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Optimal feedback for perturbed bilinear control problems

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Optimal Feedback for Perturbed Bilinear Control Problems.

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1. Introduction.

The optimal bilinear control problems have a recent background. They represent an intermediary class between linear optimal control problems and the nonlinear ones.

It has been ascertained, that they are a suitable model for some biological or mechanical problems (see [8], [4], [7], [11], [12], [14]).

In [1], [2] we have established necessary optimality conditions for bilinear control problems for which the cost function is convex or Lipschitz. The present paper treats the case in which the cost function is Lipschitz and the state equation is obtained by perturbation with a maximal monotone operator of a bilinear equation.

More exactly the problem we shall be concerned with is the following: find a pair of functions (y, u) that minimizes the functional

$$\int_0^T \Big(\varphi(y(t)) + h\big(u(t)\big)\Big)\,dt + \psi(y(T)) \tag{1.1}$$

subject to the state equation

$$\begin{cases} y' - \Delta y + \beta(y) \ni uBy & \text{a.e. in } Q = \Omega \times]0, T[\,, \\ y(x, 0) = y_0(x) & \text{a.e. } x \in \Omega\,, \\ y(x, t) = 0 & \text{for } (x, t) \in \sum = \Gamma]0, T[\,, \end{cases} \tag{1.2}$$

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where Ω is a bounded and open subset of R^N having a sufficiently smooth boundary Γ; $\varphi, \psi\colon L^2(\Omega) \to R$ are nonnegative and locally lipschitzian functions, and $h\colon R \to \bar{R}$ is a proper, lower semicontinuous, convex function satisfying the growth condition

$$(1.3) \qquad h(x) \geqslant C_1 x^2 + C_2\,, \qquad C_1 > 0\,, \qquad \text{for all } x \in R\,.$$

In (1.2) y' is the strong derivative with respect to t of the function $y\colon Q \to R$ as a function of t from $[0, T]$ to $L^2(\Omega)$, Δ is the Laplace operator in $L^2(\Omega)$, and β is a maximal monotone graph in $R \times R$ such that $0 \in D(\beta)$. By shifting the range of β we may assume without loss of generality, that $0 \in \beta(0)$. $B\colon L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator, and u is a scalar function from $L^2(0, T)$.

Throughout the following we shall denote by H the space $L^2(\Omega)$ endowed with the usual inner product $\langle .,. \rangle$ and norm $|\cdot|$.

By $\partial\varphi\colon H \to 2^H$ we shall denote the generalized gradient of φ in the sense of Clarke [9] if φ is locally lipschitzian, respectively in the sense of convex analysis for φ convex.

Let $L\colon H \to H$ be the operator

$$Ly = -\Delta y + \beta(y) \qquad \text{for } y \in D(L)$$

where

$$D(L) = \{y \in H_0^1(\Omega) \cap H^2(\Omega);\ \exists w \in H \text{ such that } w(x) \in \beta(y(x))\,, \text{ a.e. } x \in \Omega\}\,.$$

It is easily seen that L is the subdifferential of the lower semicontinuous convex function l defined as follows

$$l(y) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla y|^2\, dx + \int_\Omega j(y)\, dx & \text{if } y \in H_0^1(\Omega) \text{ and } j(y) \in L^1(\Omega), \\ +\infty \text{ otherwise} \end{cases}$$

where j is such that $\partial j = \beta$.

In this way the equation (1.2) may be written as

$$(1.2)' \qquad \begin{cases} y' + \partial l(y) \ni uBy & \text{a.e. on }]0, T[\,, \\ y(0) = y_0\,. \end{cases}$$

Firstly, let us consider the equation (1.2) with $u \in L^1(0, T)$ and $y_0 \in \overline{D(l)}$.

DEFINITION 1.1. A strong solution for the equation (1.2) is a function $y \in W^{1,1}(0, T; H) \cap C([0, T]; \overline{D(l)})$ satisfying $y(0) = y_0$ and $y' + \partial l(y) \ni uBy$ *a.e.* on $]0, T[$.

DEFINITION 1.2. The function $y \in C([0, T]; H)$ is called an integral solution of the Cauchy problem (1.2)′ if $y(0) = y_0$ and the inequality

$$\tfrac{1}{2}|y(t) - x|^2 \leqslant \tfrac{1}{2}|y(s) - x|^2 + \int_s^t \langle u(\tau) By(\tau) - z,\ y(\tau) - x\rangle \, d\tau$$

holds for each $[x, z] \in \partial l$ and $0 \leqslant s \leqslant t \leqslant T$.

