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Generalized Solutions by Cauchy's Method of Characteristics.

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SUMMARY - The classical Cauchy's Method of Characteristics is extended to obtain nonlocal generalized solutions of First Order Partial Differential Equations that in the case of the Hamilton-Jacobi Equation of Dynamic Programming associated to an optimal control problem coincide with the so called « value (Bellman's) function » and provides the optimal trajectories and the optimal controls.

1. Introduction.

The aim of this paper is to use Cauchy's Method of Characteristics to obtain nonlocal generalized solutions for boundary value problems of the form:

$$F(x, Du(x), u(x)) = 0\,, \qquad x \in X \subset R^n \tag{1.1}$$

$$u(x) = u_0(x) \qquad (\forall)\, x \in X_0 \subset X \tag{1.2}$$

defined by functions $F(\cdot, \cdot, \cdot)\colon A \subset R^n \times R^n \times R \to R$ and $u_0(\cdot)\colon X_0 \to R$ that may not be even differentiable.

Following the classical Cauchy's Method of Characteristics we introduce two « marginal » functions (i.e. defined by « min » or « max » operations) in terms of the components of the « characteristic flow » which we call *marginal characteristic solutions* of the problem (1.1)-(1.2)

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and prove that under some hypotheses these functions are ***a.e. solutions*** that are almost everywhere differentiable satisfying equation (1.1) almost everywhere (a.e.) and satisfying also (1.2) and the condition:

$$Cl(\mathrm{Int}\,(X)) = Cl(X)\,. \tag{1.3}$$

Under some other type of hypotheses on the components of the characteristic flow we prove that the marginal functions mentioned above are ***stratified solutions*** that ***are « weakly C^1-stratified functions »*** (in a sense to be defined in Section 3) ***that verify the boundary condition*** (1.2) ***and the following property***: ***for any*** $x \in X$ ***there exists*** $p_x \in R^n$ ***such that***:

$$\langle p_x, \bar{x}\rangle = Du(x)\cdot\bar{x} \quad \text{for any } \bar{x} \in T_x X \tag{1.4}$$

$$F(x, p_x, u(x)) = 0\,. \tag{1.5}$$

Since on open strata of X a stratified solution satisfies (1.1) in the classical sense and, on the other hand, lower dimensional strata have zero Lebesgue measure, a stratified solution whose set of definition, X, is n-dimensional and satisfies (1.3) is a particular type of a.e. solution that has properties (1.4)-(1.5) at the points at which does not verify (1.1) in the classical sense.

If the data, $F(\cdot, \cdot, \cdot)$ and $u_0(\cdot)$, of the problem (1.1)-(1.2) are only weakly C^1-stratified (as it is the case for the Carathéodory-Bellman-Isaacs equation of Dynamic Programming in Optimal Control Theory and Calculus of Variations for some classes of problems) then a « stratified characteristic orientor field » that becomes the well known characteristic vector field in the case $F(\cdot, \cdot, \cdot)$ is differentiable is introduced and the existence of a « piecewise smooth characteristic flow » is postulated; the possibility of using more general « contingential » or « peritangential » characteristic orientor fields using extreme contingential derivatives (e.g. [17]) and, respectively, Clarke's generalized gradient ([6]) is also discussed.

It is well known that the classical Cauchy's Method of Characteristics ([2], [7], [11], [12], [15], etc.) provides only local solutions (of class C^2) in the case $A \subset R^{2n+1}$ is open, $X_0 \subset R^n$ is a $(n-1)$-dimensional differentiable manifold of class C^2 and the functions $F(\cdot, \cdot, \cdot)$ and $u_0(\cdot)$ are of class C^2 and satisfy certain « compatibility » and « transversality » conditions while in fields such as Calculus of Variations, Optimal Control Theory, Theoretical Mechanics, there are many problems that require global (or, at least non-local, maximal with respect to the domain of definition) solutions for problems of the form (1.1)-(1.2) defined by functions $F(\cdot, \cdot, \cdot)$ and $u_0(\cdot)$ that do not enjoy the above mentioned properties. Moreover, the so called « value (Bellman's) functions » for important classes of Optimal Control and Calculus of Variations problems are shown to be differentiable in some regions but not so, and even discontinuous at some other points ([9], [21], etc.) and all the same may be interpreted as « solutions » of the so called Carathéodory-Bellman-Isaacs equation of Dynamic Programming which is of the form (1.1)-(1.2) ([3]-[5], [9], [10], [13], [19]-[21] etc.). The value functions of an important class of variational problems are proved in [10] to be locally Lipschitz and the optimal control problems studied in the literature show that for significant classes of such problems ([3]-[5], etc.) the value functions, while not necessarily continuous, are stratified functions that satisfy the equation of Dynamic Programming in a certain generalized sense (e.g., (1.4)-(1.5)) even at points at which it is not (Fréchet) differentiable.

On the other hand, an increasing number of papers (see [2], [8], [10], [15] and their references) are dealing with generalized solutions that usually ***are locally Lipschitz functions satisfying equation*** (1.1) ***almost everywhere*** but it seems that no attempt has been made so far to obtain such solutions using Cauchy's Method of Characteristics.

In Section 2, as a direct generalization of the Method of Characteristics, we use the above mentioned general procedure to characterize a.e. solutions for problems of the form (1.1)-(1.2) that are « classical » in the sense that the data $(F\cdot, \cdot, \cdot)$ and $u_0(\cdot)$ are of class C^2 and have an additional property implied (locally) by the well known compatibility and transversality conditions. The main result in this section is Theorem 2.11 in which the marginal characteristic solutions defined in terms of the characteristic flow are proved to be a.e. solutions.

In Section 3 we consider more general, « stratified » problems of the form (1.1)-(1.2) defined by functions $F(\cdot)$ and $u_0(\cdot)$ that are weakly C^1-stratified and have some additional properties that allow the introduction of the characteristic orientor field and the proof of the same type of results as in the case of the classical problems. The flow of the characteristic vector field is replaced in this case by a certain « piecewise smooth characteristic flow » so that the marginal characteristic solutions defined in the same way as for the classical problems are proved to be stratified solutions.

