslant 2$. Again by $(+)$ and the above result we have $r = n + 1 = 3$, $m = 4$, and $\mathfrak{X}(p^s) \simeq L_3(4)$. By 2.10 (a) this case is impossible.

Assume $M = (l+1, p^s + 1) > s$. Again by $(+)$ and the result of Landazuri Seitz we have, $r = n + 1 = 3$, $m = 4$, and $\mathfrak{X}(p^s) \simeq U_4(2)$. Again this contradicts 2.10 (a).

Finally assume $M = 3$. By the discussion of the previous cases we only have to consider the cases $\mathfrak{X} = D_4$, ${}^2D_4$, or ${}^3D_4$, which however do not have nontrivial representation of degree $\leqslant 4$.

(4) $G = F^*(G)$.

Now $F^*(G)$ is as in (3) and we denote by l the Lie rank of $\mathfrak{X}(q_0^t)$. Suppose $R = \langle x \rangle \in \mathcal{S}$, $x \in G - F^*(G)$. By Steinbergs theorem x induces an automorphism on $\mathfrak{X}(q_0^t)$ of the form $idfg$, where i denotes an inner, d a diagonal, f a field, and g a graph automorphism.

Suppose first $g \neq 1$. Then $r = 3 = n+1$ and $\dim V \leqslant 4$, which is of course impossible.

Assume that r divides the order of the group of diagonal automorphisms of $\mathfrak{X}(q_0^t)$. Then r divides $l+1$, $l \geqslant 2$, and $\mathfrak{X} = A_l$ or ${}^2A_l$ respectively. Now r divides $q_0^t - 1$ respectively $q_0^t + 1$, and elementary abelian r-subgroups in G have rank at most 2. In particular $\mathfrak{X}(q_0^t)$ contains no subgroup of type (r, r). Thus $l = 2$, $r = n+1 = 3$. By 2.10 (*b*) $\dim V \leqslant 3$. Hence $GF(q)$ contains $GF(q_0^t)$ respectively $GF(q_0^{2t})$ and V is the natural module for $A_2(q_0^t)$ respectively ${}^2A_2(q_0^t)$ read as a $GF(q)$-module. But then r divides $q-1$, a contradiction.

Thus we may assume, that x induces an automorphism of the form if.

Suppose $f \neq 1$. If r does not divide $|E(G)|$ we may even assume $i = 1$. If $y \in C_{E(G)}(x)$ is a q_0-element, then $\dim V^y \leqslant \dim V_R - 1$, and $\dim V^X < \dim V$ for $X = \langle z, C_{E(G)}(x)\rangle$ where $z \in y^G$. By 2.11 we can pick z in such a way, that X convers $E(G)/Z$, a contradiction.

Thus r divides $|E(G)|$ and G contains an abelian subgroup of type (r, r), $\dim V = 2n$. Pick $y \in E(G)$, $|y| = r$, such that xy induces f on $\mathfrak{X}(q_0^t)$. If $\langle xy\rangle$ is 1-irreducible, we are done as before. Thus $V^{xy} = V$ and as $C_{E(G)}(xy)$ involves a group $\mathfrak{X}(q_0^{t/r})$, we have $\mathfrak{X} = A_1$. As G/Z is isomorphic to $L_2(q_0^t)$ extended by an field automorphism, there is a S_{q_0}-subgroup $Q \subseteq E(G)$ being normalized by $\langle xy\rangle$. Thus Q and every nontrivial element in Q acts quadratically on V. Hence V is the direct sum of some copies of the natural $SL(2, q_0^t)$-module read as a $GF(q)$-module, and $E(G) \simeq SL(2, q_0^t)$, $N_G(Q) = Q.K.\langle xy\rangle$, where K is cyclic of order $(q_0^t - 1)$ faithfully on the n-dimensional space V_Q, and $\langle xy\rangle$ normalizes K. By 2.9 (*a*)-(*c*) $K\langle xy\rangle$ is abelian, a contradiction.

Thus x induces an inner automorphism of $E(G)$. By (1) and $Z \subseteq E(G)$ we have $R \subseteq E(G)$, the final contradiction.

3. Solvable normal subgroups.

We denote as in section 2 by G an irreducible subgroup in $GL(V)$, where V is a finite dimensional $GF(q)$-space, $G = \langle \mathfrak{S}\rangle$, and $\mathfrak{S}$ is a collection of 1-irreducible groups of order r, r a prime dividing $\varphi_n^*(q)$. We will determine in this section the groups G of theorem 2.9, which have solvable $F^*(G)$. Of course we only have to consider cases (*b*) or (*c*) of theorem 2.9.

3.1. *Let N be an abelian, noncyclic, normal subgroup and G/N acts faithfully on the homogenous N-components as a permutation group. Then $r = n + 1$ and:*

(i) *If* $\dim V > r$, *then G/N is at least 2-primitive.*

(ii) *If* $\dim V = r$, *then G/N is either 2-transitive or solvable.*

PROOF. Suppose $\dim V = n + k$ and $R \in \mathcal{S}$. R induces an r-cycle on its action on the homogenous components of N. Obviously G/N is primitive. By a theorem of Jordan G/N is k-primitive . Then (i) and (ii) follow by a theorem of Burnside [11; V, 21.3].

3.2 PROPOSITION. *Assumptions as in* 3.1 *and assume* $\dim V = n + k$. *Then one of the following assertions is true:*

(i) $k = 1$ *and G/N is isomorphic to:*

(*a*) Z_r, *a cyclic group of order r.*

(*b*) A_r.

(*c*) $L_l(q)$, $(q^l - 1)/(q - 1) = r$ *and l is a prime.*

(*d*) $r = 11$ *or* 23 *and* $G/N \simeq L_2(11)$, M_{11}, *or* M_{23} *respectively.*

(ii) $k = 2$ *and G/N is isomorphic to:*

(*a*) A_{r+1}.

(*b*) $L_2(r)$.

(*c*) $O_2(G/N)$ *is elementary abelian of orde* $2^m = r + 1$ *and*

$$(G/N)/O_2(G/N) \simeq Z_r \text{ or } SL(m, 2) .$$

(*d*) $r = 11$ *or* 23 *and* $G/N \simeq M_{11}$, M_{12}, *or* M_{24} *respectively.*

(iii) $k \geq 3$ *and* $G/N \simeq A_{n+k}$.

PROOF. Use the list of 2-transitive permutation groups given in [1]. Suppose $k = 1$, then (i) follows by 3.1 and [11; XII, 10.11].

Suppose next, that $k = 2$ and G/N has no regular, normal subgroup. As $2^{d-1} + 2^{d-1} - 1$ and $2^{d-1} - 2^{d-1} - 1$ are not prime, we have $G/N \not\simeq SP(2d, 2)$. Cases (*a*), (*b*), (*d*) of (ii) follow.

Assume that $k = 2$ and E/N is a regular, normal subgroup of G/N. Then $|E/N| - 1 = r$ and $|E/N| = 2^m$. Now (ii) (*c*) follows. Finally if $k \geq 3$ (iii) must hold.

3.3. *Let W be a $2a$-dimensional, symplectic space over $GF(2)$ $r = 2^a - 1$ a prime (i.e. a is a prime too). Suppose $X \subseteq Sp(W)$ is irreducible and X is generated by subgroups of order r. Then:*

(i) *X is simple or $a = 2$.*

(ii) *If X is a Chevalley group of even characteristic, then one of the following is true:*

(*a*) $X \simeq L_2(2^a)$, $Sp(2a, 2)$, $SO^+(2a, 2)$.

(*b*) $a = 2$.

(iii) *If X is a Chevalley group of odd characteristic, alternating or sporadic, then one of the following is true:*

(*a*) $a = 3$, $X \simeq A_7$, $U_3(3)$.

(*b*) $a = 2$.

PROOF. We assume $a > 2$.

(i) Let $R \subseteq X$, $|R| = r$. Then we have an R-decomposition $W = W_1 \oplus W_2$ into isotropic R-invariant spaces. W_1, W_2 are dual as R-spaces. In particular $Z(X) = 1$. By $r = 2^a - 1 > 2a$ and Cliffords theorem, X can not have an abelian, noncyclic $2'$-group, which is normalized by R. Hence $F^*(X) = E(X)$ and $R \subseteq E(X)$. Finally $X = E(X)$ can only have one component, as R is a Sylow r-subgroup of $Sp(W)$.

(ii) The case $X \simeq L_d(2^m)$: Then $m.d \leqslant 2a$ and r divides $|X|_{2'}$. Thus a divides m or some i for $1 \leqslant i \leqslant d$.

If a divides m, then $a = m$, $d = 2$, and $X \simeq L_2(2^a)$.

If a divides i for some $1 \leqslant i \leqslant d$, then $m \leqslant 2$. If $m = 2$, then $i = a = d$ and $X \simeq L_d(4)$. Now X contains a cyclic group of order $(4^d - 1)/3(d, 3)$. Cyclic subgroups of $Sp(W)$ have order at most $2^a + 1$ and therefore $a = 3$. However $Sp(6, 2)$ does not contain $L_3(4)$. So $m = 1$, $d \geqslant a \geqslant d/2$, and $X \simeq SL(d, 2)$. By the proof of [2; 4.2] we have for $a \geqslant 5$: $a = d$ and W posseses an X-invariant isotropic space of dimension d, a contradiction.

Therefore $a \leqslant 4$. If $a = 3$, we have $X \simeq SL(4,2) \simeq SO^+(6, 2)$, as $SL(5, 2) \not\subseteq Sp(6, 2)$ and $SL(3, 2)$ has no irreducible 6-dimensional representation over $GF(2)$. If $a = 4$, again $SL(5, 2) \not\subseteq Sp(8, 2)$ and $X \not\simeq SL(4, 2)$, as $SL(4, 2) \simeq A_8$ has no irreducible representation of degree 8 over $GF(2)$.

The case $X \simeq Sp(2d, 2^m)$: Then $2m \cdot d \leqslant 2a$ and $2^a - 1$ divides $|X|_{2'}$. Thus $a = d$, $m = 1$, and $X \simeq Sp(2a, 2)$.

The case $X \simeq PSO^+(2d, 2^m)$: Then $(d-1)m \leqslant a$, $2^a - 1$ divides $|X|_{2'}$ and $|X|_{2'}$ divides $|Sp(W)|_{2'}$. This yields $m = 1$, $d = a$.

The case $X \simeq PSO^-(2d, 2^m)$: As usual we get $m = 1$, $d = a$. However $2^a - 1$ does not divide $|X|$.

The case $X \simeq U_d(2^m)$: Here $2md \leqslant 2a$ if d is odd and $2m(d-1) \leqslant 2a$ if d is even. As usual $m = 1$. Inspecting the order of $U_d(2)$, we see at once, that $2^a - 1$ does not divide $|X|$.

The case $X \simeq G_2(2^m)$: Here $a \geqslant 3m$ and $2^a - 1$ divides $(2^{6m} - 1) \cdot (2^{2m} - 1)$. Now $m = 1$, $a = 3$. However $G_2(2)' \simeq U_3(3)$, which will fall under (iii).

The case $X \simeq F_4(2^m)$: Here $a \geqslant 6m$ and $2^a - 1$ divides $|X|_{2'}$. As a does not divide m, we have $a = 3$, a contradiction.

In the same manner one rules out the cases $Sz(2^m)$, ${}^2F_4(2^m)$, $E_6(2^m)$, ${}^2E_6(2^m)$, $E_7(2^m)$, and $E_8(2^m)$.

(iii) Suppose first that X is a Chevalley group of odd characteristic and denote by $m(X)$ the minimal degree of an irreducible, nontrivial representation of X over $GF(2)$, i.e. $m(X) \leqslant 2a$. We use for $m(X)$ the bounds of Landazuri and Seitz [14].

The case $X \simeq L_m(q)$, $m \geqslant 3$: Then $2^a - 1 = r \leqslant q^m - 1$ and $q^{m-1} - 1 \leqslant 2a$. As $q \geqslant 3$, $m \geqslant 3$ and a is a prime, we have $a \leqslant 7$. Therefore $q^{m-1} \leqslant 15$ and $q = m = 3$. Hence $2^a \leqslant 27$ and $a = 3$, in contradiction to $6 < 3^2 - 1$.

The case $X \simeq L_2(q)$: Now $2^a - 1 \leqslant q$ and $2a \geqslant (q-1)/2$. Therefore $a = 3$, $r = 7$, and $q \leqslant 13$. Now $L_2(13) \not\subseteq Sp(6, 2)$ and $L_2(7)$, $L_2(9)$ have no irreducible representations of degree 6 over $GF(2)$.

The case $X \simeq PSp(2m, q)$: Here we have $2a \geqslant (q^m - 1)/2$ and $2^a - 1 \leqslant q^m + 1$. This yields $a = 3$, $X \simeq PSp(4, 3) \simeq SO^-(6, 2)$, which falls under (ii).

The case $X \simeq U_m(q)$, $m \geqslant 3$, m odd: Now $2a \geqslant q(q^{m-1} - 1)/(q+1)$ and $2^a - 1 \leqslant q^m + 1$. As $q \geqslant 3$ we have $q^{m-1}/2 \leqslant q(q^{m-1} - 1)/q + 1)$ and $a \leqslant 7$. The above inequalities yield now a contradiction. Similar goes the case $X \simeq U_m(q)$, $m \geqslant 4$, m even, $(m, q) \neq (4, 3)$. Thus we are left with the exceptional case $m = 4$, $q = 3$, where $a = 3$, $r = 7$. However $Sp(6, 2)$ can not contain a Sylow 3-subgroup of $U_4(3)$.

The other cases with Chevalley groups of odd characteristic give in the same manner contradictions, with the exception of $X \simeq ({}^2G_2(3))'$. However ${}^2G_2(3) \simeq \mathrm{Aut}\,(L_2(8))$ and we can exclude this case too.

Suppose now that X is alternating of degree d. Then $2a \geqslant m(X) \geqslant d-2$ for $d \geqslant 9$ and $d \geqslant 2^a - 1$. Thus $a \leqslant 3$, a contradiction. Hence $d \leqslant 8$ and as $2^a - 1$ divides $|X|$ we have $a \leqslant 3$. Thus $X \simeq A_6, A_7, A_8$. However A_6 has no irreducible representation of degree 6 over $GF(2)$ and $A_8 \simeq SO^+(6, 2)$ falls under (ii).

Finally assume that X is sporadic. If $a = 3$, then $|X|$ divides $|Sp(6, 2)|$, which implies $X \simeq J_2$. However considering the 2-local structure of J_2 one observes $J_2 \not\subseteq Sp(6, 2)$.

If $a = 5$, then $r = 31$. But if $31 \,\|\, |X|$, X sporadic, then $|X| \nmid |Sp(10, 2)|$. Primes of the form $2^a - 1$, $a > 5$, never divide the order of a sporadic simple group.

3.4 Proposition. *Suppose $F^*(G) = O_2(G) \times Z$, Z a cyclic $2'$-group in $Z(G)$ and $O_2(G)$ is of symplectic type. Then $r = n + 1$ and one of the following is true*:

(*a*) $\dim V = n$, $r = 2^a + 1$, $a = 2^b$, *and $G/F(G)$ is isomorphic to one of the following groups*: $Sp(2\bar{n}, \bar{q})$, $SO^-(2\bar{n}, \bar{q})$, ($\bar{q}^{2\bar{n}} = 2^{2a}$) *or* A_6 ($a = 2$), $L_2(17)$ ($a = 4$).