REMARK 1.1. If $y_0 \in \overline{D(l)}$ we obtain as a simple consequence of Th. 1 in [10] that the equation (1.2)′ has a unique integral solution.

In addition if $y_0 \in D(l)$ and $u \in L^2(0, T)$ then y is a strong solution for (1.2)′ and satisfies the condition $y' \in L^2(0, T; H)$.

Using the compactness property of the map $u \to y(t, 0, y_0, u)$ in (1.2)′, it follows that the problem (1.1), (1.2) has at least one optimal pair.

Let us denote by $\Phi\colon [0, T] \times H \to R$ the optimal value function associated with the control problem (1.1), (1.2) defined by

$$(1.4) \qquad \Phi(t, y_0) = \inf\left\{ \int_t^T \Big(\varphi\big(y(s, t, y_0, u)\big) + h\big(u(s)\big)\Big) ds + \psi\big(y(T, t, y_0, u)\big); \quad u \in L^2(t, T)\right\}$$

which attains its infimum for every $(t, y_0) \in [0, T] \times H$.

In section 2 we shall derive necessary optimality conditions for problem (1.1), (1.2) while in section 3 we shall prove the existence of a feedback law for the problem considered above.

2. The approximating control process. First order necessary conditions.

This section concerns first-order necessary conditions of optimality for problem (1.1), (1.2). The main tool used in this analysis is an approximation by mollifiers developed by Barbu in [5]. Namely, one considers the control problem with pay-off

$$\int_0^T \left(\varphi^\varepsilon(y(t)) + h_\varepsilon(u(t)) + \tfrac{1}{2}(u(t) - u^*(t))^2\right) dt + \psi^\varepsilon(y(T)) \tag{2.1}$$

and state system

$$\begin{cases} y' - \Delta y + \beta^\varepsilon(y) = uBy & \text{a.e. in } Q\,, \\ y(0) = y_0 & \text{a.e. in } \Omega\,, \\ y = 0 & \text{in } \Sigma\,, \end{cases} \tag{2.2}$$

where

$$h_\varepsilon(u) = \inf\left\{\frac{(u-v)^2}{2\varepsilon} + h(v); \quad v \in R\right\} \tag{2.3}$$

$$\beta^\varepsilon(r) = \int_{-\infty}^{+\infty} \left(\beta_\varepsilon(r - \varepsilon^2\theta) - \beta_\varepsilon(-\varepsilon^2\theta)\right) \varrho(\theta)\, d\theta \tag{2.4}$$

$\beta_\varepsilon = \varepsilon^{-1}(1 - (1 + \varepsilon\beta)^{-1})$, ϱ is a C_0^∞-mollifier on R, *i.e.* $\varrho \,\varepsilon\, C^\infty(R)$, $\varrho(r) = \varrho(-r)$, $\operatorname{supp} \varrho \in [-1, 1]$, $\int_{-\infty}^{+\infty} \varrho(\tau)\, d\tau = 1$.

As regards φ^ε (and ψ^ε) they are defined as follows (see [5]).

Let $\{l_i\}$ be an orthonormal basis in H and let H_n be the linear space generated by $\{l_i\}_{i=1}^n$.

For $n = [\varepsilon^{-1}]$ we define

$$\varphi^\varepsilon(x) = \int_{R^n} \varphi(P_n x - \varepsilon \Lambda_n \tau)\, \varrho_n(\tau)\, d\tau\,, \qquad x \in H\,, \tag{2.5}$$

where $P_n \colon H \to H_n$ is the projection operator on H_n, ϱ_n is a C_0^∞ mollifier in R^n and $\Lambda_n \colon R^n \to H_n$ is the operator

$$\Lambda_n \tau = \sum_{i=1}^n \tau_i l_i; \qquad \tau = (\tau_1, \tau_2, \ldots, \tau_n)\,.$$

The functions φ^ε, ψ^ε, h_ε are lipschitzian, Fréchet differentiable and with Fréchet differentials lipschitzian on H and R, respectively.

In addition since φ and ψ are locally lipschitzian, the gradients $\nabla\varphi^\varepsilon$ and $\nabla\psi^\varepsilon$ are bounded (uniformly with respect to ε) on every bounded subset.

Let $(y_\varepsilon, u_\varepsilon)$ be an optimal pair for problem (2.1), (2.2).

LEMMA 2.1. *For every* $\varepsilon \to 0$, *we have*

$$u_\varepsilon \to u^* \text{ strongly in } L^2(0, T)\,, \tag{2.6}$$

$$y_\varepsilon \to y^* \text{ strongly in } C([0, T]; H)\,. \tag{2.7}$$

PROOF. (See also [5]).