The main shortcoming of the method, consisting in the « implicit » definition of the solutions in terms of the characteristic flow, may be overcome for particular classes of problems for which the characteristic flow may be more thoroughly described. The results in [21]-[22] concerning subanalytic marginal functions lead to the existence theorems like the following one: *if* $F(\cdot, \cdot, \cdot)$ *and* $u_0(\cdot)$ *are analytic functions and if the first component of the characteristic flow is a proper mapping then the problem* (1.1)-(1.2) *has two stratified solutions* (which are subanalytic functions).

Similarly, for the Cauchy problem in [10] considered as Example 4.1, in Section 4 of this paper, it is proved, under apparently weaker hypotheses that the minimal characteristic solution is globally defined, locally Lipschitz and coincides with the so called « variational solution ». Three other examples illustrating the results in the paper and also the fact that discontinuous a.e. or stratified solutions may have a « physical » meaning are studied in Section 4.

As already remarked, the marginal characteristic solutions introduced in this paper are generalized solutions in the sense considered in [2], [10], etc., whenever they are locally Lipschitz on open subsets of R^n; moreover, in the case the « partial functions » $F(x, \cdot, v)$ are either convex or concave for any $x \in \mathrm{pr}_1 A$, $v \in \mathrm{pr}_3 A$, one may prove that the marginal characteristic solutions are « viscosity solutions » in the sense of [8], [15], etc., whenever they are continuous on open subsets of R^n.

As it is proved in [10], [15] and other recent papers, the value functions of some classes of problems in Optimal Control, Calculus of Variations and Differential Games are locally Lipschitz functions that are a.e. and also « viscosity solutions » of the equation of Dynamic Programming but, on the other hand, as Example 4.5 shows, this equation may have yet other solutions (even in the classical sense, hence a.e. or viscosity solutions) than the value function.

While more precise relations with other types of solutions and results in the literature remain objectives of further research, the marginal characteristic solutions in this paper seem to be the only ones in the literature that lead directly to the value function of optimal control problems as generalized solutions of the Carathéodory-Bellman-Isaacs equation of Dynamic Programming; moreover, the generalized Cauchy's Method of Characteristics thus employed provides not only the value function but also the optimal trajectories (and sometimes, the optimal controls, too) of the corresponding optimal control problem.

The results in this paper extend to general boundary value problems of the form (1.1)-(1.2) and improve previous results of the author concerning the Carathéodory-Bellman-Isaacs equation of Dynamic Programming in Optimal Control Theory ([18]-[20]).

2. A.E. characteristic solutions.

In what follows R^n denotes the n-dimensional Euclidian space and $\langle \cdot, \cdot \rangle$ the scalar product in such a space; if $X \subset R^n$ then $\mathrm{Cl}(X)$ denotes the closure and $\mathrm{Int}(X)$ the interior of the subset X in the usual topology. If $X \subset R^n$ is a differentiable submanifold (e.g. [1], [14], etc.) then $T_x X \subset R^n$ denotes the tangent space of X at the point $x \in X$ considered as a subspace of R^n. If the mapping $f(\cdot)$: $X \subset R^n \to R^m$ is (Fréchet) differentiable at $x \in \mathrm{Int}(X)$ then $Df(x) \in \in L(R^n, R^m)$ denotes the derivative of $f(\cdot)$ at x considered as a linear mapping (coinciding with the usual matrix of partial derivatives when coordinate systems are considered in R^n and R^m); if $X \subset R$ then we denote $f'(x) = Df(x) \cdot 1$ and if $f(\cdot, \cdot, \cdot, \cdot, \cdot)$ is a mapping depending on several (possibly vector) variables then $D_i f(\cdot, \cdot, \cdot \cdot, \cdot)$ denotes the i-th partial derivative of $f(\cdot, \cdot, \cdot \cdot, \cdot)$.

We recall that in classical Cauchy's Method of Characteristics (e.g. [2]. [7], [11], [12], etc.) the data of the problem (1.1)-(1.2) are assumed to satisfy the following hypotheses:

HYPOTHESIS 2.1. *$A \subset R^n \times R^n \times R = R^{2n+1}$ is open, $X_0 \subset R^n$ is a $(n-1)$-dimensional differentiable submanifold of class C^2 and $F(\cdot)$: $A \to R$, $u_0(\cdot)$: $X_0 \to R$ are of class C^2.*

HYPOTHESIS 2.2. *There exists $x_0 \in X_0$ and $p_0 \in R^n$ such that the « compatibility » conditions:*

$$(2.1) \qquad F(x_0, p_0, u_0(x_0)) = 0 \,, \qquad z_0 = (x_0, p_0, u_0(x_0)) \in A$$

$$(2.2) \qquad \langle p_0, \bar{x} \rangle = Du_0(x_0) \cdot \bar{x} \quad (\forall) \bar{x} \in T_{x_0} X_0$$

as well as the « transversality » conditions:

$$(2.3) \qquad D_2 F(x_0, p_0, u_0(x_0)) \notin T_{x_0} X_0$$

are verified at the point $z_0 = (x_0, p_0, u_0(x_0)) \in A$.