(*b*) $\dim V = n + 2$, $r = 2^a - 1$, *a is a prime, and one of the following holds*:

(i) *$O_2(G)$ contains a normal, noncyclic, abelian subgroup of G. Assertion* (*c*) *of* 3.2 (ii) *holds.*

(ii) $a = 2$ *and $G/F^*(G)$ is isomorphic to* A_6, A_5, A_4, $Z_3 \times Z_3$, *or* Z_3.

(iii) $a \geqslant 3$ *and $G/F^*(G)$ is isomorphic to one of the following groups*: $L_2(2^a)$, $Sp(2a, 2)$, $SO^+(2a, 2)$, *or* $a = 3$ *and* $G/F^*(G) \simeq A_7$, $U_3(3)$.

Proof. (*a*) is a result of Hering [8; 4.2]. If $a \geqslant 3$ and $G/F^*(G)$ is not irreducible on $F^*(G)/Z(G)$, then there is a normal abelian noncyclic subgroup in $O_2(G)$ and assertion (*b*) (*i*) holds. So let $F^*(G)/Z(G)$ be a faithful irreducible symplectic $GF(2)$-module of dimension $2a$ for $G/F^*(G)$. Assertion (*b*) (ii) and (iii) follow by 3.3 and inspection of $Sp(4, 2) \simeq S_6$.

4. $F^*(G)$ is nonsolvable of Chevalley type and has characteristic different from q_0.

We have the same general assumptions on G, V, and $\mathcal{S}$ as in sections 2 and 3. Moreover we assume that $F^*(G)/Z(G)$ is a Chevalley group over $GF(\bar{q})$, $(\bar{q}, q_0) = 1$. We also make in this section the additional assumption $\dim V \leqslant 2n$. By theorem 2.12 we may exclude the cases where $E(G) \neq G$.

4.1. PROPOSITION. *Suppose $G = E(G)$ and $\bar{G} = G/Z(G)$ is a Chevalley group $GF(\bar{q})$ with $(\bar{q}, q_0) = 1$. Then one of the following assertions is true*:

(1) $\bar{G} \simeq G_2(3)$, $G_2(4)$, $r = n + 1 = 7$ *or* 13.

(2) $\bar{G} \simeq {}^2G_2(\bar{q})$, $r = n + 1$ *or* $r = 2n + 1 = \bar{q}^2 - \bar{q} + 1$.

(3) $\bar{G} \simeq {}^2D_4(2)$, $r = n + 1 = 17$.

(4) $\bar{G} \simeq Sz(8)$, $r = n + 1 = 5, 13$ *or* $r = 2n + 1 = 13$.

(5) $\bar{G} \simeq PSp(2m, \bar{q})$, $\bar{q}$ *odd*, $r = n + 1$, *or* $r = 2n + 1 = (\bar{q}^m + 1)/2$.

(6) $\bar{G} \simeq PSp(4, 4)$, $r = n + 1 = 17$.

(7) $\bar{G} \simeq PSp(6, 2)$, $r = n + 1 = 5, 7$.

(8) $\bar{G} \simeq U_4(3)$, $r = n + 1 = 5, 7$.

(9) $\bar{G} \simeq U_4(2)$, $r = n + 1 = 5$.

(10) $\bar{G} \simeq U_m(\bar{q})$, m *odd* $\geqslant 3$, $r = n + 1$ *or* $r = 2n + 1 = (\bar{q}^m + 1)/(\bar{q} + 1)$.

(11) $\bar{G} \simeq L_m(2)$, $L_m(3)$, $m \geqslant 3$, $r = 3n + 1 = 2^m - 1$, $(3^m - 1)/2$ *respectively*.

(12) $\bar{G} \simeq L_m(\bar{q})$, $m \geqslant 3$, $r = n + 1 \geqslant (\bar{q}^{m-1} - 1)/2$ *or* $r = 2n + 1 = (\bar{q}^m - 1)/(\bar{q} - 1)$.

(13) $\bar{G} \simeq L_3(2)$, $r = 7 = n + 1$, $2n + 1$, $3n + 1$.

(14) $\bar{G} \simeq L_3(4)$, $r = 7 = n + 1$, $2n + 1$, $3n + 1$.

(15) $\bar{G} \simeq L_2(\bar{q})$, $\bar{q}$ *even*, $r = \bar{q} + 1 = n + 1$ *or* $2n + 1$.

(16) $\bar{G} \simeq L_2(\bar{q})$, $\bar{q}$ *odd* $\geqslant 3$, $\bar{q} \neq r$, $r = n + 1$ *or* $r = 2n + 1 = (\bar{q} + 1)/2$.

(17) $\bar{G} \simeq L_2(r)$, $r = n + 1$, $2n + 1$, $3n + 1$, *or* $4n + 1$.

PROOF. Denote by $m(\bar{G})$ the minimal degree of a nontrivial projective representation of $\bar{G}$ over a field K with Char $K \neq$ Char $GF(q)$. By our assumption $2n \geqslant m(\bar{G})$. Further r has the form $bn + 1$, $b \geqslant 1$. Thus

$$(+) \qquad r \geqslant (b \cdot m(\bar{G})/2) + 1 > b \cdot m(\bar{G})/2 \geqslant m(\bar{G})/2 .$$

We use the table of Landazuri and Seitz [14] for lower bounds for $m(\bar{G})$ and the table of Hering for upper bounds for r[8; tab. 2].

The case $\bar{G} \simeq E_6(\bar{q})$: Here $\bar{q}^3 + \bar{q}^6 + 1 \geqslant r > \bar{q}^9(\bar{q}^2 - 1)/2$, a contradiction.

In a similar way the cases ${}^2E_6(\bar{q})$, $E_7(\bar{q})$, $E_8(\bar{q})$, $F_4(\bar{q})$, ${}^2F_4(\bar{q})$, and ${}^3D_4(\bar{q})$ are ruled out.

The case $\bar{G} \simeq \bar{G}_2(\bar{q})$: Here $\bar{q}^2 + \bar{q} + 1 > (\bar{q}^3 - \bar{q})/2$ for $\bar{q} \geqslant 5$, a contradiction. If $\bar{q} = 3, 4$ then $r \leqslant 13$ and $m(\bar{G}) = 14$ or 12 respectively. Thus $r = n + 1 = 13$ for $\bar{q} = 3$ and $r = n + 1 = 7, 13$ for $\bar{q} = 4$. The case $G_2(2)' \simeq U_3(3)$ is treated later.

The case $\bar{G} \simeq {}^2G_2(\bar{q})$: Here $\bar{q}^2 - \bar{q} + 1 \geqslant r > b(\bar{q}^2 - \bar{q})/2$. If $b = 2$, then $r = 2n + 1 = \bar{q}^2 - \bar{q} + 1$, otherwise $r = n + 1$.

The case $\bar{G} \simeq {}^2D_m(\bar{q})$, $m \geqslant 4$: Then $\bar{q}^m + 1 > (\bar{q}^{m-1} + 1)(\bar{q}^{m-2} - 1)/2$. Therefore $m = 4$ and $\bar{q} = 2$. Thus $17 \geqslant r > b27/2$. $b = 1$ and $r = 17$ follows.

The case $\bar{G} \simeq D_m(\bar{q})$, $m \geqslant 4$, $\bar{q} \neq 2, 3, 5$: Then $(\bar{q}^m - 1)/(\bar{q} - 1) > (\bar{q}^{m-1} - 1)(\bar{q}^{m-2} + 1)/2$, a contradiction.

The case $\bar{G} \simeq D_m(\bar{q})$, $m \geqslant 4$, $\bar{q} = 2, 3, 5$: Now $(\bar{q}^m - 1)/(\bar{q} - 1) > \bar{q}^{m-2}(\bar{q}^{m-1} - 1)/2$. Then $m = 4$ and $\bar{q} = 2$. Now $r = 7$ and $7 < 4(8 - 1)/2$, a contradiction.

The case $\bar{G} \simeq Sz(\bar{q})$: Suppose $\bar{q} = 2^{2c+1}$, $c > 1$. Then $\bar{q} + 2^{c+1} + 1 > 2^{c-1}(\bar{q} - 1)$, a contradiction. For $Sz(8)$ we get $13 \geqslant r = bn + 1 \geqslant 4$. Thus $r = n + 1$ or $r = 2n + 1 = 13$.

The case $\bar{G} \simeq PSp(2m, \bar{q}) \simeq C_m(\bar{q})$, $\bar{q}$ odd: Here $(\bar{q}^m + 1)/2 > b(\bar{q}^m - 1)/4$. Thus $b \leqslant 2$ and if $r = 2n + 1$, then $r = (\bar{q}^m + 1)/2$.

The case $\bar{G} \simeq PSp(2m, \bar{q}) \simeq C_m(\bar{q})$, $\bar{q}$ even, $(m, \bar{q}) \neq (2, 2)$, $(3, 2)$: Then $\bar{q}^m + 1 > \bar{q}^{m-1}(\bar{q}^{m-1} - 1)(\bar{q} - 1)/4$. Thus $m = 2$, $\bar{q} = 4$ and $r = n + 1 = 17$.

We do not consider the case $\bar{G} \simeq PSp(4, 2)' \simeq L_2(9)$ here. Finally $\bar{G} \simeq PSp(6, 2)$. Here $r = n + 1 = 5$ or 7.

The case $\bar{G} \simeq P\Omega(2m + 1, \bar{q}) \simeq B_m(q)$, $\bar{q}$ odd > 5, $m \geqslant 3$: Then $(\bar{q}^m + 1)/2 > (\bar{q}^{2m-2} - 1)/2$, a contradiction.

Suppose next $\bar{q} = 3, 5$ but $(m, \bar{q}) \neq (3, 3)$. Then $\bar{q}^m + 1 > \bar{q}^{m-1} \cdot (\bar{q}^{m-1} - 1)$ a contradiction. Finally if $\bar{G} \simeq P\Omega(7, 3)$, then $r \geqslant 13$, but $m(\bar{G}) \geqslant 27$, in contradiction to $(+)$.

The case $\bar{G} \simeq U_m(\bar{q})$, m odd $\geqslant 3$: Now $(\bar{q}^m + 1)/(m, \bar{q} + 1)(q + 1) > b\bar{q}(\bar{q}^{m-1} - 1)/(\bar{q} + 1)2$. Clearly $(m, \bar{q} + 1) = 1$. If $b = 2$ then $r = 2n + 1 = (\bar{q}^m + 1)/(\bar{q} + 1)$.

The case $\bar{G} \simeq U_m(\bar{q})$, m even $\geqslant 4$, $(m, \bar{q}) \neq (4, 2)$, $(4, 3)$: Suppose first $m \geqslant 6$. Then $(+)$ becomes $\bar{q}^{m-1} + 1 > b(\bar{q}^m - 1)/2$. Therefore $\bar{q} = 2$ and $b = 1$. Thus $2n \leqslant (2^m - 4)/3$. On the other hand $2n \geqslant m(G) =$

$= (2^m - 1)/3$, a contradiction. For $m = 4$ we have $\bar{q}^2 + 1 > (\bar{q}^4 - 1)/ /2(\bar{q} + 1)$, contradicting $\bar{q} \geqslant 4$.

The remaining unitary cases: If $\bar{G} \simeq U_4(3)$, then $2n \geqslant 6$ and $r = = bn + 1 \leqslant 7$. Thus $r = 2n + 1 = 7$ or $r = n + 1$. If $\bar{G} \simeq U_4(2)$, then $2n \geqslant 4$ and $r = bn + 1 \leqslant 5$. Thus $r = 2n + 1 = 5$ or $r = n + 1$.

The case $\bar{G} \simeq L_m(\bar{q})$, $m \geqslant 3$, $(m, \bar{q}) \neq (3, 2)$, $(3, 4)$: Here we get $(\bar{q}^m - 1)/(m, \bar{q} - 1)(\bar{q} - 1) > b(\bar{q}^{m-1} - 1)/2$. This implies $b(m, \bar{q} - 1) \leqslant 3$.

Suppose $b(m, \bar{q} - 1) = 3$. Then the above inequality yields $\bar{q} \leqslant 3$ and therefore $(m, \bar{q} - 1) = 1$, $b = 3$.

Suppose $b(m, \bar{q} - 1) = 2$. If $b = 2$, then $r \leqslant (\bar{q}^{m-1} - 1)/(\bar{q} - 1)$ gives the contradiction $\bar{q} \leqslant 1$. Thus r divides $(\bar{q}^m - 1)/(\bar{q} - 1)$. Moreover $(+)$ implies in this case $r = 2n + 1 = (\bar{q}^m - 1)/(\bar{q} - 1)$. Suppose now $(m, \bar{q} - 1) = 2$ i.e. $b = 1$. Thus $\bar{q}$ is odd and $m \geqslant 4$. As before $r = = n + 1 = (\bar{q}^m - 1)/2(\bar{q} - 1)$.

Finally assume $b(m, \bar{q} - 1) = 1$. Then as usual $r \geqslant (\bar{q}^{m-1} - 1)/2 \cdot \cdot (\bar{q} - 1)$. If $\bar{G} \simeq L_3(2)$ obviously $r = n + 1$ or $r = 7 = 2n + 1$, $3n + 1$ and if $\bar{G} \simeq L_3(4)$, then $m(\bar{G}) = 4$ and $b \leqslant 3$.

The case $\bar{G} \simeq L_2(\bar{q})$, $\bar{q}$ even: Then $\bar{q} + 1 \geqslant bn + 1 = r$ and $2n > > \bar{q} - 1$. Thus $r = n + 1$ or $r = 2n + 1 = \bar{q} + 1$.

The case $\bar{G} \simeq L_2(\bar{q})$, $\bar{q}$ odd $\geqslant 3$: If $\bar{q} \neq r$, we have $\bar{q} > b(\bar{q} - 1)/2$. Thus $b \leqslant 2$ and $r = 2n + 1 = (\bar{q} + 1)/2$ or $r = n + 1$. If $\bar{q} = r$, then $\bar{q} = bn + 1$ and $2n \geqslant (\bar{q} - 1)/2$. Hence $b \leqslant 4$, and if $b = 4$, we have $\bar{q} = r = 4n + 1$.

As a corollary of 4.1 we have:

4.2 *Same assumption on* G *as in* 4.1. *Assume in addition* $\dim V \leqslant \leqslant n + 1$. *Then one of the following is true*:

(1) $\bar{G} \simeq \bar{G}_2(4)$, $r = n + 1 = 13$.

(2) $\bar{G} \simeq {}^2G_2(\bar{q})$, $r = n + 1 = \bar{q}^2 - \bar{q} + 1$.

(3) $\bar{G} \simeq Sz(8)$, $r = n + 1 = 13$.

(4) $\bar{G} \simeq PSp(2m, \bar{q})$, $\bar{q}$ *odd*, $r = n + 1$ *divides* $\bar{q}^m + 1$ *or* $\bar{q}^m - 1$.

(5) $\bar{G} \simeq PSp(6, 2)$, $r = n + 1 = 7$.

(6) $\bar{G} \simeq U_m(\bar{q})$, m *odd* $\geqslant 3$, $r = n + 1 = (\bar{q}^m + 1)/(\bar{q} + 1)$.

(7) $\bar{G} \simeq U_4(3)$, $r = n + 1 = 7$.

(8) $\bar{G} \simeq U_4(2)$, $r = n + 1 = 5, 7$.