Since $(y_\varepsilon, u_\varepsilon)$ is an optimal pair we have

$$\int_0^T \Big(\varphi^\varepsilon(y_\varepsilon(t)) + h_\varepsilon(u_\varepsilon(t)) + \tfrac{1}{2}(u_\varepsilon(t) - u^*(t))^2\Big)dt + \\ + \psi^\varepsilon(y_\varepsilon(T)) \leqslant \int_0^T (\varphi^\varepsilon(z_\varepsilon) + h(u^*))\,dt + \psi^\varepsilon(z_\varepsilon(T)) \tag{2.8}$$

where z_ε satisfies the system

$$\begin{cases} z_\varepsilon' - \Delta z_\varepsilon + \beta^\varepsilon(z_\varepsilon) = u^* B z_\varepsilon\,, & \text{a.e. in } Q\,, \\ z_\varepsilon(0) = y_0\,. \end{cases}$$

But since $z_\varepsilon \to y^*$ strongly in $C([0, T]; H)$ it follows

$$\lim_{\varepsilon\to 0} \int_0^T \Big(\int_{R^n} \varphi(P_n z_\varepsilon - \varepsilon \Lambda_n \tau)\,\varrho_n(\tau)\,d\tau\Big)\,dt = \int_0^T \varphi(y^*)\,dt\,. \tag{2.9}$$

and

$$\lim_{\varepsilon\to 0} \psi^\varepsilon(z_\varepsilon(T)) = \psi(y^*(T))\,. \tag{2.10}$$

Now (2.9), (2.10) and (1.3) implies that $\{u_\varepsilon\}$ is bounded in $L^2(0, T)$ and so there exists $\tilde{u} \in L^2(0, T)$ such that

$$u_\varepsilon \to \tilde{u} \text{ weakly in } L^2(0, T)\,, \quad y_\varepsilon \to \tilde{y} \text{ strongly in } C([0, T]; H)\,.$$

The last convergence implies

$$\lim_{\varepsilon\to 0}\int_0^T \varphi^\varepsilon(y_\varepsilon(t))\,dt = \int_0^T \varphi(\tilde{y}(t))\,dt$$

and

$$\lim_{\varepsilon\to 0}\psi^\varepsilon(y_\varepsilon(T)) = \psi(\tilde{y}(T))\,.$$

Fatou's lemma gives now

$$\liminf_{\varepsilon\to 0}\int_0^T h_\varepsilon(u_\varepsilon(t))\,dt \geqslant \int_0^T h(\tilde{u}(t))\,dt$$

which together with the preceding relations implies (2.6) and (2.7).

Since the functions φ^ε, ψ^ε, h_ε and β^ε are differentiable we may write the approximating optimality system corresponding to (2.1), (2.2):

$$p'_\varepsilon + \Delta p_\varepsilon - p_\varepsilon\nabla\beta^\varepsilon(y_\varepsilon) + u_\varepsilon B^* p_\varepsilon = \nabla\varphi^\varepsilon(y_\varepsilon)\,, \quad \text{a.e. in } Q\,, \tag{2.11}$$

$$p_\varepsilon(T) + \nabla\psi^\varepsilon(y_\varepsilon(T)) = 0\,, \quad \text{a.e. in } \Omega\,, \tag{2.12}$$

$$\langle B^* p_\varepsilon, y_\varepsilon\rangle = \nabla h_\varepsilon(u_\varepsilon) + u_\varepsilon - u^*\,. \quad \text{a.e. on }]0, T[\,. \tag{2.13}$$

Making the remark that $\{y_\varepsilon\}$ is bounded in $C([0, T]; H)$ and $\nabla\varphi^\varepsilon$, $\nabla\psi^\varepsilon$ are bounded on bounded subsets it follows that $\nabla\varphi^\varepsilon(y_\varepsilon)$ is bounded in $C([0, T]; H)$ and $\nabla\psi^\varepsilon(y_\varepsilon(T))$ is bounded in H.

Writting (2.11) in integral form we obtain by Gronwall's lemma that $\{p_\varepsilon\}$ is bounded in $C([0, T]; H)$ hence there exists $p \in L^\infty(0, T; H)$ to which, on a subsequence, p_ε converges weakly star in $L^\infty(0, T; H)$.

The above convergences imply

$$\langle By_\varepsilon(t), p_\varepsilon(t)\rangle \to \langle By^*(t), p(t)\rangle\,, \quad \text{a.e. } t \in [0, T]\,,$$

so, by the Lebesgue dominated convergence theorem

$$\langle By_\varepsilon, p_\varepsilon\rangle \to \langle By^*, p\rangle\,, \quad \text{strongly in } L^2(0, T)\,.$$

This is conjunction with (2.6) and Th. 1.2 in [6] yields

$$\langle By^*(t), p(t)\rangle \in \partial h(u^*(t)), \quad \text{a.e. } t \in]0, T[.$$

We denote by $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$ and corresponding norms by $\|\cdot\|$ and $\|\cdot\|_*$.