The main tools of Cauchy's Method of Characteristics are the characteristic vector field, $\zeta_F(\cdot): A \to R^{2n+1}$, the characteristic flow, $Z_F(\cdot, \cdot): B \subset R \times A \to A$, the initial characteristic multifunction $P_0(\cdot)$: $X_0 \to \mathcal{P}(R^n)$ and the «initial characteristic strip» $A_0 \subset A$ defined as follows:

DEFINITION 2.3. *If* $F(\cdot): A \subset R^{2n+1} \to R$ *and* $u_0(\cdot): X_0 \subset R^n \to R$ *defining the problem* (1.1)-(1.2) *satisfy Hypothesis* 2.1 *then* $\zeta_F(\cdot): A \to$ $\to R^{2n+1}$ *defined by*:

$$(2.4) \qquad \zeta_F(x, p, v) = \big(D_2F(x, p, v), -D_1F(x, p, v) - \\ - pD_3F(x, p, v), \langle p, D_2F(x, p, v)\rangle\big) \qquad z = (x, p, v) \in A\,,$$

is said to be the characteristic vector field of the problem, the mapping

$$Z_F(\cdot\,;\,\cdot) = \big(X(\cdot\,;\,\cdot), P(\cdot\,;\,\cdot), V(\cdot\,;\,\cdot)\big): B \subset R \times A \to A$$

for which

$$Z_F(\cdot\,;\,z): I(z) = \big(t^-(z), t^+(z)\big) \subset R \to A$$

is the maximal (non-continuable) *solution of the initial value problem*:

$$(2.5) \qquad z' = \zeta_F(z)\,, \qquad z(0) = z = (x, p, v) \in A$$

for any $z \in A$ *and* $B = \{(t, z);\, z \in A, t \in I(z)\}$ *is said to be the characteristic flow of the problem, the multifunction* $P_0(\cdot): X_0 \to \mathcal{P}(R^n)$ *defined by*:

$$(2.6) \qquad P_0(y) = \{q \in R^n;\, z = (y, q, u_0(y)) \in A, F(z) = 0, \\ \langle q, \bar{y}\rangle = Du_0(y)\cdot\bar{y}\ (\forall)\bar{y} \in T_yX_0\} \qquad (\forall)y \in X_0\,,$$

is said to be the initial characteristic multifunction and the subset $A_0 \subset A$ *defined by*:

$$(2.7) \qquad A_0 = \{z = (y, q, u_0(y)) \in A;\, y \in X_0,\, q \in P_0(y)\}$$

is said to be the initial characteristic strip of the problem (1.1)-(1.2).

Since the characteristic vector field $\zeta_F(\cdot)$ in (2.4) is of class C^1 (if Hypothesis 2.1 is satisfied), from the general theory of Ordinary Differential Equations (e.g. [1], [11], [12], [14]) it follows that for any $z = (x, p, v) \in A$ there exists a unique maximal (noncontinuable) solution, $Z_F(\cdot\,;\,z): I(z) \to A$, of the problem (2.5), the interval $I(z)$

is open, the subset $B = \{(t, z);\ z \in A,\ t \in I(z)\} \subset R \times A$ is open and the characteristic flow, $Z_F(\cdot, \cdot)\colon B \to A$ in Definition 2.3 is of class C^1 with respect to both variables and of class C^2 with respect to the first one verifying the relations:

$$(2.8)\qquad D_1 D_2 Z_F(t; z) = D_2 D_1 Z_F(t; z) = \\ = D\zeta_F(Z_F(t; z)) \cdot D_2 Z_F(t; z) \qquad (\forall)(t, z) \in B\,.$$

On the other hand, if Hypothesis 2.2 is also satisfied then from the implicit functions theorem ([1], [14], etc.) it follows the existence and uniqueness of a C^1-mapping, $p_0(\cdot)\colon X_1 \to R^n$ defined on a neighbourhood X_1 of x_0 in X_0 that is a selection of the multifunction $P_0(\cdot)$ (defined in (2.6)) i.e. it verifies:

$$(2.9)\qquad p_0(y) \in P_0(y) \qquad (\forall) y \in X_1 \subset X_0$$

in which case the subset $A_1 \subset A$ defined by:

$$(2.10)\qquad A_1 = \{(y, p_0(y), u_0(y));\ y \in X_1\}$$

is a $(n - 1)$-dimensional submanifold of class C^1 in R^{2n+1} and moreover, there exists an open interval $I_2 \subset R$ containing the origin and a relatively open neighbourhood $A_2 \subset A_1$ of the point $z_0 = (x_0, p_0, u_0(x_0))$ such that $I_2 \times A_2 \subset B$, the restriction mapp $X(\cdot\,;\ \cdot)|I_2 \times A_2$ is a C^1-diffeomorphism and the function $u(\cdot)\colon \tilde{X} \subset R^n \to R$ defined by:

$$(2.11)\qquad u(x) = V(t; z) \quad \text{if } x = X(t; z) \in \tilde{X} = X(I_2 \times A_2)$$

is the unique (local) solution of the problem (1.1)-(1.2) that satisfies the additional condition:

$$(2.12)\qquad Du(x_0) \cdot \bar{x} = \langle p_0, \bar{x} \rangle \qquad (\forall) \bar{x} \in T_{x_0} X_0\,.$$

In fact, since $u(\cdot)$ verifies condition:

$$(2.13)\qquad Du(x) = P(t; z) \quad \text{if } x = X(t; z) \in \tilde{X}$$

relation (2.12) is verified at each point $y \in X_2 = pr_1 A$ (p_0 being replaced, obviously, by $p_0(y)$).

Moreover, for any subset $B_1 \subset \{(t, z) \in B;\ z \in A_1,\ t \in I(z)\}$ containing the points of the form $(0, z)$ and for which the restriction mapp

$X(\cdot\,;\,\cdot)|B_1$ is a C^1-diffeomorphism, the corresponding function, $u(\cdot)$, defined as in (2.11) is a solution of the problem (1.1)-(1.2) and satisfies (2.12)-(2.13).

However, very simple examples ([2], [7], [12], etc.) show that generally, the first component, $X(\cdot\,;\,\cdot)$, of the characteristic flow fails to be a diffeomorphism on larger subsets of B and actually global (i.e. solutions defined on $X = pr_1 A$) classical solutions of problems of the form (1.1)-(1.2) do not exist except for some very special classes of problems.

Some examples in Optimal Control Theory as well as the results to follow justify the following extension of the class of locally Lipshitz generalized solutions in the literature:

Definition 2.4. *A function* $u(\cdot)\colon X \subset R^n \to R$ *is said to be an a.e. solution of the problem* (1.1)-(1.2) *if it satisfies* (1.2), X *has property* (1.3), $u(\cdot)$ *is a.e.* (almost everywhere) *differentiable and satisfies equation* (1.1) *a.e. on* Int (X).