(9) $\bar{G} \simeq L_m(2)$, $m \geqslant 3$, $r = 2n + 1 = 2^m - 1$.

(10) $\bar{G} \simeq L_m(2)$, $m \geqslant 3$, $r = n + 1 = 2^{m-1} - 1$.

(11) $\bar{G} \simeq L_m(\bar{q})$, $m \geqslant 3$, $r = n+1 = (\bar{q}^m - 1)/(\bar{q}-1)$.

(12) $\bar{G} \simeq L_2(\bar{q})$, $\bar{q}$ *even*, $r = n+1 = q+1$.

(13) $\bar{G} \simeq L_2(\bar{q})$, $r = \bar{q} = 2n+1$ *or* $r = n+1 = (\bar{q}+1)/2$.

(14) $\bar{G} \simeq L_3(2)$, $r = 3n+1 = 7$.

(15) $\bar{G} \simeq L_3(4)$, $r = 2n+1 = 7$.

5. $F^*(G)$ is nonsolvable of Chevalley type and has characteristic q_0.

Again we have the same general assumptions on G, V, and S as in the sections before. This time we assume that $F^*(G)/Z(G)$ is a Chevalley group of characteristic q_0 and that $\dim V \leqslant 2n$. In view of 2.12 we restrict our attention to the case $G = E(G)$. We need some additional notation:

Let $X = \mathfrak{X}(q_0^a)$ be a universal, nontwisted Chevalley group over $GF(q_0^a)$ of rank l and let $\Pi = \{r_1, \ldots, r_l\}$ be a set of fundamental roots. Denote by $\{\lambda_1, \ldots, \lambda_l\}$ the corresponding set of fundamental dominant weights and set $\Lambda = \{\Sigma c_i \lambda_i : c_i \in \mathbb{Z},\ 0 \leqslant c_i < q_0\}$. To each $\lambda \in \Lambda$ corresponds an absolutely irreducible $GF(q_0^a)$-module $M = M(\lambda)$. If X induces on M the matrix representation $X \ni x \to ((D_\lambda(x)_{ij})$ we get more representations by $X \ni x \to (D_\lambda(x)_{ij}\theta^k)$, $(0 \leqslant k < a)$, where θ is the Frobenius automorphism of $GF(q_0^a)$. Denote by $M^{(0)}(\lambda), \ldots, M^{(a-1)}(\lambda)$ the corresponding $GF(q_0^a)$-modules and call the set of modules $M^{(i)}(\lambda)$ $(\lambda \in \Lambda,\ 0 \leqslant i < a)$ the basic modules. Also we call the $GF(q_0^a)$-module M basic of type $M(\lambda)$ if M is algebric conjugate to some $M^{(i)}(\lambda)$. By Steinbergs tensor product theorem the set $M^{(0)}(\lambda_1) \otimes \ldots \otimes M^{(a-1)}(\lambda_{a-1})$; $\lambda_1, \ldots, \lambda_{a-1} \in \Lambda$ is the set of absolutely irreducible modules of X in characteristic q_0.

Suppose next $X = \mathfrak{X}(q_0^a)$ is a twisted universal Chevalley group. If X is of type ${}^2B_2$, ${}^2F_4$, or ${}^2G_2$, then $GF(q_0^a)$ is a splitting field for X. The basic modules of are obtained as restrictions to X of some basic modules of $B_2(q_0^a)$, $F_4(q_0^a)$, or $G_2(q_0^a)$ respectively, and the tensor product theorem holds too.

Suppose next $\mathfrak{X}$ has a diagram with a symmetry σ of order t (*i.e.* $t = 2$ for A_l, D_l, E_6 respectively $t = 3$ for D_4). Then σ induces a permutation on Λ. Again we obtain the basic modules as restrictions of the corresponding basic modules for the nontwisted Chevalley group over $GF(q_0^{at})$. If $\lambda \in \Lambda$ and $\lambda^\sigma = \lambda$ then $GF(q_0^a)$ is the field of definition for $M^{(i)}(\lambda)$ (as a X-module) and otherwise $GF(q_0^{at})$ is the field of definition. For these facts see for instance [15], [20].

Finally define for $X = \mathfrak{X}(q_0^a)$ (now twisted or not) a number $\mu = \mu(X)$ as follows:

(1) There is a q_0-primitive divisor $\bar{r}$ dividing $|X|$ and $q_0^{\mu a} - 1$.

(2) If $s \neq q_0$ is a prime dividing $|X|$, then s divides $q_0^m - 1$ for some $1 \leqslant m \leqslant \mu a$.

For $X = \mathfrak{X}(q_0^a)$ we keep the following notation:

$t = 1$ for $\mathfrak{X} = {}^2B_2$, ${}^2F_4$, ${}^2G_2$ or if $\mathfrak{X}$ is nontwisted;

$t = 2$ for $\mathfrak{X} = {}^2A_l$, ${}^2D_l$, ${}^2E_6$;

$t = 3$ for $\mathfrak{X} = {}^3D_4$.

If X is simple of type $\mathfrak{X}(q_0^a)$ and M is a projective, irriducible X-module in charactaristic q_0, then M affords an irreducible module of the universal Chevalley group $\mathfrak{X}(q_0^a)$ by a result of Griess (see [6; p. 302]).

Return to G and V. Assume $q = q_0^b$ and $\dim V = n + k$, $k \leqslant n$ and $G/Z(G)$ is a Chevalley group of type $\mathfrak{X}(q_0^a)$. Suppose $GF(q_0^{\bar{a}})$ is a spiitting field for G with $GF(q_0^b) \subseteq GF(q_0^{\bar{a}})$ and $V^* = V \otimes GF(q_0^{\bar{a}}) = V_1 \oplus \oplus \ldots \oplus V_s$ is a decomposition into absolutely irreducible modules. Denote by $GF(q_0^c)$ the field of definition for V_1 (i.e. $c|at$). By 2.2 s $= c/(b, c)$, r divides $\varphi^*_{n/s}(q_0^c)$ and if M is an irreducible $GF(q_0^c)$-module with $V_1 \simeq M \otimes GF(q_0^a)$, then $R \in \mathcal{S}$ induces a 1-irreducible group on M. Set $n' = n/s$, $k' = k/s$, i.e. $\dim M = n' + k'$ and $\dim M^R = n'$. By our assumptions

$$(+) \qquad n'c \leqslant \mu a \leqslant (n' + k')c, \quad \text{i.e.} \quad |M| \leqslant q_0^{2\mu a}.$$

Throughout this section we will keep this notation. The following results are consequences of some work of Liebeck [15], [16].

5.1. *Notations as above. Suppose M is not basic, then one of the following holds*:

(*a*) *If $\mathfrak{X}$ is not twisted or of type ${}^2B_2$, ${}^2F_4$, ${}^2G_2$, then*

(i) $\mathfrak{X} = A_2, A_3, C_2$ *and M is algebraic conjugate to $V_0 \otimes V_0^\sigma$ where $V_0 \simeq M(\lambda_1)$ is the standard module and σ is the involution in* Aut $(GF(q_0^a))$, $a = 2c$.

(ii) $\mathfrak{X} = A_1$ *and M is algebraic conjugate to $V_0^{\sigma_1} \otimes \ldots \otimes V_0^{\sigma_p}$ $a = cp (p = 2, 3, 4)$ and $\{\sigma_1, \ldots, \sigma_p\}$ is a cyclic subgroup of order p in* Aut $(GF(q_0^a))$, *$V_0 \simeq M(\lambda_1)$ is the standard module.*

(*b*) *Suppose* $\mathfrak{X} = {}^2A_l$, ${}^2D_l$, ${}^2E_6$, *or* ${}^3D_4$. *Then* $\mathfrak{X} = {}^2A_2$ *and* M *is algebraic conjugate to* $V_0 \otimes V_0^\sigma$, *where* σ *is the involution in* Aut $(GF(q_0^{2a}))$ *and* V_0 *is the standard module.*

Proof. (*a*) Set $m = m_G$ for the minimal degree of a nontrivial, irreducible, projective representation of G over $GF(q_0^a)$. By [15] $2\mu a \geqslant cm^{a/c}$. By [8] and [15] $\mu \leqslant m$ and thus $2a/c \geqslant m^{(a/c)-1}$. So if $m \geqslant 5$ we have $a = c$. However the degree m is only obtained for basic modules and thus by our assumptions dim $M \geqslant m^2$, a contradiction. We remain with the cases A_l ($l \leqslant 3$) and C_2. Since M is not basic, we have $a > c$ for $m > 2$ by the tensor product theorem. For A_3, C_2 we have $2a/c \geqslant 4^{(a/c)-1}$. Thus $a = 2c$ and M is as in (*a*) (i). For A_2 we have $2a/c \geqslant 3^{(a/c)-1}$ and again $a = 2c$. As A_2 has no basic module of dimension 4 again assertion (*a*) (i) follows. (*a*) (ii) follows by a well known theorem of Brauer and Nesbitt and (+).

(*b*) As before denote by m the minimal degree of a nontrivial, projective, absolutely irreducible representation of G over $GF(q_0^{at})$ and by m_1 the minimal degree of a nontrivial, projective, absolutely irreducible representation of G over $GF(q_0^a)$. By [8] and [15] we have the following table:

	μ	m_1	m
${}^2A_l$	$2(l+1)$, l even; $2l$, l odd	$l(l+1)$ $(l \geqslant 6)$ 20 $(l = 4, 5)$ 6 $(l = 2, 3)$	$l+1$
${}^2D_l$	$2l$	$2l$	$2l$
${}^2E_6$	18	72	27
${}^3D_4$	12	24	8

We have $c < ta$ by the same argument as above.

Case 1. $a = c$. Then $2\mu \geqslant m_1$. Thus ${}^2E_6$ falls out. By [16] for ${}^3D_4$ the value m_1 is only obtained for basic modules and so this case can not occur too. For ${}^2D_l$ we have dim $M \geqslant (2l)^2$ as M is not basic, which is impossible. In the case ${}^2A_l$ for $l \geqslant 4$ the value m_1 is only obtained for basic modules [16]. Thus $l \leqslant 3$, $\mu = 6$, and dim $M \geqslant (l+1)^2$. Thus $l = 2$ and assertion (*b*) holds.

Case 2. $a > c$. Then $2\mu a \geqslant cm^{ta/c}$ or $2\mu a \geqslant cm_1^{a/c}$. Hence $2\mu a/c \geqslant m^{ta/c}$ or $2\mu a/c \geqslant m_1^{a/c}$, which is impossible by the above table.

5.2. THEOREM. *Let G, V, M be as in the introduction of this section. Then the following assertions are true*:

(i) $\mathfrak{X} \neq E_6$, ${}^2E_6$, E_7, E_8, ${}^2F_4$.

(ii) *If M is not basic, then M is as in* 5.1.

(iii) *M is basic and furthermore*:

(1) *If $\mathfrak{X} = A_l$ $(l \geqslant 1)$, then M is basic of type $M(\lambda_1)$, $M(\lambda_l)$ or $M(\lambda_2)$, $M(\lambda_{l-1})$ $(l = 3, 4)$ or $M(2\lambda_1)$, $M(2\lambda_2)$ $(l = 1,2)$ or $M(3\lambda_1)$, $M(4\lambda_1)$ $(l = 1)$, $q_0 \geqslant 5$.*

(2) *If $\mathfrak{X} = B_l$ $(l \geqslant 3)$, q_0 odd, then M is basic of type $M(\lambda_1)$ or $M(\lambda_l)$ $(l \leqslant 4)$.*

(3) *If $\mathfrak{X} = C_l$ $(l \geqslant 2)$, then M is basic of type $M(\lambda_l)$ or $M(\lambda_1)$ $(l \leqslant 4)$.*

(4) *If $\mathfrak{X} = D_l$ $(l \geqslant 4)$, then M is of type $M(\lambda_1)$ or $M(\lambda_l)$ $(l = 4)$.*

(5) *If $\mathfrak{X} = G_2$, then M is basic of type $M(\lambda_1)$ or $q_0 = 3$ and M is of type $M(\lambda_2)$.*

(6) *If $\mathfrak{X} = {}^2A_l$ $(l \geqslant 2)$ and $c = 2a$ then M is a restriction of a module from* (1) *of $A_l(q_0^{2a})$ to ${}^2A_l(q_0^a)$. If $a = c$, then M is basic of type $M(\lambda_1 + \lambda_2)$ $(l = 2)$, $M(\lambda_2)$, $M(\lambda_1 + \lambda_3)$ $(l = 3)$, or $M(\lambda_3)$ $(l = 5)$.*

(7) *If $\mathfrak{X} = {}^2B_2$, then M is basic of type $M(\lambda_1)$.*

(8) *If $\mathfrak{X} = {}^2D_l$ $(l \geqslant 4)$, then M is a restriction of a module of* (4) *from $D_l(q_0^{2a})$ to ${}^2D_l(q_0^a)$.*

(9) *If $\mathfrak{X} = {}^3D_4$, then M is basic of type $M(\lambda_1)$, $M(\lambda_3)$, or $M(\lambda_4)$.*

(10) *If $\mathfrak{X} = {}^2G_2$, then M is described in* (5) *(restriction to the subgroup ${}^2G_2(q_0^a)$ of $G_2(q_0^a)$).*

PROOF. By 5.1 we may assume that M is basic and that c divides at. Now $c = ai$, where $i = 1$ or t as M is basic. Thus $i \cdot \dim M \leqslant 2\mu$, The bounds for the minimal degree of a nontrivial, irreducible. projective representation of G over $GF(q_0^{ai})$ [15] and the values of μ exclude the cases E_6, ${}^2E_6$, E_7, E_8, F_4, ${}^2F_4$: (i) follows. Next assume $\mathfrak{X} = A_l$ or C_l. [16; theorem 2.2 and 2.6] give assertions (iii) (1) and (3). (2), (4), and (8) follow by [15; theorem 1.1]. (5) and (10) are [16; theorem 2.10]. For $\mathfrak{X} = {}^2A_l$ use that absolutely irreducible modules are obtained as restrictions of absolutely irreducible $A_l(q_0^{2a})$-modules to ${}^2A_l(q_0^a)$. Also use $\dim M(2(\lambda_1+\lambda_2)) \geqslant 12$ for $l=2$ (see proof of theorem 2.2 of [16]). This shows (6)·(7) follows from a theorem of Martineau [17]. For (9) finally use that every absolutely irreducible $GF(q_0^a)$-module has dimension $\geqslant 26$ [16].

6. Translation planes and 1-irreducible groups.

Let V be a finite dimensional vectorspace over $GF(p)$, p a prime, $W = V \oplus V$, and denote by π a spread on W with components $V_\infty, V_0, V_1, \ldots$ such that $\mathcal{T} = (W, \pi)$ becomes a translation plane. We will call a subgroup R of the collineation group of $\mathcal{T}$ 1-component irreducible, if there is an R-invariant component V^* such that R is 1-irreducible and faithful on V^* and W/V^*. Let r be a p-primitive divisor of $p^n - 1$ and set $\mathcal{S}^* = \{R \mid |R| = r,\ R$ is 1-component irreducible$\}$. We will consider subgroups of Aut $(\mathcal{T})$ generated by subsets of $\mathcal{S}^*$ and also always assume $n \geqslant 2$.