Now we take the scalar product of (2.11) by $p_\varepsilon(t)$ and integrate on $[t, T]$.

Since $\langle \Delta p_\varepsilon, p_\varepsilon\rangle \leqslant 0$ and $\dot{\beta}^\varepsilon \geqslant 0$ we obtain through Gronwall's inequality,

$$\text{(i)} \quad |p(t)|^2 + \int_0^T \|p_\varepsilon(s)\|^2 ds \leqslant C .$$

Let ζ be a smooth and monotone approximation of signum function with $\zeta(0) = 0$ (see for example [6], p. 80).

Multiplying (2.11) by $\zeta(p_\varepsilon)$ and integraring on Q, it results

$$\int_Q \dot{\beta}^\varepsilon(y_\varepsilon)\cdot p_\varepsilon \cdot \zeta(p_\varepsilon)\, dx\, dt \leqslant C$$

and then letting ζ tend to sign we get

$$\text{(ii)} \quad \int_Q \dot{\beta}^\varepsilon(y_\varepsilon)|p_\varepsilon|\, dx\, dt \leqslant C , \qquad \text{hence } \dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon \text{ is}$$

is bounded in $L^1(0, T; L^1(\Omega))$. Above we used the fact that

$$\int_Q \Delta p_\varepsilon \cdot \zeta(p_\varepsilon)\, dx\, dt \leqslant 0 \qquad \text{and} \qquad \int_Q (p_\varepsilon)_t \zeta(p_\varepsilon)\, dx\, dt \leqslant C .$$

Now since Δ is a linear bounded operator from V to V', from (2.11) it follows that $\{\dot{\beta}^\varepsilon(y_\varepsilon) p_\varepsilon\}$ is bounded in $L^1(0, T; L^1(\Omega) + V')$. From Sobolev theorem it follows that $H^s(\Omega) \subset C(\bar{\Omega})$, for $s > N/2$, hence $L^1(\Omega) \subset (H^s(\Omega))'$, which implies that $\{p_\varepsilon'\}$ is bounded in $L^1(0, T; Y^*)$ where $Y^* = (H^s(\Omega))' + V'$ is the dual of $Y = H^s(\Omega) \cap V$.

Since the injection $H \subset Y^*$ is compact and the set $\{p_\varepsilon(t)\}$ is bounded in H for every $t \in [0, T]$, by the vectorial Helly theorem we may infer that there exists a function $p \in BV([0, T]; Y^*)$ such that, on a subsequence $\varepsilon_n \to 0$, $p_{\varepsilon_n}(t) \to p(t)$ strongly in Y^* for energy $t \in [0, T]$.

On the other hand by (ii) it follows that $p_{\varepsilon_n} \to p$ weak star in $L^\infty(0, T; H)$ and weakly in $L^2(0, T, V)$ and since the inclusion $V \subset H$ is compact it follows (see [13]) that for every $\lambda > 0$ there exists $C(\lambda)$ such that

$$|p\varepsilon_n(t) - p(t)| \leqslant \lambda \|p\varepsilon_n(t) - p(t)\| + C(\lambda)\|p\varepsilon_n(t) - p(t)\|_*$$

for all ε_n and $t \in [0, T]$.

This implies

$p_{\varepsilon_n} \to p$ strongly in $L^2(0, T; H)$ and

$p_{\varepsilon_n}(t) \to p(t)$ weakly in H, for every $t \in [0, T]$.

Finally (ii) implies that there exist $\mu \in (L^\infty(Q))^*$ such that on a generalized subsequence λ

$$\dot{\beta}^\lambda(y_\lambda)p_\lambda \to \mu \text{ weakly star in } (L^\infty(Q))^* .$$

Now the boundedness property of $\{\nabla\psi^\varepsilon\}$ and Lemma 2 in [5] allows us to assume that on a subsequence, again denoted by ε_n

$$\nabla\psi^{\varepsilon_n}(y\varepsilon_n(T)) \to q \text{ weakly in } H,\ q \in \partial\psi(y^*(T))$$

and similarly

$$\nabla\varphi^{\varepsilon_n}(y\varepsilon_n) \to \eta \text{ weak star in } L^\infty(0, T; H)$$

where $\eta \in \partial\varphi(y^*(t))$ *a.e.* $t \in]0, T[$.