If, in addition, $u(\cdot)$ *is locally Lipschitz* (continuous) *then it is said to be a Lipschitzian* (respectively, continuous) *a.e. solution.*

Remark 2.5. In [2], [10] and in many other papers, a « generalized solution » of the problem (1.1)-(1.2) is a Lipschitzian a.e. solution in the sense of Definition 2.4; a « viscosity solution » in the sense of [8], [15] and some other recent papers is a continuous function $u(\cdot)$ defined on an open subset $X \in R^n$ satisfying the boundary condition (1.2) and the relations:

$$\begin{cases} F(x, p, u(x)) \leqslant 0 & (\forall)x \in X\,,\ p \in D^+u(x) \\ F(x, p, u(x)) \geqslant 0 & (\forall)x \in X\,,\ p \in D^-u(x) \end{cases} \tag{2.14}$$

where the (Fréchet) « semi-differentials » $D^+u(x)$ (« super-differential ») and $D^-u(x)$ (« sub-differential ») are defined by:

$$\begin{cases} D^+u(x) = \\ = \{p \in R^n;\ \limsup\limits_{y\to x} \big(u(y) - u(x) - \langle p, y - x\rangle\big)/\|y - x\| \leqslant 0\}\,, \\ D^-u(x) = \\ = \{p \in R^n;\ \liminf\limits_{y\to x} \big(u(y) - u(x) - \langle p, y - x\rangle\big)/\|y - x\| \geqslant 0\}\,. \end{cases} \tag{2.15}$$

Due to the well known Rademacher's theorem according to which a locally Lipschitz mapping is a.e. differentiable and to the obvious fact that $u(\cdot)$ is differentiable at x iff $D^+u(x) = D^-u(x) = Du(x)$, a locally Lipschitz viscosity solution is also a lipschitzian a.e. solution in the sense of Definition 2.4; on the other hand, an a.e. solution in the sense of Definition 2.1 may be discontinuous (and this makes it applicable to some classes of optimal control problems with discontinuous value functions ([9], [21], etc.)) hence it may not be a viscosity solution; since conditions (2.14) must be verified at each point, even a lipschitzian a.e. solution may not be a viscosity solution as simple examples ([15]) show and, on the other hand, since there exist nowhere differentiable continuous functions, a viscosity solution may not be an a.e. solution in the sense of Definition 2.4.

The a.e. solutions in Definition 2.4 may be too general as no other regularity properties are required so nothing is known about the behaviour of these solutions at the points at which they are not differentiable; except the continuity property, this is also the case of viscosity solutions at the points $x \in X$ at which $D^+u(x) = D^-u(x) = \emptyset$.

On the other hand, we shall consider only a.e. « characteristic » solutions defined in terms of the characteristic flow and this should imply certain properties at the points at which such a solution does not satisfy equation (1.1) in the classical sense; for stratified characteristic solutions such a property is given by (1.4)-(1.5) but for a.e. characteristic solutions a property of this kind is yet to be found and may be expressed, perhaps, in terms of the extreme contingential (Dini) derivatives (e.g. [17]).

Condition (1.3) for the a.e. solutions in Definition 2.4 not only eliminates the trivial case in which Int $(X) = \emptyset$ and $u(\cdot)$ is nowhere differentiable but also requires every point in X to be a limit of points at which equation (1.1) is satisfied in the classicael sense. Condition (1.3) may be droped in the case of stratified solutions to be studied in Section 3 since in this case equation (1.1) is satisfied in the generalized sense (1.4)-(1.5) on sets that may have empty interior.

The *general procedure* to obtain generalized « characteristic solutions » of the problem (1.1)-(1.2) has the following two steps:

I) One considers the « initial characteristic multifunction » $P_0(\cdot)$ defined in (2.6), the « initial characteristic strip », $A_0 \subset A$ in (2.7) and the restriction of the characteristic flow, $Z_F(\cdot\,;\,\cdot) = (X(\cdot\,;\,\cdot), P(\cdot\,;\,\cdot), V(\cdot\,;\,\cdot))$ at A_0 restricting (possibly) the interval $I(z) = (t^-(z), t^+(z)) \subset R$

on which $Z_F(\cdot\,; z)$ is defined such that:

$$X(t; z) \notin X_0 \quad \text{for any } t \in I(z) \setminus \{0\}\,, \quad z \in A_0 \tag{2.16}$$

II) One defines the subset $X \subset R^n$ and the multifunctions $Q(\cdot)$ and $W(\cdot)$ as follows:

$$X = \{X(t; z);\, z \in A_0,\, t \in I(z)\} \tag{2.16}$$

$$Q(x) = \{P(t; z);\, z \in A_0,\, t \in I(z),\, X(t; z) = x\}\,, \quad x \in X \tag{2.18}$$

$$W(x) = \{V(t; z);\, z \in A_0,\, t \in I(z),\, X(t; z) = x\}\,, \quad x \in X \tag{2.19}$$

then one chooses an a.e. differentiable selection $u(\cdot)\colon X \to R$ of the multifunction $W(\cdot)$ whose derivative, $Du(\cdot)$, is an a.e. selection of the multifunction $Q(\cdot)$ satisfying the conditions:

$$u(x) \in W(x) \quad \text{for any } x \in X \tag{2.20}$$

$$Du(x) = P(t; z) \in Q(x) \quad \text{if } x = X(t; z) \in X\,, \quad u(x) = V(t; z) \tag{2.21}$$

Since (2.16) implies: $X_0 \subset X$, $Q(x) = P_0(x)$ and $W(x) = \{u_0(x)\}$ for any $x \in X_0$, from the fact that $F(\cdot, \cdot, \cdot)$ is a first integral of the characteristic vector field $\zeta_F(\cdot, \cdot, \cdot)$ ([2], [7], etc.) hence verifies:

$$F(Z_F(t; z)) = F(X(t; z), P(t; z), V(t; z)) = 0 \qquad \text{for any } z \in A_0\,, \quad t \in I(z) \tag{2.22}$$

it follows immediately:

PROPOSITION 2.6. *If the data $F(\cdot, \cdot, \cdot)$ and $u_0(\cdot)$ satisfy Hypothesis* 2.1, *the initial characteristic multifunction $P_0(\cdot)$ in* (2.6) *has non-empty values at each point in X_0 and $X \subset R^n$ defined by* (2.17) *satisfies* (1.3) *then any a.e. differentiable selection $u(\cdot)\colon X \to R$ of $W(\cdot)$ satisfying* (2.20)-(2.21) *is an a.e. solution of the problem* (1.1)-(1.2) *in the sense of Definition* 2.4.