Starting point for these considerations are investigations of Iha, Kallaher, Hiramine and others (see for instance [10], [12], [13]) on so called « (G, Γ, n, q)-translation planes ». These are planes $\mathcal{T} = (W, \pi)$ with kern $GF(q)$, q a power of p, which satisfy the following hypothesis:

HYPOTHESIS 6.0. $\mathcal{T}$ posesses a collineation group G, which fixes on l_∞ a set Δ of $q+1$ points and acts transitively on the remaining points $\Gamma = l_\infty - \Delta$.

Thus one has two problems:

(*a*) Determine the structure of G.

(*b*) Determine the isomorphism type of $\mathcal{T}$.

We use the results of the previous sections as a natural approach to question (*a*). It is useful for applications to weaken hypothesis 6.0.

HYPOTHESIS 6.1. Let $\mathcal{T}$ be a translation plane of order p^{n+k}. G is a collineation group of $\mathcal{T}$ fixing a set Δ of p^k+1 points on l_∞ and acting transitively on $l_\infty - \Delta$.

HYPOTHESIS 6.2. Let $\mathcal{T}$ be a translation plane of order p^{n+k}. Let $G = \langle R\colon R \in \mathcal{S}^*,\ R \subseteq G\rangle$ be a collineation group of $\mathcal{T}$. G fixes a set Δ of p^k+1 points and every G-orbit on $l_\infty - \Delta$ has a lenght divisible by r.

HYPOTHESIS 6.3. Let $\mathfrak{T}$ be a translation plane of order p^{n+k}. Let $G = \langle R\colon R \in \mathcal{S}^*,\ R \subseteq G\rangle$ be a collineation group of $\mathfrak{T}$. G fixes a component, say V^*.

Obviously hypothesis 6.0 implies hypothesis 6.1. Moreover:

6.1. (*a*) *Suppose $\mathfrak{T}$ and G satisfy hypothesis* 6.1, $k < n$, *and* $n \neq 2k$. *Then G contains a normal subgroup G_0, such that $\mathfrak{T}$ and G_0 satisfy hypothesis* 6.2. *Moreover Δ is fixed by G_0 pointwise.*

(*b*) *Suppose $\mathfrak{T}$ and G satisfy hypothesis* 6.2, $k < n$, *and* $n \neq 2k$. *Then $\mathfrak{T}$ and G satisfy hypothesis* 6.3. *Moreover Δ is fixed by G pointwise.*

PROOF. (*a*) Let G satisfy hypothesis 6.1 and pick $\tilde{R} \in \mathrm{Syl}_r(G)$. As $r \nmid p^k+1$ there is a fixed component V^* belonging to Δ. As $k \neq n$, $\dim (V^*)_{\tilde{R}}$, $\dim (W/V^*)_{\tilde{R}} \geqslant k$, and thus $\tilde{R}$ is 1-component irreducible with $\dim (V^*)_{\tilde{R}} = \dim (W/V^*)_{\tilde{R}} = k$, i.e. $\Omega_1(\tilde{R}) \in \mathcal{S}^*$. Now $\tilde{R}$ fixes at least two components of Δ and $\dim W_{\tilde{R}} = 2k$. Thus the intersection of $W_{\tilde{R}}$ with the fixed components gives a subplane of order p^k. So $\tilde{R}$ fixes precisely p^k+1 components. In particular $|\tilde{R}|$ is the r-part of $p^n - 1$. Thus $\tilde{R}$ is semi-regular on $l_\infty - \Delta$ and Δ is fixed pointwise by $\tilde{R}$. As $\Omega_1(\tilde{R}) \in \mathcal{S}^*$ we may set $G_0 = \langle \Omega_1(\tilde{R})\colon \tilde{R} \in \mathrm{Syl}_r(G)\rangle$.

(*b*) As in (*a*) one observes, that $\tilde{R} \in \mathrm{Syl}_r(G)$ fixes a subplane of order p^k. Thus $R = \Omega_1(\tilde{R})$ has precisely the same fixed structure as $\tilde{R}$. By our assumptions R is semi-regular on $l_\infty - \Delta$ and the components of Δ are exactly the fixed components of R. Hypothesis 6.3 follows.

In the remainder of this section we do not aim for best possible results. The applications of results of the previous sections and the proofs are rather crude.

6.2. *Let V be a $(n+k)$-dimensional $GF(p)$-space, $k \leqslant n$. Let $r \mid \varphi_n^*(p)$ be a prime and as in sections* 1-5 *denote by $\mathcal{S}$ the set of 1-irreducible subgroups of $GL(V)$ of order r. Suppose $G = \langle R\colon R \in \mathcal{S},\ R \subseteq G\rangle$ and Δ is a subset of $GL(V)$ centralized by G with the properties*:

$$|\Delta| = p^k - 1\,. \tag{1}$$

$$t - t' \in GL(V) \quad \textit{for } t, t' \in,\ t \neq t'\,. \tag{2}$$

Then one of the following assertions is true:

(*a*) $n = dk$, *V can be considered as a $(d+1)$-dimensional $GF(p^k)$-module of G. Either V is irreducible or the composition factors have $GF(p^k)$-dimension* 1 *and d.*

(*b*) *$V = V_R \oplus V^R$ is a G-composition for $R \in \mathcal{S}$, $R \subseteq G$. $V_R = V_G$ and V^R can be considered as an d-dimensional $GF(p^l)$-module of G, where $n = dl$, and $l \geqslant k$.*

PROOF. Set $D = \langle \Delta \rangle \subseteq C_{GL(V)}(G)$ and $U = V^R$. By assumptions (1) and (2), $V_1 = V_R$ is an irreducible D-module. As R is irreducible on U, we have $U = V_2 \oplus \ldots \oplus V_d$, with irreducible isomorphic D-modules $V_i (2 \leqslant i \leqslant d)$.

If $V_1 \simeq V_2$, then $n = dk$, D is isomorphic to the multiplicative group of $GF(p^k)$ and assertion (*a*) follows.

If $V_1 \not\simeq V_2$, then $V = V^R \oplus V_R$ is a G-invariant decomposition. If $GF(p^l) \simeq C_{\mathrm{End}(V^R)}(G)$, then $l \geqslant k$ as $\Delta \subseteq C_{GL(V)}(G)$, and (*b*) follows.

6.3. *Let V be a finite dimensional vectorspace over $GF(q)$, q odd. Suppose $r = n + 1$ is a prime and $r \mid \varphi_n^*(q)$. Let R be a* 1*-irreducible subgroup of order r in $GL(V)$, P an elementary abelian* 2*-group in $GL(V)$ with $P = [R, P]$ and* $\dim U_x \leqslant (\dim U)/2$ *for $x \in P - 1$, where $U = V^X$ and $X = PR$. Then* $\dim U = r = 2^l - 1$, *for* $|P| = 2^l$. *Moreover* $\dim U_X = 2^{l-1} - 1$ *for $x \in P - 1$.*

PROOF. By an easy induction one proves the following lemma:

Let P be an elementary abelian subgroup of $GL(U)$, U a m-dimensional vectorspace over $GF(q)$. Suppose $|P| = 2^l$ and $\dim U_X \leqslant m/2$ *for $x \in P - 1$. Then*

(*a*) $m \geqslant 2^{l-1}$.

(*b*) *If* $U_P = 0$, $\dim U_x = \dim U_y = s \neq 0$ *for* $x, y \in P - 1$, *then* $m = b(2^l - 1)$ *and* $s = b(2^{l-1} - 1)$ *for some* $b \geqslant 1$.

Now consider $X = PR$ and set $U = V^X$. By 2.2 $\dim U = n + 1 = r$ and $U_P = 0$. By (*a*) of the lemma $r \geqslant 2^{l-1}$, where $|P| = 2^l$. By Maschkes theorem r divides $2^l - 1$. Thus $r = 2^l - 1$ and R is transitive on $P - 1$. Thus part (*b*) of the lemma applies and we are done.

6.4. PROPOSITION. *Let G satisfy hypothesis* 6.3 *and let V_1 be a fixed component of G. Set $P = C_G(V_1, W/V_1)$. Then the following assertions hold*:

(*a*) (i) *Either* G *has on* V_1 *a* G*-invariant chain* $0 \subseteq V_1^* \subset V_1^{**} \subseteq V_1$ *such that* $V_1^* = (V_1)_G \cap (V_1)^G$, $V_1^{**} = (V_1)^G$ *and the pair* V_1^{**}/V_1^*, $G/C_G(V_1^{**}/V_1^*)$ *satisfies the general assumptions of section* 1 (*with* $q = q_0 = p$) *or*

(ii) $\dim V_1 = 2n$ *and* V_1 *posesses an irreducible* G*-subspace* V_1^* *of dimension* n *and* V_1/V_1^* *is irreducible.*

(*b*) *The analogous statement of* (*a*) *holds with* W/V_1 *in the role of* V_1.

(*c*) (i) *Either* G/P *is faithful both on* V_1 *and* W/V_1 *or*

(ii) *There is a component* $V_2 \neq V_1$ *and a subgroup* $1 \neq H \subseteq G$ *with* $H = H_1 \times H_2$, $H_i = C_H(V_i)(i = 1, 2)$ *is a group of homologies.* $G = G_0\, P$ *is a semidirect product with* $G_0 = N_G(H)$. *Finally* G_0 *is generated by* 1*-component irreducible groups of order* r.

PROOF. (*a*) and (*b*) follow by 2.1. Set $P_1 = C_G(V_1)$, $P_2 = C_G(W/V_1)$, and $N = P_1P_2$. If $P_1 = P_2 = P$ assertion (*c*) (i) holds. Let H be a complement of P in N. By the modular law we have $P_i = H_iP$ with $H_i = P_i \cap H(i = 1, 2)$. Thus $H = H_1 \times H_2$. Clearly there are precisely two components-say V_1 and V_2-fixed by H and H_i is a group of homologies. Set $G_0 = N_G(H)$. By a Frattini argument we have $G = G_0P$. As $P \cap G_0$ must fix V_1 and V_2 we have $P \cap G_0 = 1$.

6.5. *Assumptions and notation as in* 6.4. *Then one of the following assertions are true*:

(*a*) G *has on* V_1 *and* W/V_1 *precisely one nontrivial irreducible composition factor.*

(*b*) G *has on* V_1 *and* W/V_1 *precisely two nontivial irreducible composition factors. Moreover one of the following holds*:

(i) G/P *is faithful on* V_1 *and* W/V_1 *and* $G/0_p(G) \simeq A \times B$, *where* A, B *are isomorphic to one of the following groups*: Z_r, $SL(2, 3)$, *or* $SL(2, 5)$.

(ii) $H \neq 1$, $G_0 \simeq Z_r \times Z_r \times P_1Z_r$, *where* P_1 *is an elementary abelian* p*-group and* P_1Z_r *is a Frobenius group.*

(*c*) G *has one nontrivial irreducible composition factor on* V_1 *and two on* W/V_1. *Moreover one of the following holds*:

(i) G/P *is a semidirect product of an elementary abelian* p*-group* P_1 *with* $Z_r \times Z_r$. *Moreover* G/P *is faithful on* W/V_1 *and induces a Frobenius group of order* $|P|\ r$ *on* V_1.

(ii) $\dim V_1 = 4$, $H_2 \simeq Q_8$ *and* $G_0/H_2 \simeq Z_3 \times Z_3$.

(iii) $\dim V_1 = 4$, $H_2 \simeq Q_8 \times Z_3$, *and* $G_0/H_2 \simeq Z_3 \times Z_3$.

(iv) $\dim V_1 = 2n$, $n = 2$ *or* 4, $r = 3$ *or* 5 *respectively*, $H_2 \simeq SL(2, 3)$ *or* $SL(2, 5)$, $G_0/H_2 \simeq Z_r \times Z_r$, ($r = 5$ *only if* $H_2 \simeq SL(2, 5)$).

(d) *Exchange the roles of* V_1 *and* W/V_1 *in* (c). *The analogous statements of* (c) *hold.*

PROOF. Assume first that we are in situation (d). Thus we have a G-invariant chain $0 \subset \tilde{V}_1 \subset V_1$, where $\tilde{V}_1$, $V_1/\tilde{V}_1$ are irreducible modules of dimension n. Further we assume, that G has on W/V_1 precisely one nontrivial irreducible composition factor.

Case 1. $H \neq 1$. In this case we assume $G = G_0$. First assume $0_p(G/H_1) \neq 1$. Then there is a p-group $\tilde{P} \subseteq G$, $\tilde{P}H_1 \lhd G$. $|\tilde{P}| \geqslant p^n$ as $R \in \mathcal{S}^*$, $R \subseteq G$ acts fixed-point -free on $\tilde{P}H_1/H_1$. H_1 is fixed-point-free on V_2. So every Sylow subgroup of H_1 is cyclic or generalized quaternion. Considering the group $H_1\tilde{P}R$, we conclude $H_1\tilde{P} = H_1 \times \tilde{P}$, and $\tilde{P} \trianglelefteq G$.

Thus $0 \subset \tilde{V}_2 \subset V_2$ is a G-invariant chain with $\tilde{V}_2 = (V_2)_{\tilde{P}}$. Suppose for instance, that $\tilde{V}_2$ is the irreducible nontrivial composition factor of G on V_2. Then every subgroup of type (r, r) induces on $\tilde{V}_2$ a group of order r, (i.e. $r| \, |H_2|$). Thus G/H_2 is a Frobenius group of order $|\tilde{P}|r$, $H = H_2$, and $|H| = r$. Now we have assertion (d) (i).

So from now on we assume $0_p(G/H_1) = 1$. By a symmetric argument as above we have $0_p(G/H_2) = 1$ too. Set $\mathcal{S}_1 = \{R \subseteq G: R \in \mathcal{S}^*, \tilde{V}_1 = (V_1)_R\}$, $\mathcal{S}_2 = \{R \subseteq G: R \in \mathcal{S}^*, \tilde{V}_1 = (V_1)^R\}$ and $G_i = \langle \mathcal{S}_i \rangle$ ($i = 1, 2$). Then $G_i \trianglelefteq G = G_1G_2$. Now $[G_1, G_2]$ stabilizes the chain $0 \subset \tilde{V}_1 \subset V_1$ and thus $G/H_1 \simeq G_1H_1/H_1 \times G_2H_1/H_1$ and V_1 has a G-decomposition of the form $V_1 = \tilde{V}_1 \oplus U_1$.

Pick $R_i \in \mathcal{S}_i$ ($i = 1, 2$) and set $X = \langle R_1, R_2 \rangle$. Now $[R_1, R_2] \subseteq H_1$. Thus $X/X_1 \simeq Z_r \times Z_r$, where $X_1 = X \cap H_1$.

If for all choices of R_i we have $[R_1, R_2] = 1$, then $[R_1, G_2] = 1$, which implies $G \simeq Z_r \cdot Z_r$, and G induces a group of order r on V_2.