Thus we have proved the following theorem.

THEOREM 2.1. *If* (y^*, u^*) *is an optimal pair for problem* (1.1), (1.2), *then there exists the functions* $p \in BV([0, T]; Y^*) \cap L^2(0, T; V) \cap \cap L^\infty(0, T; H)$ *and* $\mu \in (L^\infty(Q))^*$ *such that* $p' + \Delta p - \mu - u^* B^* p \in \in L^\infty(0, T; H)$ *and*

$$\begin{cases} p' + \Delta p - \mu - u^* B^* p \in \partial\varphi(y^*), & a.e. \text{ in } Q\,, \\ p(T) + \partial\psi(y^*(T)) \ni 0, & \text{in } \Omega\,, \\ \langle B^* p, y^* \rangle \in \partial h(u^*), & a.e. \text{ in }]0, T[\,. \end{cases}$$

3. Optimal feedback control and Hamilton-Jacobi equation.

Let Φ be the function defined by (1.4). As we have already seen the infimum defined by Φ is attained for all $(t, y) \in [0, T] \times H$.

PROPOSITION 3.1. *For every* $t \in [0, T]$ *the function* $y_0 \to \Phi(t, y_0)$ *is locally Lipschitz and for each* $y_0 \in D(\partial l)$ *the function* $t \to \Phi(t, y_0)$ *is Lipschitz on* $[0, T]$.

PROOF. The proof is similar with those of Lemma 5.8 in [6], so we only schetch it.

Since the operator—$\Delta + \beta$ is monotone on H—it results.

$$|y(s, t, y_0, u) - y(s, t, \tilde{y}_0, u)| \leqslant |y_0 - \tilde{y}_0| \cdot \exp\left[c_1 \int_s^t |u(\tau)|\, d\tau\right].$$

Now multiplying (1.2)′ with $y_0 \in D(\partial l)$ and integrating on $[t, s]$, after some manipulations we obtain

$$|y(s, t, y_0, u) - y_0| \leqslant c_2 \cdot \exp\left[\int_t^s |u(\tau)|\, d\tau\right]$$

where c_1, c_2 are independent of u. The rest of the proof goes as in Lemma 5.8 in [6] (see also Lemma 3.1 in [2]).

LEMMA 3.1. For all $t \in [0, T]$ and $y_0 \in H$ we have

$$\Phi(0, y_0) = \inf\left\{\int_0^t \Big(\varphi(y(s, 0, y_0, u)) + h\big(u(s)\big)\Big) ds + \right.$$
$$\left. + \Phi\big(t, y(t, 0, y_0, u)\big); \qquad u \in L^2(0, t)\right\}.$$

PROOF. The proof involves simple manipulations of the infimum definition.

THEOREM 3.1. Let (y^*, u^*) be an optimal pair for problem (1.1), (1.2) with $y_0 \in D(l)$. Then

$$u^*(t) \in \partial h^*\Big(-\langle By^*(t), \partial\Phi\big(t, y^*(t)\big)\rangle\Big), \qquad \text{a.e. } t \in]0, T[\,. \tag{3.1}$$

PROOF. The formula (3.1) follows from Lemma 3.1 and optimality conditions. Its proof goes step by step as those of Th. (5.6) in [6].

If in addition to the assumptions from the beginning of the paper we impose

i) $\psi \equiv 0$, $h^* \in C^1(R)$,

ii) The map $(t, y) \to \partial\varphi(t, y)$ is weakly star upper semicontinuous on $[0, T] \times H$,

one can prove, in the same way as in [2], that the Bellman function Φ satisfies a Hamilton-Jacobi equation of the form below

$$(3.2) \quad \begin{cases} \Phi_t(t, y) - h^*(-\langle By, \Phi_y(t, y)\rangle) + \langle \Phi_y(t, y), \partial l(y)\rangle + \varphi(y) = 0, \\ \qquad\qquad \text{a.e. } (t, y) \in]0, T[\times H, \\ \Phi(T, y) = 0, \quad \forall y \in H. \end{cases}$$

or making backward transformation

$$(3.3) \quad \begin{cases} \psi(t, y = \Phi(T - t, y), \\ \psi_t(t, y) + h^*(-\langle By, \psi_y(t, y)\rangle) - \langle \psi_y(t, y), \partial l(y)\rangle - \varphi(y) = 0, \\ \qquad\qquad \text{a.e. } (t, y) \in]0, T[\times H, \\ \psi(0, y) = 0, \quad \forall y \in H: \end{cases}$$

We mention that a direct treatment of (3.2) as well as its relationhip with control theory is made in [3].

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