Obvious candidates for a.e. differentiable selections satisfying conditions (2.20)-(2.21) are the « marginal » selections:

$$u_m(x) = \min\{v;\, v \in W(x)\}\,, \quad x \in X_m \subset X \tag{2.23}$$

$$u_M(x) = \max\{v;\, v \in W(x)\}\,, \quad x \in X_M \subset X \tag{2.24}$$

where X_m (respectively, X_M) is the subset of all points $x \in X$ at which the minimum in (2.23) (respectively, the maximum in (2.24)) exists; in view of the results to follow we shall call $u_m(\cdot)$ and $u_M(\cdot)$ *marginal characteristic solutions* of the problem (1.1)-(1.2).

Since for $u_m(\cdot)$ and $u_M(\cdot)$ the same type of results are valid we shall consider in what follows only the « *minimal* » *characteristic solution* $u_m(\cdot)$.

We shall prove next that for the « classical » problems in Proposition 2.6 for which $P_0(\cdot)$ is single valued and of class C^1, the function $u_m(\cdot)$ in (2.23) (and also $u_M(\cdot)$ defined in (2.24)) is a.e. differentiable and satisfies (2.20)-(2.21) and therefore, since (2.16) implies the fact that $u_m(\cdot)$ verifies (1.2), it remains to prove for more particular classes of problems that its domain, X_m, satisfies (1.3) or, at least, the weaker condition $\operatorname{Int}(X_m) \neq \emptyset$ to prove that $u_m(\cdot)$ is indeed an a.e. solution of the problem (1.1)-(1.2) in the sense of Definition 2.4.

In the remaining of this section we shall assume that the problem (1.1)-(1.2) is « classical » in the following sense:

DEFINITION 2.7. *The problem* (1.1)-(1.2) *is said to be classical if its data* $F(\cdot, \cdot, \cdot)$ *and* $u_0(\cdot)$ *satisfy Hypothesis* 2.1 *and the initial characteristic multifunction* $P_0(\cdot)$ *defined in* (2.6) *is single valued and of class* C^1.

As already mentioned, in actual classical problems only a local selection $p_0(\cdot)$ of $P(\cdot)$, of class C^1, was obtained under the additional compatibility and transversality conditions (2.1)-(2.3); in fact, a boundary value problem it is not well defined only by the conditions (1.1)-(1.2) but also by condition (2.12) defined by a fixed selection $p_0(\cdot)$ of the multifunction $P_0(\cdot)$. It follows that for the same data $F(\cdot, \cdot, \cdot)$ and $u_0(\cdot)$ one may consider a boundary value problem of the form (1.1)-(1.2), (1.8) for every selection $p_0(\cdot)$ of $P_0(\cdot)$, of class C^1, which explains many non-uniqueness cases in the literature.

We note that if the problem (1.1)-(1.2) is classical in the sense of Definition 2.3 then the initial characteristic strip A_0 defined in (2.7) is a $(n-1)$-dimensional differentiable submanifold of class C^1 of the open subset $A \subset R^n \times R^n \times R$ and the subset $B_0 \subset R \times A$ defined by:

$$B_0 = \{(t, z);\ z = (y, q, w) \in A_0,\ t \in I(z)\} \tag{2.25}$$

is a n-dimensional differentiable submanifold of class C^1 of the open subset $R \times A \subset R \times R^{2n+1}$. From (2.7) and (2.25) it follows that the tangent space of A_0 at $z = (y, P_0(y), u_0(y)) \in A_0$, $y \in X_0$, is given by: $T_z A_0 = \{(\bar{y}, DP_0(y)\cdot\bar{y}, Du_0(y)\cdot\bar{y});\ \bar{y} \in T_y X_0\}$ and the tangent space of B_0 at the point $(t, z) \in B_0$ is given by: $T_{(t,z)}B_0 = R \times T_z A_0$.

To prove the main result of this section we need two preliminary results.

In view of its extension to a more general case in Section 3, we give the proof of the next lemma which is a global version of a result that is usually hidden inside the proof of the Cauchy's existence theorem ([7], [11], [12], etc.).

LEMMA 2.8. *If $Z_F(\cdot\,;\cdot) = (X(\cdot\,;\cdot), P(\cdot\,;\cdot), V(\cdot\,;\cdot))\colon B \subset R \times A \to A$ is the flow of the characteristic vector field $\zeta_F(\cdot,\cdot,\cdot)$ defined in* (2.4) *then for any $z = (x, p, v) \in A$, $\bar{t} \in R$, $\bar{z} = (\bar{x}, \bar{p}, \bar{v}) \in R^n \times R^n \times R$ the function $r(\cdot\,; z, \bar{t}, \bar{z})\colon I(z) \to R$ defined by*:

$$(2.26)\qquad r(t; z, \bar{t}, \bar{z}) = DV(t; z)\cdot(\bar{t}, \bar{z}) - \langle P(t, z), DX(t; z)\cdot(\bar{t}, \bar{z})\rangle\,, \quad t \in I(z)$$

satisfies:

$$(2.27)\qquad r(t; z, \bar{t}, \bar{z}) = D_2 V(t; z)\cdot\bar{z} - \langle P(t; z), D_2 X(t; z)\cdot\bar{z}\rangle \quad \text{for any } t \in I(z)$$

and is the unique solution of the initial value problem:

$$(2.28)\qquad r' = DF(Z_F(t; z))\cdot(D_2 Z_F(t; z)\cdot\bar{z} - (0, 0, r))$$

$$(2.29)\qquad r(0) = \bar{v} - \langle p, \bar{x}\rangle \quad \text{if } z = (x, p, v) \in A\,,\quad \bar{z} = (\bar{x}, \bar{p}, \bar{v}) \in T_z A = R^n \times R^n \times R\,.$$