So assume $X_1 \neq 1$. Suppose first $(r, |H_1|) = 1$, i.e. $(r, |X_1|) = 1$. If $T \in \mathrm{Syl}_t(X)$ for a prime t, we have $\tilde{R} \subseteq N_X(T)$ for some $\tilde{R} \in \mathrm{Syl}_r(X)$. Apply the well known structure theorem of Zassenhaus on Frobenius complements. Thus T is cyclic or generalized quaternion. In the first case $[T, \tilde{R}] = 1$ (see 2.8) and in the second case $[T, \tilde{R}] \neq 1$ can only hold if $T \simeq Q_8$, and $r = 3$. We conclude $X_1 \simeq Q_8$, $r = 3$, and $\dim V_1 = 4$, as $X_1 \nsubseteq Z(X)$. In particular H_1 must be solvable by the result of Zassenhaus. $G_i(i=1,2)$ is solvable as G_i contains a subgroup of index 3 centralizing $X_1/Z(H_1)$. Now G is solvable. Apply 2.9 and conclude $H_1 = X_1 \simeq Q_8$. $G_i \simeq Q_8 Z_3$ or $(Q_8 * Q_8)Z_3$, because $H_1 \subseteq G_i$ and G_i is faithful on V_2. However in the second case G_i would induce on V_1 the group $(Q_8 * Q_8)Z_3/Q_8 \simeq A_4$, a contradiction. Hence $G \simeq$ $\simeq Q_8(Z_3 \times Z_3)$. Assertion (ii) of (*d*) follows.

Suppose now $r \| H_1|$. First assume $r \| 0(H_1)|$. It is trivial to verify the following observation:

(+) *Let U be a $2n$-dimensional $GF(p)$-vectorspace and Y be a subgroup of $GL(U)$ acting fixed-point-free on U. Let $\tilde{R} \in \mathrm{Syl}_r(Y)$ (as usual r is a prime dividing $\varphi_n^*(p)$). If Y is a Z-group then R* char *Y.*

By (+) $\tilde{R} \trianglelefteq G$ for $\tilde{R} \in \mathrm{Syl}_r(0(H_1))$. As S_r-subgroups of G are abelian we have $\tilde{R} \leqslant Z(G)$. So if $T \in \mathrm{Syl}_t(H_1)$ for aprime t, we have $\Omega_1(R^*) \subseteq$ $\subseteq N_G(T)$ for a suitable $R^* \in \mathrm{Syl}_r(G)$. Thus $[\Omega_1(R^*), T] = 1$ if t is odd. We conclude $0(H_1) \subseteq Z(G)$, and as H_1 is not central in G, we have $Q \simeq Q_8$ for $Q \in \mathrm{Syl}_2(H_1)$. Moreover by the result of Zassenhaus H_1 is solvable. Since G is generated by elements of order 3 we conclude in the same manner as above, that $H_1 \simeq Z_3 \times Q_8$, $G/H_1 \simeq Z_3 \times Z_3$ showing (iii) of (*d*).

Finally assume $r \| H_1|$ but $(r, |0(H_1)|) = 1$. By the result of Zassenhaus H_1 has a subgroup Y of index 1 or 2 of the form $Y = M \times S$, where $(|M|, |S|) = 1$, $S \simeq SL(2,3)$ or $SL(2,5)$ $(r = 3$ or $5)$ and M is a Z-group. As usual $M \subseteq Z(G)$, which implies even $M = 1$. Now $R \in \mathcal{S}^*$, $R \subseteq G$ induces an inner automorphism on S. Thus $G_i H_1 =$ $= H_1 * K_i$ $(i = 1, 2)$ and K_i is faithfully induced on V_1. Hence $|K_i|$ is odd implying $K_i \simeq Z_r$. Therefore $H_1 \simeq SL(2,3)$ or $SL(2,5)$ and $G/H_1 \simeq Z_r \times Z_r$. (*d*) (iv) follows.

Case 2. $H = 1$. As $\bar{G}$ is faithful on V_1 and W/V_1 and irreducible on W/V_1, we have $O_p(\bar{G}) = 1$. Now $\bar{G} = \bar{G}_1 \times \bar{G}_2$, where $\bar{G}_i$ is defined analogous as above. Hence $(W/V_1)^{\bar{G}_1} = W/V_1$. But then $(W/V_1)^{\bar{R}} =$ $= W/V_1$ for $R \in \mathcal{S}^*$, $\bar{R} \subseteq \bar{G}_1$, a contradiction.

Now (*c*) and (*d*) follow.

Assume next, that G has both on V_1 and W/V_1 two nontrivial composition factors.

Case 1. $H \neq 1$. Again we assume $G = G_0$. Then we have for the fixed components G-invariant chains of the form $0 \subset \tilde{V}_i \subset V_i$ $(i = 1, 2)$. Set $P^* = O_p(G)$. Then P^* stabilizes both of the above chains and

$HP^* = H_1 \times H_2 \times P^*$. Set

$$\mathcal{S}_1 = \{R \subseteq G \colon R \in \mathcal{S}^*,\ \tilde{V}_1 = V_1^R\}\,, \qquad \mathcal{S}_2 = \{R \subseteq G \colon R \in \mathcal{S}^*\,,\ \tilde{V}_1 = (V_1)_R\}$$

and $G_i = \langle \mathcal{S}_i \rangle$ $(i = 1, 2)$.

Suppose G_1 has on V_2 two nontrivial irreducible composition factors. By parts (c) and (d) we have that $G_1P^*H/H_1 \simeq Z_r$, and $G_1P^*H/H_2 \simeq \simeq Z_r \times Z_r$. If we define now

$$\tilde{\mathcal{S}}_1 = \{R \subseteq G \colon R \in \mathcal{S}^*,\ \tilde{V}_2 = V_2^R\}\,, \qquad \tilde{\mathcal{S}}_2 = \{R \subseteq G \colon R \in \mathcal{S}^*,\ \tilde{V}_2 = (V_2)_R\}$$

and $\tilde{G}_i = \langle \tilde{\mathcal{S}}_i \rangle$ $(i = 1, 2)$, then we see immediately, that for $i = 1$ or 2 $\tilde{G}_i$ also has two nontrivial composition factors on V_1. Thus $G/P^* \simeq Z_r \times Z_r \times Z_r$.

So assume now, that G_i $(i = 1, 2)$ have precisely one nontrivial composition factor on V_2. Hence G_i is faithful both on V_1 and V_2. But then G is faithful both on V_1 and V_2, a cotradiction.

Case 2. $H = 1$. Now we have G-invariant chains $0 \subset \tilde{V}_1 \subset V_1$ and $0 \subset \tilde{W}^* \subset W^*$, where $W^* = W/V_1$. Define $\mathcal{S}_i$ and G_i $(i = 1, 2)$ analogous as in case 1. The arguments of case 1 show, that G_i has precisely one nontrivial composition factor on W/V_1. Thus for $p \neq s$, s a prime, the groups G_i can not contain abelian subgroups of type (s, s), as otherwise there is a nontrivial s-element x with $\dim W_x > 2n$. For $i = 1$ or 2 consider $X = G_i/O_p(G_i)$, which has a faithful representation of degree n over $GF(p)$. Suppose $X \not\simeq Z_r$. By [7; 4.2, Th. B] $X/F(X)$ is simple. Moreover $F(X)$ contains no elementary abelian subgroup of rank $\geqslant 2$. If $F(X) \not\subseteq Z(X)$ we conclude from [7; Th. A], that $X \simeq \simeq SL(2, 3)$.

So assume $F(X) \subseteq Z(X)$, *i.e.* X is quasisimple. Let $\{i, j\} = \{1, 2\}$ and pick $R \in \mathcal{S}_j$. Then R centralizes the factor X.

By Glaubermans fixed point theorem we find a subgroup $Y \subseteq G_i$, $Y = Y'$, $Y \subseteq C_G(R)$, such that Y covers the factor X. Now V_1^R is Y-invariant and we have $O_p(Y) \subseteq C_{O_p(G)}(V_1) \cap C_{O_p(G)}(W^*) = P$. As $Y = Y'$ and Y induces a p-group on W^R we even have $W^R \subseteq W_Y$. Hence $\dim W_{O_p(Y)} > 2n$, and thus $O_p(Y) = 1$. Moreover Y centralizes W^R and acts fixed-point-free on W_R. Hence $Y \simeq SL(2, 5)$ i.e. $X \simeq \simeq SL(2, 5)$.

6.6. *Assumptions and notation as in* 6.4. *Assume further* $H \neq 1$ *and* G_0 *has precisely one nontrivial irreducible composition factor on*

V_1 and V_2. Then one of the following holds:

(i) $H \subseteq Z(G_0)$.

(ii) $\dim V_1 = r = n + 1$, *H_i $(i = 1, 2)$ is cyclic, and* $G_0/H \simeq Z_r$.

(iii) $\dim V_1 = n$. *Then either* $G_0 = H_1 \times H_2$, $H_1 \simeq Z_r$, $SL(2, 3)$ $(r = 3)$, $SL(2, 5)$ $(r = 5)$, *or* $r = 3$, $G_0/H \simeq Z_3$, *and H_i is a subgroup of* Q_8 $(i = 1, 2)$.

(iv) $\dim V_1 = 2n$. *Moreover one of the following is true:*

(α) $r = 5$. $H \simeq SL(2, 5)$, $Z_5 \times SL(2, 5)$, *or* $SL(2, 5) \times SL(2, 5)$, *and* $G_0/H \simeq Z_5$, $SL(2, 5)$, *or G_0/H is isomorphic to a quasisimple subgroup of* $SL(2, p^2)$.

(β) $r = n + 1 = 3$. *If for $i = 1$ or 2 H_i is not central in G_0, then* $H_i \simeq Q_8$, $Q_8 \times Z_3$, $SL(2, 3)$, *or* $SL(2, 5)$. *If H_j is central for $j \neq i$, then $H_j = 1$ or Z_3. Further* $G_0/H \simeq Z_3$, $Z_3 \times Z_3$, $SL(2, 3)$, $Q_8(Z_3 \times Z_3)$, $SL(2, 5)$ *or G_0/H is isomorphic to a quasisimple subgroup of* $SL(2, p)$.

PROOF. With the usual notation we assume for convenience $G = G_0$ and $H_2 \not\subseteq Z(G)$.

Case 1. $n < \dim V_1 < 2n$. As H_2 acts fixed-point-free on V_1 we have $(r, |H_2|) = 1$. So if T is any Sylow subgroup of H_2, then $R \subseteq N_G(T)$ for some $R \in \mathrm{Syl}_r(G)$.

Suppose such a T of odd order is not centralized by R. Then as T is cyclic we have $\dim V_1 = n + 1 = r$. As G induces on V_1 a subgroup of $SL(V_1)$, we have $|H_2| \equiv 1 \pmod 2$. As G is generated by elements of order r, we have $|R| = r$. By 2.9 H_2 is cyclic and G induces on V_1 a group of order $|H_2| r$. Now assertion (ii) follows.

Assume now, that all odd order Sylow subgroups of H_2 are central in G. Then as usual $Q \simeq Q_8$, $r = 3$, $n = 2$ for $Q \in \mathrm{Syl}_2(H_2)$. But then $\dim V_1 = 3$, contradicting the fixed-point-free action of Q on V_1.

Case 2. $\dim V_1 = n$. If $r \,\|H_2|$, then G induces H_2 on V_1. By the theorem of Zassenhaus we conclude $H_2 \simeq Z_r$, $SL(2, 3)$, $SL(2, 5)$. Now $r \,\|H_1|$ too, and $G = H_1 \times H_2$, H_1 is isomorphic to one of the three groups above.

Suppose next $(r, |H_2|) = 1$. Then as in case 1 $T = O(H_2) \subseteq Z(G)$ and $H_2 = T \times Q$, $Q \simeq Q_8$, $r = n + 1 = 3$. Hence $T = 1$. Now $(r, |H_1|) = 1$ and thus $G/H \simeq Z_3$, $H_1 \simeq Q_8$ or 1. Now (iii) follows.

Case 3. $\dim V_1 = 2n$.

Case 3.1. $(r, |H_2|) = 1$. Then $R \subseteq N_G(T)$ for any Sylow subgroup T of H_2 and a suitable $R \in \mathrm{Syl}_r(G)$. As $r = an + 1$ for some $a \geqslant 1$, we have $[R, T] = 1$ if $|T|$ is odd. As $H_2 \nsubseteq Z(G)$, we have $H_2 = O(H_2) \times Q$, $Q \simeq Q_8$, $r = n + 1 = 3$. If G/H_1 is solvable, then by 2.9 we have $H_2 \simeq Q_8$ and $G/H_1 \simeq (Q_8 * Q_8)(Z_3 \times Z_3)$, $(Q_8 * Q_8) Z_3$, $Q_8(Z_3 \times Z_3)$, or $Q_8 Z_3$. If G/H_1 is nonsolvable again by 2.9 $H_2 \simeq Q_8$ and $G/H_1 \simeq X * * SL(2, 3)$, where X is isomorphic to a quasisimple subgroup of $SL(2, p)$.

Case 3.2. $r \| O(H_2)|$. By (+) in the proof of 6.5 we have $\tilde{R} \trianglelefteq Z(G)$ for $\tilde{R} \in \mathrm{Syl}_r(O(H_2))$. As in the proof of 6.5 $H_2 = O(H_2) \times Q$, $Q \simeq Q_8$, $O(H_2) \subseteq Z(G)$, and $r = n + 1 = 3$. Apply 2.9 to conclude $H_2 \simeq Q_8 \times Z_3$ and $G/H_1 \simeq Q_8(Z_3 \times Z_3)$.

Case 3.3. $(r, |O(H_2)|) = 1$, $r \| H_2|$. By the theorem of Zassenhaus H_2 has a subgroup Y of index 1 or 2 with $Y = M \times S$, $(|M|, |S|) = 1$, and $S \simeq SL(2, 3)$ ($r = n + 1 = 3$) or $SL(2, 5)$ ($r = 3$ or 5), $M \subseteq Z(G)$. Again $M = 1$ and $H_2 = S$. By 2.9 $G/H_1 \simeq S * C$, $C \simeq Z_r$, $SL(2, 3)$, $SL(2, 5)$ or a quasisimple subgroup of $SL(2, p)$ or $SL(2, p^2)$.

We collect the informations of cases 3.1-3:

Let H_2 be as in case 3.1. If 3 does not divide $|H_1|$, then $H_1 \simeq 1$ or Q_8. If 3 divides $|H_1|$ and $H_1 \subseteq Z(G)$, then $9 \nmid |G/H|$ and $H_1 \simeq Z_3$. If $H_1 \nsubseteq Z(G)$ and 3 divides $|O(H_1)|$, then $G/H \simeq Z_3$, $H_1 \simeq Q_8 \times Z_3$. If $3 \| H_1|$, $(3, |O(H_1)|) = 1$, and $H_1 \nsubseteq Z(G)$, then $H_1 \simeq SL(2, 3)$, $SL(2, 5)$.

Next let H_2 be as in case 3.2. If $H_1 \subseteq Z(G)$, then $H_1 = 1$, as $G/H \simeq Z_3$. If $H_1 \nsubseteq Z(G)$, $3 \| O(H_1)|$, then $H_1 \simeq Q_8 \times Z_3$, and if $(3, |O(H_1)|) = 1$, $3 \| H_1|$, then $H_1 \simeq SL(2, 3)$, $SL(2, 5)$.

Finally let H_1 be as in case 3.3. Then $r = 3$ or 5. If $H_1 \subseteq Z(G)$, then $H_1 \simeq 1$ or Z_r. If $H_1 \nsubseteq Z(G)$ and $(r, |O(H_1)|) = 1$, $r \| H_1|$, then $H_1 \simeq SL(2, 3)$ $(r = 3)$ or $SL(2, 5)$ $(r = 3, 5)$. Assertion (iv) follows.