In particular, if the problem (1.1)-(1.2) *is classical in the sense of Definition* 2.7 *then*:

$$(2.30)\qquad r(t; z, \bar{t}, \bar{z}) = 0 \quad \text{for any } z \in A_0\,,\ t \in I(z)\,,\ (\bar{t}, \bar{z}) \in T_{(t,z)}B_0 = R \times T_z A_0$$

PROOF. – We shall use the already mentioned fact that $F(\cdot,\cdot,\cdot)$ is a first integral of the characteristic vector field $\zeta_F(\cdot,\cdot,\cdot)$ (i.e. satisfies (2.22)) and the well known properties of the flow of a smooth vector field (e.g. [1], [14]) among which we mention relations (2.8) and the fact that: $D_2 Z_F(0; z)\cdot\bar{z} = \bar{z}$ for any $z \in A$ and $\bar{z} \in T_z A = R^{2n+1}$.

We note first that from (2.4) it follows:

$$(2.31) \qquad D_1 V(t; z) = \langle P(t; z), D_1 X(t; z)\rangle \quad \text{for any } (t, z) \in B$$

and therefore, since $DZ_F(t; z)\cdot(\bar{t}, \bar{z}) = D_1 Z_F(t; z)\cdot\bar{t} + D_2 Z_F(t; z)\cdot\bar{z}$, from (2.26) and (2.31) it follows (2.27).

Further on, from (2.27) it follows: $r(0; z, \bar{t}, \bar{z}) = \bar{v} - \langle p, \bar{x}\rangle$ hence $r(\cdot; z, \bar{t}, \bar{z})$ verifies the initial condition (2.29); to prove that this function is a solution of the « affine » differential equation (2.28) we use (2.27), (2.8), (2.31) and the rules of differentiation to write successively:

$$\begin{aligned}
r'(t; z, \bar{t}, \bar{z}) &= D_1 D_2 V(t; z)\cdot\bar{z} - \langle D_1 P(t; z), D_2 X(t; z).\bar{z}\rangle - \\
&\quad - \langle P(t; z), D_1 D_2 X(t; z).\bar{z}\rangle = \frac{\partial}{\partial z}\big(\langle P(t; z), D_2 F(Z_F(t; z))\rangle\big).\bar{z} - \\
&\quad - \Big\langle P(t; z), \frac{\partial}{\partial z} D_2 F(Z_F(t; z)).\bar{z}\Big\rangle + \langle D_1 F(Z_F(t; z)), D_2 X(t; z), \bar{z}\rangle + \\
&\quad + \langle P(t; z), D_2 X(t; z).\bar{z}\rangle\cdot D_3 F(Z_F(t; z)) = \\
&\quad = \langle D_2 F(Z_F(t; z)), D_2 P(t; z).\bar{z}\rangle + \langle D_1 F(Z_F(t; z)), D_2 X(t; z).\bar{z}\rangle + \\
&\quad + \langle P(t; z), D_2 X(t; z).\bar{z}\rangle\cdot D_3 F(Z_F(t; z)) = \\
&\qquad\qquad = DF(Z_F(t; z))\cdot\big(D_2 Z_F(t; z)\cdot\bar{z} - (0, 0, r(t; z, \bar{t}, \bar{z}))\big).
\end{aligned}$$

If (1.1)-(1.2) is a classical problem in the sense of Definition 2.7 then from (2.6)-(2.7) it follows that for any $z = (x, p, v) \in A_0$ and $\bar{z} = (\bar{x}, \bar{p}, \bar{v}) \in T_z A_0$ one has: $\bar{v} = \langle p, \bar{x}\rangle$ hence $r(0; z, \bar{t}, \bar{z}) = 0$; further on, since A_0 and B_0 are differentiable manifolds, from (2.22) it follows that $DF(Z_F(t; z))\cdot DZ_F(t; z)\cdot(\bar{t}\cdot\bar{z}) = 0$ for any $(t, z) \in B_0$, $(\bar{t}, \bar{z}) \in T_{(t,z)} B_0$ hence in this case the function $r(\cdot; z, \bar{t}, \bar{z})$ defined in (2.26) is a solution of the linear differential equation: $r' = -DF(Z_F(t; z))\cdot(0, 0, r)$ and therefore from the fact that $r(0; z, \bar{t}, \bar{z}) = 0$ it follows (2.30) and the lemma is completely proved.

An essential tool for the results to follow is given in the next lemma caracterizing the derivatives of the « marginal functions » of the form (2.23) and (2.24):

LEMMA 2.9. *Let* $Y \subset R^m$ *be an n-dimensional differentiable submanifold of class* C^1, $n \leqslant m$, *let* $h(\cdot): Y \to R^n$, $g(\cdot): Y \to R$ *be of class* C^1 *and let* $f(\cdot): X \subset h(Y) \to R$ *be defined by*:

$$f(x) = \min\{g(y);\ y \in Y, h(y) = x\}\ , \quad x \in X \tag{2.32}$$

(i) *If* $x \in \mathrm{Int}(X)$, $y \in Y$, $h(y) = x$, $g(y) = f(x)$, $f(\cdot)$ *is differenrentiable at* x *and* $Dh(y) \in L(T_y Y, R^n)$ *is surjective then the derivative of* $f(\cdot)$ *at* x *is given by*:

$$Df(x)\cdot\bar{x} = Dg(y)\cdot\bar{y} \qquad \text{for any } \bar{x} \in R^n\ , \quad \bar{y} \in T_y Y \tag{2.33}$$

$$\text{for which } Dh(y)\cdot\bar{y} = \bar{x}$$

(ii) *If* $\mathrm{Int}(X) \neq \emptyset$ *and* $f(\cdot)$ *in a.e. differentiable on* $\mathrm{Int}(X)$ *then its derivative*, $Df(\cdot)$, *is given by* (2.33) *at almost all points* $x \in \mathrm{Int}(X)$.

(iii) *If* $h(\cdot)$ *is proper* (i.e. $h^{-1}(K) \subset Y$ is compact whenever $K \subset R^n$ is compact) *then* $X = h(Y)$, $f(\cdot)$ *is lower semicontinuous at every point* $x \in X$ *and locally Lipschitz at almost all points in* X *hence, if in addition* $\mathrm{Int}(h(Y)) \neq \emptyset$ *then* $f(\cdot)$ *is a.e. differentiable and satisfies* (2.33) *a.e. on* $\mathrm{Int}(X) = \mathrm{Int}(h(Y))$.

(iv) *If* $h(\cdot)$ *is proper and its derivative is surjective at each point* $y \in Y$ *then* $X = h(Y) \subset R^n$ *is open and* $f(\cdot)$ *is locally Lipschitz at every point* $x \in X$ *hence a.e. differentiable and satisfies* (2.33) *a.e. on* X.

PROOF. (i) Since $\dim(Y) = n$, if $Dh(y) \in L(T_y Y, R^n)$ is surjective then it is a linear isomorphism hence from the inverse functions theorem it follows that there exists an open neighbourhood U of $y \in Y$ such that the restriction mapp $h_U(\cdot) = h(\cdot)|U$ is a C^1-diffeomorphism; if $c(\cdot): (-\gamma, \gamma) \to R^n$ is of class C^1 and satisfies: $c(0) = x$, $c'(0) = \bar{x}$ then there exists $0 < r \leqslant \gamma$ such that $c(t) \in h(U)$ for any $t \in (-r, r)$ and $k(t) = h_U^{-1}(c(t))$, $t \in (-r, r)$ satisfies: $k(0) = y$, $k'(0) = \bar{y}$ (since $Dh(y)\cdot\bar{y} = \bar{x}$). Since $f(x) = g(y)$, $f(c(t)) \leqslant f(k(t))$ for any $t \in (-r, r)$ and $f(\cdot)$ is assumed to be differentiable at x one has: $Df(x)\cdot\bar{x} = \lim_{t\to 0}\big(f(c(t)) - f(x)\big)/t$; from the above inequality it follows that for any $t \in (0, r)$ one has: $\big(f(c(t)) - f(x)\big)/t \leqslant \big(g(k(t)) - g(y)\big)/t \to Dg(y)\cdot\bar{y}$ as $t \to 0^+$ hence $Df(x)\cdot\bar{x} \leqslant Dg(y)\cdot\bar{y}$; reasoning in the same way for $t \in (-r, 0)$ one obtains the reversed inequality and (2.33) is proved.

(ii) From Sard's theorem (e.g. [1], § 5) it follows that almost all points $x \in h(Y)$ are regular values of $h(\cdot)$ i.e. $Dh(y)$ is surjective at any point $y \in h^{-1}(x)$ hence if Int $(X) \neq 0$ and $f(\cdot)$ is a.e. differentiable then at almost all points $x \in \text{Int}(X) \subset h(Y)$ $f(\cdot)$ is differentiable and $Dh(y)$ is surjective at any point $y \in h^{-1}(x)$ and therefore the statement follows from (i).

(iii) If $h(\cdot)$ is proper than for any $x \in h(Y)$ the subset $h^{-1}(x) \subset Y$ is compact hence the minimum in (2.32) exists.

To prove that $f(\cdot)$ is lower semicontinuous at $x \in X = h(Y)$ it suffices to show that for any sequence $x_k \to x$ for which $f(x_k) \to a$ one has: $a \geqslant f(x)$; since $h(\cdot)$ is proper and $g(\cdot)$ is continuous it follows that for any $k \in N$ there exists $y_k \in h^{-1}(x_k)$ such that $f(x_k) = g(y_k)$ and also that $\{y_k\}$ is bounded hence it has a convergent subsequence, say $y_{k_m} \to y \in Y$. Since $h(\cdot)$ and $g(\cdot)$ are continuous, one has: $h(y) = x$ and $a = \lim_{k_m \to \infty} f(x_{k_m}) = \lim_{k_m \to \infty} g(y_{k_m}) = g(y) \geqslant f(x)$.

We prove now that there exists a subset $A \subset R^n$ of zero Lebesgue measure such that for any $x \in X \setminus A$ the function $f(\cdot)$ defined in (2.32) is locally Lipschitz on a neighbourhood of x; we take A to be the set of critical values of $h(\cdot)$ (i.e. $x \in A$ iff there exists $y \in h^{-1}(x)$ such that $Dh(y)$ is not surjective) which, according to Sard's theorem has measure zero in R^n.

Since $x_0 \in X \setminus A$ is a regular value of $h(\cdot)$ and $h(\cdot)$ is proper, there exists a compact neighbourhood G of x_0 in R^n such that each point $x \in G$ is a regular value of $h(\cdot)$ (otherwise it would exist a sequence $y_k \to y_0 \in h^{-1}(x_0)$ such that $\det(Dh(y_k)) = 0$ for any $k \in N$ hence $\det(Dh(y_0)) = 0$ contradicting the fact that x_0 is a regular value). Since $h(\cdot)$ is proper, the subset $h^{-1}(G) \subset Y$ is compact hence it may be covered by a finite family of open subsets $Y_1, Y_2, \ldots, Y_k \subset Y$ such that the restrictions $h_j(\cdot) = h(\cdot)|Y_j$, $j = 1, 2, \ldots, k$ are C^1 diffeomorphisms and therefore the function $f(\cdot)$ in (2.32) is given on G by: $f(x) = \min\{g(h_j^{-1}(x)); j = 1, 2, \ldots, k\}$, $x \in G$. Since the functions $g(h_j^{-1}(\cdot))$, $j = 1, 2, \ldots, k$ are of class C^1 function $f(\cdot)$ turns out to be Lipschitzian on an open neighbourhood $G_0 \subset G$ of x_0 and according to Rademacher's theorem is a.e. differentiable.

(iv) If $Dh(y)$ is surjective at each point $y \in Y$ then the same proof as above shows that $X = h(Y) \subset R^n$ is open and $f(\cdot)$ is locally Lipshitz at each point $x \in X$.