6.7. *Let V be a $GF(p)$-vectorspace, $p > 2$, and G be an irreducible subgroup of $SL(V)$ generated by 1-irreducible subgroups of order r such that the following properties hold* (*see* 2.9 (*b*)):

(1) *$V = V_1 \oplus \dots \oplus V_d$ and G permutes the set $\Omega = \{V_1, \dots, V_d\}$ of 1-dimensional subspaces transitively.*

(2) *Let N be the normal subgroup of G fixing Ω elementwise. Then $(G/N, \Omega)$ is a 2-transitive permutation group and either $d = r$ and $G/N \simeq L_2(11)$, M_{11}, M_{23}, $L_l(q)$ ($r = (q^l - 1)/(q - 1)$, l a prime) or $d =$*

$= r+1$ and $G/N \simeq L_2(r)$, $E_{2^m}Z_r$ $(r = 2^m - 1)$, $E_{2^m}SL(m, 2)$ $(r = 2^m - 1)$, M_{11}, M_{12}, M_{24}.

(3) $|N/(N \cap Z(G))|$ *is odd.*

Suppose $x \in G - N$ is an involution and x fixes precisely the subspaces $V_1, \ldots, V_k$. Then x induces on $U = V_1 \oplus \ldots \oplus V_k$ the identity or $G/N \simeq L_2(r)$ and x induces on U the map -1 or $G/N \simeq E_8SL(3, 2)$ and $|N \cap Z(G)|$ is even.

PROOF. *Case* 1. $d = r$. Now $|N \cap Z(G)|$ and thus $|N|$ is odd, as $G \subseteq SL(V)$. Using Glaubermans fixed-point-theorem, we see that $C_G(x)$ covers $C_{G/N}(x)$. So $C_G(x)$ acts either transitive on $\Gamma = \{V_1, \ldots, V_k\}$ or Γ splits into one orbit of length $k-1$ and one orbit of length 1. In the first case clearly x induces a scalar on U. As x induces on $\Delta = \{V_{k+1}, \ldots, V_d\}$ precisely $(d-k)/2$ 2-cycles and $(d-k)/2$ is even, $\det x = 1$ implies that x induces in the second case a scalar on U. Now if x induces -1 on U we have $\det x = (-1)^k(-1)^{(r-k)/2} = (-1)^r = -1$, a contradiction.

Case 2. $d = r+1$. Assume first, that $|N|$ is odd. Again $C_{G/N}(x)$ is covered by $C_G(x)$ and with the notation of case 1, we see that $C_G(x)$ is transitive on Γ in the cases $L_2(r)$, M_{11}, M_{12}, or M_{24}, i.e. x induces a scalar on U. Note that in all cases except $L_2(r)$ there are 4-groups $\langle x, y\rangle$, $x \sim y \sim xy$ in G such that $\langle x, y\rangle$ leaves at least one element in Γ fixed. This forces, that x induces 1 on U.

We do not have to consider the case $G/N \simeq E_{2^m}Z_r$ as involutions act fixed-point-free on Ω. Finally if $G/N \simeq E_{2^m}SL(m, 2)$, then set $M/N = O_2(G/N)$. Again involutions in $M - N$ are fixed-point-free on Ω. If $x \in G - M$, then $|C_{M/N}(x)| = 2^l$ implies that $C_M(x)$ acts fixed-point-free on the $2^l = k$ fixed components of x and again x induces a scalar on U. The above argument even yields, that x induces 1.

Assume now $|N \cap Z(G)|$ is even. If G/N is simple, then G has a nontrivial 2-part in the Schur multiplier, i.e. for $G - N$, $G/N \simeq L_2(r)$ there are no involutions. In the case $G/N \simeq M_{12}$ as usual x induces a scalar on U. However as $y \sim yz$ in $\hat{M}_{12}$ for $1 \neq z \in Z(\hat{M}_{12})$ and an involution $y \in \hat{M}_{12} - Z(\hat{M}_{12})$, this case can not happen.

Suppose $G/N \simeq E_{2^m}Z_r$. Then $|N|$ is odd as G is generated by elements of order r. So finally we have to consider $G/N \simeq E_{2^m}SL(m, 2)$. Set $M/N = O_2(G/N)$. By our assumptions it is clear, that $M/O(M)$ is elementary abelian of order 2^{m+1}, and G/M acts indecomposable on $M/O(M)$ or G contains a subgroup G_0 such that $G_0/O(M)$ is a perfect central extension of Z_2 by $SL(m, 2)$ and $G_0 \cap M \subseteq N$. This yields in any case $m = 3$. If G acts decomposable on $M/O(M)$, then there are no involutions in $G - M$ and we are done. If $G/O(M)$ contains an $SL(2, 7)$, this group would permute 7 elements of Ω and fix precisely one. Hence $Z(G/O(M))$ would induce 1 on V_1, a contradiction. Thus $G/O(M)$ is a split extension of E_{16} by $SL(3, 2)$.

6.8. *Assumptions and notation as in* 6.4. *Assume further that* G *has on* V_1 *and* W/V_1 *precisely one nontrivial, irreducible composition factor,* $\bar{H} \subseteq Z(\bar{G})$, *and* $F(\bar{G}) = F^*(\bar{G})$, $\bar{G}/F(\bar{G})$ *is nonsolvable* (*with* $\bar{G} = G/P$, $P = C_G(V_1, W/V_1)$). *Then the following holds*:

(*a*) *Suppose* $H = 1$. *Denote by* U *the irreducible composition factor of* G *on* V_1. *Then* $\bar{G}/O_p(\bar{G})$ *is faithful on* U.

(i) *Suppose* $G/C_G(U)$ *contains a noncyclic, normal, abelian subgroup* $N/C_G(U)$. *If* $\dim U = r$, *then* $\dim V_1 = r + 1 = n + 2$ *and* $\bar{G}/\bar{N} \simeq L_3(2)$, $L_5(2)$ *and* $|\bar{N}|$ *is even. If* $\dim U = r + 1$, *then* $U = V_1$ *and* $r = n + 1$, $\bar{G}/\bar{N} \simeq L_2(r)$, $E_8SL(3, 2)$, $E_{32}SL(5, 2)$.

(ii) *Suppose* $F(\bar{G})$ *contains a normal subgroup of symplectic type* (see 2.9 (*c*)), *but no normal, noncyclic abelian subgroup. Then* $U = V_1$ *and*

(α) $\dim U = n$, $r = n + 1 = 2^a + 1$, a *a 2-power. Moreover* $\dim U = 4$, *or* $\bar{G}/F(\bar{G}) \simeq SL(2, \bar{q})$, $\bar{q} = 2^a$, *or* $a = 4$, $\bar{G}/F(\bar{G}) \simeq L_2(17)$.

(β) $\dim U = n + 2$, $r = n + 1 = 2^a - 1$, a *a prime. Moreover* $\dim U \leqslant 8$, *or* $\bar{G}/F(\bar{G}) \simeq SL(2, 2^a)$.

(*b*) *Suppose* $H \neq 1$ *and use the notation of* 6.4.

(i) G_0/H_i *contain noncyclic, normal, abelian subgroups* N_i/H_i ($i = 1, 2$).

(α) *If* $N_2 \subset N_1$, *then* $\dim V_1 = n + 2 = r + 1$ *and* $G_0/N_2 \simeq E_8SL(3, 2)$ *or* $E_{32}SL(5, 2)$.

(β) *If* $N_2 \not\subseteq N_1$, $N_1 \not\subseteq N_2$, *then* $\dim V_1 = n + 2 = r + 1$ *and* G_0 *contains subgroups* G_1, G_2 *with* $G_0 = G_iN_1N_2$ ($i = 1, 2$), $G_1 \cap G_2 = N_1 \cap N_2$, $G_i/N_i \simeq E_8SL(3, 2)$ *or* $E_{32}SL(5, 2)$.

(γ) *If* $N = N_1 = N_2$, *then* $\dim V_1 = n + 2 = r + 1$, *nontrivial composition factors of* G *on* V_1, V_2 *have degree* r, $G_0/N \simeq SL(3, 2)$, $SL(5, 2)$, *or* $G_0/N \simeq L_2(11)$ (*where* $\dim V_1 = 12$, *on* $V_1 G_0$ *has a composition factor of degree* 11, *while* V_2 *is irreducible* (*or vice versa*)) *or* $G_0/N \simeq L_2(r)$ (V_1, V_2 *are irreducible*) *or* $G_0/N \simeq E_8 SL(3, 2)$, $E_{32}SL(5, 2)$ (V_1, V_2 *are irreducible*).

(ii) *If* G_0/H_1 *contains a noncyclic, normal, abelian subgroup but* G_0/H_2 *does not, then* $G_0/F(G_0) \simeq A_7$, A_8, *and* $\dim V_1 = 8$.

(iii) *If* G_0/H_1 *nor* G_0/H_2 *contain noncyclic, normal, abelian subgroups, then* $G_0/F(G_0)$ *is as in* (*a*) (ii).

PROOF. We use the notation of 6.4 i.e. we assume $G = G_0$ for convenience if $H \neq 1$. Set $\dim V_1 = n + k$ and we observe

$$(+) \qquad \dim W_x = n + k \quad \text{for involutions } x \notin Z(G)\,;$$

moreover

$$\dim (V_1)_x = (n + k)/2 \quad \text{if } x \text{ is not a homology}\,.$$

We use the bar convention for homomorphic images modulo P, i.e. $\bar{G} = G/P$ and $\bar{G} = G$ if $H \neq 1$. Denote by U the irreducible, nontrivial composition factor of G on V_1.

If $O_p(\bar{G}/\bar{H}) \neq 1$, then of course $O_p(\bar{G}) \neq 1$. In this case $H = 1$, as we have one nontrivial composition factor on V_1. Using a Frattini argument we find a subgroup G_0 generated by elements in S^*, such that $G_0 \cap O_p(G) = P$ and $G = G_0 O_p(G)$. In this case we work with G_0 instead with G. So we assume $O_p(\bar{G}/\bar{H}_1) = 1$.

Case 1. G induces on U a group which satisfies assertion 2.9 (*b*). Thus there is a normal subgroup $\bar{N} \trianglelefteq \bar{G}$ such that $\bar{N}/\bar{H}_1$ is noncyclic and abelian and $\bar{G}/\bar{N}$ permutes the homogeneous components of $\bar{N}$ on U transitively. Moreover we have a G-decomposition $V_1 = U \oplus U_0$.

Case 1.1. $H = 1$. Here $\bar{G}$ is faithful on V_1 and W/V_1.

Let $x \in G - P$ such that $x^2 \in P$, $\bar{x} \notin Z(\bar{G})$. If $p > 2$, then $\dim (V_1)_x = (n + k)/2$ by $(+)$. If $p = 2$ and $x^2 = 1$ then again $\dim (V_1)_x = (n + k)/2$. If $x^2 \neq 1$, then $W^{x^2} = V_1$ forcing again $\dim (V_1)_x = (n + k)/2$. By 3.2 we have the following possibilities:

$\dim U = r = n + 1$, $G/N \simeq A_r$, $L_f(q)$ $\left(r = (q^f - 1)/(q - 1)\right)$, $L_2(11)$, M_{11}, M_{23} or

$$\dim U = r + 1 = n + 2, \quad G/N \simeq A_{r+1}, \quad L_2(r), \quad E_{2^m}SL(m, 2),$$

M_{11}, M_{12}, M_{24} or $\dim U \geqslant n+3$, G/N is alternating of degree $\dim U$.

Assume $G/N \simeq A_m$ $(m \geqslant r)$. Let $\bar{p}$ denote a prime $\bar{p} \neq p$ such that there is a normal, noncyclic, elementary abelian $\bar{p}$-subgroup $\bar{E} \subseteq \bar{N}$. Then $\bar{E}/(\bar{E} \cap Z(\bar{G}))$ can be considered as the irreducible part of the permutation module of A_m over $GF(\bar{p})$. One finds an element $\bar{y} \in \bar{E}$ with $\dim (V_1)^{\bar{y}} = \dim (W/V_1)^{\bar{y}} = 2$. Thus y fixes a subplane of order p^{n+k-2} forcing $n+k \leqslant 4$, a contradiction.

Case 1.1.1. $\dim U = r = n+1$. If $|N/(N \cap Z(G))|$ is odd we apply 6.7 to obtain an involution $x \in G$ with $\dim U_x > (\dim U)/2$, a contradiction to $(+)$.

Thus there is an elementary abelian 2-subgroup $\bar{E} \trianglelefteq \bar{G}$, $\bar{E} \leqslant \bar{N}$, $\bar{E} \nsubseteq Z(\bar{G})$. Set $\bar{E}_0 = [\bar{E}, \bar{R}]$ for some $R \in S^*$, $R \subseteq G$. Apply 6.3. Then $\dim U = r = 2^l - 1$ and for $\bar{x} \in \bar{E}_0 - 1$ we have $\dim U_{\bar{x}} = 2^{l-1} - 1$, where $|\bar{E}_0| = 2^l$. Thus $(n+k)/2 = \dim V_1 - \dim U + \dim U_{\bar{x}}$. Hence $\dim V_1 = 2^l = r+1 = \dim U + 1$. This excludes in particular the cases $r = 11$ or 23 and forces $\bar{E} = \bar{E}_0$.

Suppose therefore $G/N \simeq L_f(\bar{q})$. First assume that $\bar{q}$ is odd. Denote by m the minimal degree of a nontrivial projective representation of $L_f(\bar{q})$ over $GF(2)$. Obviously $R \in S^*$, $R \leqslant G$, acts irreducibly on $\bar{E}$. Thus $l \geqslant m$. If $f = 2$, then $\bar{q}+1 = r$ is a prime, a contradiction. If $f > 2$, then $m \geqslant \bar{q}^{f-1} - 1$. Now $2^l - 1 = r = (\bar{q}^f - 1)/(\bar{q}-1) \leqslant$ $\leqslant m\bar{q}/(\bar{q}-1) + 1 \leqslant 3l/2 + 1$ and $l \leqslant 2$, contradicting $r \geqslant 13$.

Now assume that $\bar{q}$ is a 2-power. Then $r = 2^l - 1 = (\bar{q}^f - 1)/(\bar{q}-1)$ forces $\bar{q} = 2$ and $|\bar{E}| = 2^f$, i.e. f is a prime. If $f \neq 3, 5$ then G contains a subgroup L such that $G = LN$ and $\bar{L} \cap \bar{N} = O(\bar{N})$ (see [3]). It is now obvious, that G contains elementary abelian groups of order 2^{2f-3}, which is in conflict with $(+)$. So $f = 3$ or 5.

Case 1.1.2. $\dim U = r+1 = n+2$. Suppose first $\bar{N}/(\bar{N} \cap Z(\bar{G}))$ has odd order. By 6.7 $G/N \simeq L_2(r)$ or $E_8 SL(3,2)$. So now assume, that there is an elementary abelian 2-subgroup $\bar{E} \subseteq \bar{N}$, $\bar{E} \trianglelefteq \bar{G}$, $\bar{E} \nsubseteq Z(\bar{G})$. The same argument as in case 1.1.1 forces $|\bar{E}_0| = 2^l$, $r = 2^l - 1$ for $\bar{E}_0 = [\bar{E}, \bar{R}]$. Now $\dim V_1 = 2^l$, $U = V_1$. Again we dismiss the cases $r = 11$ or 23. Suppose $G/N \simeq E_{2^m} SL(m, 2)$. Then $l = m$. Set $M/N =$ $= O_2(G/N)$. Act with M on $\bar{E}$. Thus $|\bar{E}| = 2^{l+1}$ and $|\bar{N} \cap Z(\bar{G})|$ is even. As G contains a subgroup G_0 such that $G_0 M = G$ and $G_0 \cap M \subseteq N$, the same argument as in case 1.1.1 shows now $m = 3$ or 5.

Case 1.2. $H \neq 1$. Now $H = H_1 \times H_2$, where $H_i = C_G(V_i)$ $(i = 1, 2)$ H_i is cyclic and $H \subseteq Z(G)$. Thus $F(G)/H = F(G/H)$. Denote now by U_i $(i = 1, 2)$ the irreducible nontrivial composition factor of G on V_i.

Case 1.2.1. Assume $U_i = U_i(1) \oplus \ldots \oplus U_i(d_i)$, $\Omega_i = \{U_i(1), \ldots, \ldots, U_i(d_i)\}$, and G acts as a transitive permutation group on Ω_i $(i = 1, 2)$. Let N_i be the normal subgroup of G, which leaves all components of Ω_i fixed (i.e. $(G/H_i, U_i)$ satisfies the assertion 2.9 (b)). Note N_1, $N_2 \subseteq F(G)$. We distinguish three situations: (A) $N_2 \subset N_1$, (B) $N_2 \not\subseteq N_1$, $N_1 \not\subseteq N_2$, and (C) $N_1 = N_2$.

Suppose first, that we are in situation (A). Then N_1/N_2 is abelian, which implies $G/N_2 \simeq E_{2^m}L(m, 2)$. As N_1/H_2 is nonabelian, we have $O_2(G)H_2/H_2$ is irreducible on V_2 and $O(F(G))H_2/H_2$ is cyclic. We conclude $O(F(G)) \subseteq Z(G)$. As the Schur multiplier of $SL(m, 2)$ has 2-power order, we get $O(F(G)) = 1$ and $F(G) = O_2(G) = N_1$. As $N_1' \subseteq H_1$ and $C_{\Omega_1 N(_2)}(N_1) = \Omega_1(N_2) \cap H$ we have $|\Omega_1(N_2)/(\Omega_1(N_2) \cap H)| = 2^m$. Thus N_2/H is homocyclic. Since G/N_2 is a split extension of E_{2^m} by $SL(m, 2)$ we have $m = 3$ or 5 as in case 1.1.1.

If we are in situation (B), we have two subgroups G_1, G_2 of G with $G_i \cap F(G) = N_i$ $(i = 1, 2)$, and G_i satisfies the assertions of G in situation (A). Thus $G/F(G) \simeq SL(m, 2)$, $m = 3$ or 5 and $(N_1 \cap N_2)/H$ is homocyclic.

So finally we have (C): $N = N_1 = N_2$. As in case 1.1 the case that G/N is alternating can be dismissed.

Suppose $\dim U_1 = \dim U_2 = r$. As $H \neq 1$, we have $U_i = V_i$. Now by $(+)$ and as G induces on V_i $(i = 1, 2)$ a subgroup of $SL(V_i)$, we conclude that $|G|$ is odd, a contradiction.

Suppose $\dim U_1 = r = n + 1$ but $\dim U_2 > r$. Then $\dim U_2 = r + 1 = n + 2$, $U_2 = V_2$, and G/N is a 2-transitive group on r as well as on $r + 1$ symbols. Hence $G/N \simeq L_2(11)$ or M_{11}. Now as in case 1.1.1 we exclude that $|N/(N \cap Z(G))|$ is even. Also by 6.7 $|N \cap Z(G)|$ is even, i.e. $2\,\big|\,|H_1|$ and the Schur multiplier of G/N has even order. Thus $G/N \simeq L_2(11)$, $G/O(N) \simeq SL(2, 11)$, and on Ω_1 G has a permutation representation of degree 11, while on Ω_2 the representation of degree 11, while on Ω_2 the representation has degree 12.

So finally we have $\dim U_1 = \dim U_2 = r + 1$ and $U_i = V_i$. If $|N|$ is odd the as usual $G/N \simeq L_2(r)$. Suppose next, that $|N/(N \cap Z(G))|$ is odd but $|N \cap Z(G)|$ is even. If G/N is simple, then G/N has a Schur multiplier of even order. Thus $G/N \simeq L_2(r)$ or M_{12}. However the same argument as in the proof of 6.7 yields that the second case can not occur. So assume next $G/N \simeq E_{2^m}SL(m, 2)$. Clearly $O_2(G/O(N))$ must be abelian and as in 6.7 we see, that $O_2(G/O(N)H_i)$ $(i = 1, 2)$ is elementary abelian of order 2^{m+1}. Thus $H_1 \simeq H_2 \simeq Z_2$ and as in 6.7 we have $m = 3$ or 5.

For $|N/(N \cap Z(G))|$ even we get as in case 1.1.2 that $G/N \simeq L_2(r)$ or $E_{2^m}SL(m, 2)$ $(m = 3, 5)$.

Case 1.2.2. Now we assume that G/H_2 has no normal, noncyclic abelian subgroup, i.e. $H_1 \neq 1$ and V_2 is an irreducible G-module. Apply 2.9. We are in case (*c*) with the pair $(V_2, G/H_2)$. Thus by 3.2 and 3.4 we have $\dim V_2 = n + 2 = r + 1$, $r = 2^a - 1$, $O_2(G/H_2)$ is of syplectic type and $O(G/H_2)$ is cyclic and central in G/H_2. Finally $G/F^*(G)$ is irreducible on $O_2(G/H_2)/Z(O_2(G/H_2))$. Hence $F(G)$ induces on V_1 an abelian group such that $G/F(G)$ is a 2-transitive group. Now either the irreducible composition factor U of G on V_1 has dimension r or $r + 1$. As $G/F(G)$ is nonsolvable we have $a \geqslant 3$. By 3.2 and 3.4 we get $G/F(G) \simeq A_7$ or A_8, where $\dim U = 7$ or 8.

Case 2. G induces on U, the irreducible composition factor of G on V_1, a group which satisfies the conditions of 2.9 (*c*).

Case 2.1. $H = 1$, i.e. $\bar{G}$ is faihful on V_1 and W/V_1. Now $F(\bar{G}) = \bar{Z} \times O_2(\bar{G})$, where $\bar{Z}$ has odd order and is a cyclic subgroup of $Z(\bar{G})$. $\bar{E} = O_2(\bar{G})$ is of symplectic type. Finally as $\bar{E}$ has width $\geqslant 2$, we have that $V_1^{Z(\bar{E})} = V_1$, i.e. V_1 and W/V_1 are $\bar{G}$-irreducible. Now $\bar{E} = \bar{F} * \bar{Z}_0$, where $\bar{Z}_0$ is cyclic and $\bar{F}$ is extraspecial of order 2^{2a+1} and $r = 2^a + 1$ or $2^a - 1$, and $\dim V_1 = n$ or $n + 2$ accordingly. Let $\bar{A}^*$ be a maximal abelian subgroup in $\bar{E}$ such that $\bar{A}^*$ is normal in a S_2-subgroup of $\bar{G}$.

If possible $\bar{A}^*$ is chosen such that $\bar{A}^*$ contains an elementary abelian subgroup or order 2^{a+1}. If $\bar{E}$ is extraspecial set $\bar{A} = \bar{A}^*$. If $|\bar{Z}_0| \geqslant 4$ set $\bar{A} = \Omega_1(\bar{A}^*)$. Set $\bar{C} = \{\bar{x} \in \bar{G} | [\bar{A}, \bar{x}] \subseteq Z(\bar{F})\}$ and $\bar{C}_0 = C_{\bar{G}}(\bar{A})$. $\bar{A}$ has precisely 2^a different homogenous components on V_1, which must be fixed by $\bar{C}_0$, i.e. $\bar{C}_0$ is abelian. Now $C_{\bar{E}}(\bar{A}) = \bar{A}^*$ and $|\bar{E} : \bar{A}^*| = 2^a$ and thus every element $\bar{x} \in \bar{A} - Z(\bar{E})$ is conjugate to $\bar{x}\bar{z}$ in $\bar{E}$ for $1 \neq \bar{z} \in \Omega_1(Z(\bar{E}))$. Hence $\bar{E}$ covers $\bar{C}/\bar{C}_0$ and $\bar{E} \cap \bar{C}_0 = \bar{A}^*$. Now $\bar{C}/\bar{E}$ corresponds in $\bar{G}/\bar{E}$ to the centralizer of a maximal isotropic space. Hence $\bar{C}/\bar{E} \simeq \bar{C}_0/\bar{A}^*$, is elementary abelian. Use the notation of 3.4. For $\bar{G}/\bar{E} \simeq Sp(2\bar{n}, \bar{q})$, $SO^-(2\bar{n}, \bar{q})$, $Sp(2a, 2)$, $SO^+(2a, 2)$ we have $|\bar{C}/\bar{E}| = \bar{q}^{(\bar{n}+1)\bar{n}/2}$, $q^{\bar{n}(\bar{n}-1)/2}$, $2^{(a+1)a/2}$, $2^{a(a-1)/2}$ respectively.

Suppose $|\bar{C}_0/\bar{A}^*| \geqslant 2^{a+2}$, then $|\Omega_1(\bar{C}_0)| \geqslant 2^{a+2}$. This implies however that there is an involution $\bar{x} \in \bar{C}_0$, with $\dim (V_1)_{\bar{x}} > 2^{a-1}$, a contradiction.

Hence $|\bar{C}/\bar{E}| \leqslant 2^{a+1}$. This implies $n = 1$ if $a \geqslant 4$ for $\bar{G}/\bar{E} \simeq Sp(2\bar{n}, \bar{q})$, $\bar{n} \leqslant 3$ if $a \geqslant 4$ for $\bar{G}/\bar{E} \simeq SO^-(2\bar{n}, \bar{q})$, $a = 2$ if $G/E \simeq Sp(2a, 2)$, $a \leqslant 3$ if $\bar{G}/\bar{E} \simeq SO^+(2a, 2)$.

Case 2.2. $H \neq 1$ (i.e. $G = G_0$ is assumed). Because of case 1 we also assume, that G/H_2 has no noncyclic, normal, abelian subgroup. Wlog. $H_1 \neq 1$ and V_2 is irreducible. Thus $\dim V_2 = n$ or $n + 2$ according to wether $r = 2^a + 1$ or $2^a - 1$, where $O_2(G)/(O_2(G) \cap H_i)$ $(i = 1, 2)$ is of symplectic type and $|O_2(G)/Z(O_2(G))| = 2^a$. Hence also V_1 is irreducible.

If $|H_i|$ $(i = 1, 2)$ is odd we get of course the same results as in case 2.1. Thus we assume, that $|H_i|$ is even. Further set $E = O_2(G)$, $E_i = E \cap H_i$ $(i = 1, 2)$. Clearly $Z_i = \Omega_1(E_i) \neq 1$ for $i = 1, 2$. E' covers $\Omega_1(Z(E/E_i))$ $(i = 1, 2)$ and thus either $E' = \langle z_1 z_2 \rangle$ or $E' = Z_1 \times Z_2$, where $Z_i = \langle z_i \rangle$.

Suppose first $E' = \langle z_1 z_2 \rangle$. Then $E = E_1 \times F = E_2 \times F$, where F is a group of symplectic type faithful both on V_1 and V_2. Again denote by A^* a maximal abelian group in E, such that $A^* \trianglelefteq S$, $S \in Syl_2(G)$, and $A = \Omega_1(A^*)$ has maximal possible order. Proceed as in case 2.1 and get the analogous results.

Now assume $E' = Z_1 \times Z_2$. If $G/E \simeq Sp(2n, \bar{q})$, $SO^-(2n, \bar{q})$, $SO^+(2a, 2)$, $Sp(2a, 2)$ (notation of 3.4), we find $x \in G$, $x^2 \in E$, such that $|C_{E/Z(E)}(x)| = 2^a$. As E/E_i $(i = 1, 2)$ is of symplectic type, the counter image A of $C_{E/Z(E)}(x)$ is abelian. From the action of G/E on $E/Z(E)$ we get $|C_{G/E}(A/Z(E))| = \bar{q}^{\bar{n}(\bar{n}+1)/2}, \bar{q}^{\bar{n}(\bar{n}-1)/2}, 2^{a(a+1)/2}, 2^{a(a-1)/2}$ where $G/E \simeq Sp(2\bar{n}, \bar{q})$, $SO^-(2\bar{n}, \bar{q})$, $Sp(2a, 2)$, or $SO^+(2a, 2)$ accordingly. Furthermore any element $a \in \Omega_1(A) - Z(E)$ is conjugate in E to every element in aZ_1Z_2. Thus $C_G(\Omega_1(A))$ covers $C_{G/E}(A/Z(E))$. We get the same restrictions as above.

6.9. *Suppose G satisfies hypothesis* 6.2 *and in addition we assume, that G fixes Δ pointwise. The following assertions are true*:

(*a*) *If $O_p(G) \neq 1$, then $n = k$ and $G \simeq EZ$ is a Frobenius group, where E is elementary abelian of order p^n and Z is cyclic of order r.*

(*b*) *If $G/Z(G) \simeq \mathfrak{X}(p^a)$ is a Chevalley group of characteristic p then*:

(i) $n = k$, $\mathfrak{X}(p^a) \simeq L_2(p^n)$, $L_2(p^{n/2})$.

(ii) $n = 3$, $k = 1$, $p = 2$ *and* $\mathfrak{X}(p^a) \simeq L_3(2)$.

(*c*) *Suppose $G/Z(G) \simeq \mathfrak{X}(\bar{p}^a)$ is a Chevalley group of characteristic $\bar{p} \neq p$.*

(i) *If $n = k$, then $\mathfrak{X}(\bar{p}^a) \simeq A_5$.*

(ii) *If $\varphi_n^*(p)$ divides $|G|$ and $k < n$, then $k = 2$ and we have the following possibilities.*

(α) *V is an irreducible G-module, $\mathfrak{X}(\overline{p}^a) \simeq U_4(2)$, $L_3(4)$, and V can be considered as an irreducible 4-dimensional $GF(p^2)$-module for $p = 3$ or 5.*

(β) *V is indecomposable as a G-module, $\mathfrak{X}(\overline{p}^a) \simeq L_2(11)$, $L_2(19)$, $p = 2$ and V can be considered as a 6- or 10-dimensional module over $GF(4)$ accordingly.*

(γ) *$\mathfrak{X}(\overline{p}^a) \simeq L_2(7)$, $p = 3$ or 5, and V can be considered as a 4-dimensional $GF(p^2)$-module.*

PROOF. We choose the notation such that $W = V \oplus V$, where $\dim V = n + k$ and denote by $V_\infty = \{(v, 0): v \in V\}$, $V_0 = \{(0, v): v \in V\}$, and $V_1 = \{(v, v): v \in V\}$ fixed components of Δ. There is a subset $K(\Delta) \subseteq GL(V)$, such that $V_t = \{(vt, v): v \in V\}$ $(t \in K(\Delta))$ are the components different from V_∞, V_0 of Δ. Moreover the representations of G on V_∞, V_0, V_t $(t \in K(\Delta))$ are all isomorphic. Abusing the notation we denote by x also the map that $x \in G$ induces on a component $V \simeq V_\infty \simeq V_0 \simeq V_t$. We have $K(\Delta) \subseteq C_{GL(V)}(G)$ and G as a subgroup of $SL(V)$ is generated by 1-irreducible subgroups of order r. Apply 6.2. Hence there is a number d such that either $n = kd$, V can be considered as an indecomposable or irreducible G-module over $GF(p^k)$ or $n = ld$, $l \geqslant k$, $V = V_R \oplus V^R$ $(R \in \mathcal{S}^*$, $R \subseteq G)$ is a G-decomposition with $V_R = V_G$ and V^R can be considered as a d-dimensional $GF(p^l)$-module.

Assume $O_p(G) \neq 1$. Then V is a indecomposable $(d+1)$-dimensional $GF(p^k)$-module of G. The nontrivial elements of $O_p(G)$ induce transvections and thus $d + 1 = 2$ and (a) follows.

From now on we assume $O_p(G) = 1$, $G/Z(G)$ is quasisimple, and we note, that $r \mid \varphi_d^*(p^k)$ respectively $r \mid \varphi_d^*(p^l)$.

Suppose $d = 1$. As $G/Z(G)$ is nonsolvable, we have that V is G-irreducible, i.e. $n = k$. So $G = E(G) * Z(G)$, where $E(G)$ is an irreducible subgroup of $SL(2, p^n)$ and $r = |Z(G)|$ divides $|E(G)|$. (b) (i) and (c) (i) follow.

So from now on we assume $d > 1$. By 2.12 we have $G = E(G)$. To (b). Suppose $G/Z(G) \simeq \mathfrak{X}(p^a)$ is a Chevalley group over $GF(p^a)$.

Case 1. V is irreducible. $d + 1$ must be even, as $G = G'$ and for involutions $x \in G - Z(G)$ we have $\dim V_x = (d+1)/2$. If $\mathfrak{X} \neq A_1$ we have by 5.2 $\mathfrak{X} = A_l$, V is the standard module, $l = d$, $k = a$, or $\mathfrak{X} = {}^2A_l$, $l = d$, $k = 2a$. In both cases there are p-elements $t \in G$, which induce on the $GF(p^k)$-space V transvections, contradicting $d > 1$. Thus $\mathfrak{X} = A_1$. By 5.2 $d = 2$, $p > 2$, $k = a$. But then $d + 1 = 3$ is odd, which excludes this case too.

Case 2. V is completely reducible, i.e. for $R \in \mathcal{S}^*$, $R \subseteq G$, V^R is the nontrivial irreducible submodule of G on V. Use Herings result [8], or 5.2 and find an p-element $x \in G$, with $\dim (V^R)_x \geqslant (\dim V^R)/2$, a contradiction.

Case 3. V is indecomposable but not irreducible. Denote by U the d-dimensional $GF(p^k)$-composition factor of G on V. Apply 5.2 to the pair (G, U). As in case 1 $d+1$ is even. Suppose $\mathfrak{X} = A_l$, ${}^2B_l$, ${}^2D_l$, G_2, or C_l. Then $k|a$ and U can be considered as the natural $GF(p^a)$-module read as a $GF(p^k)$-module. Since $d+1$ is even the cases A_l (l odd), ${}^2B_l$, ${}^2D_l$, G_2, C_l fall out. If $\mathfrak{X} = A_l$ by a result of Higman [9] we get $l = 2$, $p^a = 2$.

If $\mathfrak{X} = {}^2A_l$ and U is standard, then $l \leqslant 2$ (consider the action of transvections of ${}^2A_l(p^a)$ on V). However for $l = 2$ G contains an element x of order $(p^a+1)/(3, p^a+1)$, which has eigenvalues λ, λ, λ^{p^a-1} on its 3-dimensional standard module $(|\lambda| = (p^a+1)/(3, p^a+1))$. For $y \in C_G(x)$ of order p we get $\dim V_y > (\dim V)/2$, a contradiction.

To (*c*). We now assume $G/Z(G) \simeq \mathfrak{X}(\bar{p}^a)$, $\bar{p} \neq p$, and $\varphi_n^*(p) \| G|$. We apply 4.2 and observe, that $r^2 \nmid |G|$ if $r \| G|$ and $r | \varphi_n^*(p)$. Moreover $r \leqslant 2n+1$, if $\mathfrak{X}(p^a) \neq A_1(7) \simeq A_2(2)$.

So assume first $r > 2n+1$ and $\mathfrak{X}(p^a) \simeq A_1(7)$. Then $r = 3n+1 = 7$. As $\bar{p} \neq p$ we have that V is irreducible and $d+1 = 3$, which is impossible as usual.

So we have from now on $\varphi_n^*(p) \leqslant (n+1)(2n+1)$. By Hering [7] we have one of the following cases:

$$\varphi_n^*(p) = n+1 \text{ and } p^n = 2^4,\ 2^{10},\ 2^{12},\ 2^{18},\ 3^4,\ 3^6,\ 5^6,$$

$$\varphi_n^*(p) = 2n+1 \text{ and } p^n = 2^3,\ 2^8,\ 2^{20},$$

$$\varphi_n^*(p) = (n+1)(2n+1) \text{ and } p^n = p^2,\ 3^{18},\ 17^6.$$

We pick r as the maximal prime divisor of $\varphi_n^*(p)$ and denote by m_G the minimal degree of a nontrivial, projective representation of G' over a field of characteristic p.

Case 1. $r = 2n+1$. By 4.2 we have one of the following:

$$\mathfrak{X}(\bar{p}^a) = A_l(2)\ (l \geqslant 3),\ 2n+1 = 2^{l+1}-1,\ m_G \geqslant 2^l - 1 = n,$$

$$\mathfrak{X}(\bar{p}^a) = A_1(r),\ m_G \geqslant n,$$

$$\mathfrak{X}(\bar{p}^a) = A_2(4),\ r = 7,\ m_G \geqslant 4.$$

If V is irreducible or indecomposable we have $n = dk$ and $d + 1 \geqslant \geqslant m_G$, implying $k = 1$, $d = n$. If V is completely reducible we have $d \geqslant m_G$ and again $k = l = 1$ and $d = n$. As $G = G'$, we have that $n + 1$ is even. Hence $(n, p) = (3, 2)$. Inspecting $GL(4, 2) \simeq A_8$ (note that $A_1(7) \simeq A_2(2)$ was considered under (*b*)), we see that this case does not occur.

Case 2. $r = n + 1 = \varphi_n^*(p)$. Suppose $k = 1$, then $\dim V = n + 1 = r$, which is impossible as usual. Thus $k \geqslant 2$.

$m_G \leqslant d + 1 = (n + k)/k < (r + 2)/2$ if V is irreducible or indecomposable and $m_G \leqslant d \leqslant n/l \leqslant n/k < r/2$ if V is completely reducible. Comparing with 4.2 and using the bounds for m_G [14], this yields $l = k = 2$. We have in the irreducible case the following possibilities for the pair $(\mathfrak{X}(\bar{p}^a),\ d + 1)$:

$$(L_m(2),\, 2^{m-1})\,, \qquad (L_2(r),\, (r-1)/2)\,, \qquad (L_2(r),\, (r+1)/2)\,,$$
$$(U_4(2),\, 4) \qquad \text{or} \qquad (L_3(4),\, 4)\,.$$

If V is reducible we get for the pair $(\mathfrak{X}(\bar{p}^a), d)$ the following possibilities:

$$(L_m(2),\, 2^{m-1} - 1)\,, \qquad \text{or} \qquad (L_2(r),\, (r-1)/2)\,.$$

Suppose first $G/Z(G) \simeq L_m(2)$. If $m \geqslant 5$, then $Z(G) = 1$ and G contains elementary abelian 2-subgroups of rank $(m/2)^2$ if m is even and of rank $(m^2 - 1)/4$ if m is odd. This conflicts with a result of Ostrom [18]. If $m = 4$, then $d + 1 \leqslant 4$. The result of Ostrom forces $|Z(G)| = 2$, which is however impossible as $L_4(2)$ has no proper projective module of dimension $\leqslant 4$. If $G/Z(G) \simeq L_2(7) \simeq L_3(2)$ we may assume $p \neq 2$ because of (*b*). Thus $n = 6$ and $p = 3$ or 5 and G acts on V reducibly or irreducibly.

So we assume $G/Z(G) \simeq L_2(r)$, $r \geqslant 11$, and $d + 1 = (n + 1)/2$ is even. Hence $(n, p) = (10, 2)$, $(18, 2)$. $G/Z(G) \simeq L_2(11)$, $L_2(19)$ and V can be considered as a 6-dimensional, respectively 10-dimensional $GF(4)$-module. However $L_2(11)$, $L_2(19)$ do not have irreducible 2-modular representations of degree 6 or 10 respectively (see for instance [5]). Hence V must be indecomposable as in the completely reducible case we would have $\dim V_x = 2 + \dim (V^R)_x > (\dim V)/2$ for an involution $x \in G$ and $R \in \mathcal{S}^*$, $R \subseteq G$. Suppose finally $G/Z(G) \simeq \simeq U_4(2)$ or $(L_3 4)$. Then $d + 1 = 4$, $p = 3$ or 5, and V is irreducible.

6.10. THEOREM. *Suppose G satisfies hypothesis* 6.1, $2k \neq n$, *and* $k < n$. *Set* $G_0 = \langle R | R \in \mathrm{Syl}_r(G), r | \varphi_n^*(p) \rangle$. *Then* $G_0/Z(G_0) \simeq L_2(7)$ *or* G_0 *is solvable.*

As a corollary we have:

6.11. THEOREM. *Suppose G satisfies hypothesis* 6.0 *with* $q = p^k$ *and* $\dim_{GF(q)} V = d + 1$. *Set* $G_0 = \langle R | R \in \mathrm{Syl}_r(G), r | \varphi_d^*(q) \rangle$. *Then one of the following assertions holds:*

(*a*) $d \leqslant 2$.

(*d*) G_0 *is solvable.*

(*c*) $G_0/Z(G_0) \simeq L_2(7)$.

PROOF of 6.10. Let G_0 be nonsolvable. By 6.1 G_0 fixes Δ pointwise. Clearly $G_0 \trianglelefteq G$ and $O_p(G_0) = 1$. Apply 6.2 as in the proof of 6.9. Thus we can consider G_0 as a subgroup of $SL(V)$, where V is a fixed component of dimension $n + k$. Moreover there is a number d, such that either $n = kd$, V can be considered as a $(d + 1)$-dimensional $GF(p^k)$-module (irreducible or indecomposable) or $n = ld$, $l \geqslant k$, $V = V_R \oplus V^R$ is a G_0-decomposition for $R \in S^*$, $R \subseteq G_0$, and V^R is a $GF(p^l)$ G_0-module. We apply 2.9 to G_0 and the nontrivial, irreducible G_0-factor U on V. By our assumptions the case (*d*) of 2.9 falls out. If $F(G_0) \nsubseteq Z(G_0)$ by 2.9 we have $r = n + 1$. Since $G_0 \subseteq SL(V)$, $n + k$ is even (consider the action of involutions) i.e. $k \geqslant 2$. On the other hand $C_{\mathrm{End}(U)}(G_0) \simeq GF(p)$ and $d = n$, a contradiction. So assume $F(G_0) \subseteq Z(G_0)$. By our assumptions and 2.12 we have $G_0 = E(G_0)$. If $G_0/Z(G_0)$ is a Chevalley group of characteristic p, then by 6.9 $G_0/Z(G_0) \simeq L_3(2)$, $p = 2$.

If $G_0/Z(G_0)$ is a Chevalley group of characteristic $\bar{p} \neq p$, we have by 6.9 $k = 2$ and the triple $(G_0/Z(G_0), p, n)$ is one of the following:

$$(X, p, 6) \quad (X \simeq L_2(7),\ U_4(2),\ L_3(4);\ p = 3, 5)\ ,$$
$$(L_2(11), 2, 10)\ , \quad (L_2(19), 2, 18)\ .$$

In the cases $L_2(7)$, $U_4(2)$, $L_3(4)$ $p^6 - 1$ divides the order of G. Let $T \in \mathrm{Syl}_s(G)$, where $s = 13$ for $p = 3$ and $s = 31$ for $p = 5$. Then $s > |\Delta|$ i.e. T fixes Δ pointwise. Thus T is faithful on V. However $(|\mathrm{Aut}\,(G_0)|, s) = 1$ implies $G_0 \subseteq C_G(T)$, a contradiction. If $G_0/Z(G_0) \simeq$ $\simeq L_2(11)$ or $L_2(19)$ we set $s = 31$ for $p^n = 2^{10}$ and $s = 73$ for $p^n = 2^{18}$. The same argument as above gives a contradiction.

Suppose $G_0/Z(G_0)$ is alternating of degree m, $m \geqslant 9$ or $m = 7$. If

$m = 7$, then $r = 5$ or 7. As $d > 2$, we have $\varphi_n^*(p) \leqslant (n+1)(2n+1)$. If $k = 1$, then $d = n = 3$, $r = 2n+1 = 7$, and $p = 2$ (see proof of 6.9). But there is no translation plane of order 16 with A_7 as a collineation group [4]. Thus $k \geqslant 2$ and then of course $k = l = 2$. Now $n = 6$ and $p = 3$ or 5. As above this yields a contradiction. If $m \geqslant 9$, we note that the minimal degree of a nontrivial, projective representation of A_m is $\geqslant m - 2$ [21] and $m \geqslant s$ for any prime s dividing $|G_0|$. Hence $d + 1 \geqslant m - 2 \geqslant r - 2 \geqslant n - 1$. So $k = 1$, $n = d$. Now $r = an + 1 \leqslant m \leqslant n + 3$ implies $r = n + 1$. But then $\dim V = r$, a contradiction.

Finally assume that $G_0/Z(G_0)$ is sporadic. Let $r = an + 1$ be the maximal prime dividing $\varphi_n^*(p)$ and denote by $m = m_{G_0}$ the minimal degree of a nontrivial, projective representation of G_0. Suppose s is a prime with $s^2 | \varphi_n^*(p)$. Thus $m \leqslant s$. However m can not be so small (look for instance at extraspecial or Frobenius subgroups in G_0 or use the degrees of irreducible characters for sporadic groups). Hence $\varphi_n^*(p)$ is square-free. If $a \geqslant 3$, then $m - 1 \leqslant n < r/3$. Again m will be too small. Hence $\varphi_n^*(p) \leqslant (n+1)(2n+1)$.

Suppose $r = 2n + 1$. The list in the proof of 2.9 tells us, that for the pair (p, n) we have the following possibilities: $(2, 3)$, $(2, 8)$, $(2, 20)$, $(3, 18)$, $(17, 6)$. Now $n \neq 3$ as $GL(5, 2)$ contains no sporadic group. As $n + k$ is even, we have $k \geqslant 2$. But then $d \leqslant n/2$ and $m \leqslant d + 1$. Again m is too small.

Thus $\varphi_n^*(p) = n + 1$ and $(p, n) = (2, 4)$, $(2, 10)$, $(2, 12)$, $(2, 18)$, $(3, 4)$, $(3, 6)$, $(5, 6)$. Now $\dim V = n + k$ is even, i.e. k is even. Now $k \geqslant 4$ would imply $m \leqslant (r + 3)/4$, which is clearly impossible. Thus $k = 2 = l$. As $d + 1$ is even too we have $(p, n) = (2, 10)$, $(2, 18)$, $(3, 6)$, $(3, 5)$. Again by a rough inspection of the lower bounds for m we see that only the case $G_0/Z(G_0) \simeq M_{22}$, $n = 10$, $d + 1 = 6$, V is a 6-dimensional, irreducible $GF(4)$-module can occur. However in this case there are involutions $x \in G_0$ with $\dim_{GF(4)} V_x = 4$, a contradiction.

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