REMARK 2.10. The examples to follow show that one cannot expect more regularity properties of the marginal function $f(\cdot)$ in (2.32) unless more hypotheses are added to those in statement (iii) of the Lemma 2.9.

At critical values of $h(\cdot)$ the function $f(\cdot)$ may not be continuous even if $h(\cdot)$ is proper as the following example shows: if $h(y) = y^3 + y^2$ and $g(y) = y$ for any $y \in R = Y$ then $f(\cdot)$ is given by: $f(x) = h_1^{-1}(x)$ if $x \leqslant 4/27$ and $f(x) = h_2^{-1}(x)$ if $x > 4/27$ where $h_1(\cdot)$ and $h_2(\cdot)$ are the strictly monotone restrictions of $h(\cdot)$: $h_1(\cdot) = h(\cdot)|(-\infty, -2/3)$, $h_2(\cdot) = h(\cdot)|(0, \infty)$; it is easy to see that $f(\cdot)$ is not continuous at $x_0 = 4/27$ which is a critical value of $h(\cdot)$.

On the other hand, if $h(\cdot)$ is not proper then $f(\cdot)$ may not be lower semicontinuous even at points that are regular values of $h(\cdot)$: if $h(y) = y \cdot \exp(y)$ and $g(y) = y$ for any $y \in R = Y$ then $f(\cdot)$ is given by: $f(x) = h_1^{-1}(x)$ if $x \in (-\exp(-1), 0)$, $f(x) = h_2^{-1}(x)$ if $x > 0$ and $f(0) = 0$ where $h_1(\cdot) = h(\cdot)|(-\infty, -1)$ and $h_2(\cdot) = h(\cdot)|(-1, \infty)$; $f(\cdot)$ is not lower semicontinuous at $x_0 = 0$ $(f(0-) = -\infty, f(0+) = f(0) = 0)$ which is a regular value of $h(\cdot)$.

From Lemmas 2.8 and 2.9 it follows very easily the next theorem validating the functions $u_m(\cdot)$ and $u_M(\cdot)$ defined in (2.23) and (2.24), respectively, as generalized solutions of the problem (1.1)-(1.2) under appropriate hypotheses on the components of the characteristic flow, $Z_F(\cdot\,;\,\cdot) = (X(\cdot\,;\,\cdot), P(\cdot\,;\,\cdot), V(\cdot\,;\,\cdot))$ in Definition 2.3:

THEOREM 2.11. *Let us assume that the problem* (1.1)-(1.2) *is classical in the sense of Definition* 2.7 *and the subset* $X_m \subset X$ *defined in* (2.23) *satisfies condition* (1.3).

(i) *If* $u_m(\cdot)\colon X_m \to R$ *is a.e. differentiable then it is an a.e. solution of the problem* (1.1)-(1.2) *in the sense of Definition* 2.4.

(ii) *If the restriction* $X(\cdot\,;\,\cdot)|B_0$ *of the first component of the characteristic flow is proper then* $X_m = X(B_0)$ *and* $u_m(\cdot)$ *is an a.e. solution of* (1.1)-(1.2) *which is lower semicontinuous at each point and a.e. locally Lipschitz on* X_m.

(iii) *If the restriction mapp* $X(\cdot\,;\,\cdot)|B_0$ *is proper and its derivative is surjective at each point in* B_0 *then* $X_m = X = X(B_0) \subset R^n$ *is open and* $u_m(\cdot)$ *is a Lipschitzian a.e. solution of the problem* (1.1)-(1.2) *in the sense of Definition* 2.4.

PROOF. To prove (i) we note that from statements (i) and (ii) Lemma 2.9 it follows that at almost all points $x \in \operatorname{Int}(X_m)$ the function $u_m(\cdot)$ given by (2.23) (which is of the form (2.32)) is differentiable and for any $(t, z) \in B_0$, $\bar{x} \in R^n$ and $(\bar{t}, \bar{z}) \in T_{(t,z)}B_0$ for which:

$$(2.34) \qquad u_m(x) = V(t; z), \qquad x = X(t; z), \qquad DX(t; z)\cdot(\bar{t}, \bar{z}) = \bar{x}$$

its derivative is given by:

$$(2.35) \qquad Du_m(x)\cdot\bar{x} = DV(t; z)\cdot(\bar{t}, \bar{z}).$$

From (2.26) and (2.30) in Lemma 2.8 it follows that $DV(t; z)\cdot\cdot(\bar{t}, \bar{z}) = \langle P(t; z), DX(t; z)\cdot(\bar{t}, \bar{z})\rangle$ hence (2.34)-(2.35) imply: $Du_m(x)\cdot\bar{x} = = \langle P(t; z), \bar{x}\rangle$ for any $\bar{x} \in R^n$ and any $(t, z) \in B_0$ that satisfy (2.34) and therefore $Du_m(x) = P(t; z) \in Q(x)$ a.e. on X_m which means, according to Proposition 2.6 that $u_m(\cdot)$ is an a.e. solution of the problem (1.1)-(1.2) in the sense of Definition 2.4.

Statements (ii) and (iii) in the theorem obviously follow from the statements (iii) and (iv) in Lemma 2.9, respectively.

REMARK 2.12. For the problems of the form (1.1)-(1.2) for which the characteristic flow $Z_F(\cdot\,;\,\cdot) = (X(\cdot\,;\,\cdot), P(\cdot\,;\,\cdot), V(\cdot\,;\,\cdot)) : B \to A$ may be sufficiently well characterized Theorem 2.11 furnishes as a.e. solutions the « characteristic extremal solutions » $u_m(\cdot)$ and $u_M(\cdot)$ defined in (2.23) and (2.24), respectively; other characteristic a.e. solutions may be obtained by Proposition 2.6 whenever the characteristic flow allows the characterization of other a.e. differentiable selections satisfying conditions (2.20)-(2.21).

On the other hand, results of the type of those in Lemma 2.9 for the « semi-differentials » (2.15) of the marginal functions of the form (2.32) may provide criteria under which the extremal characteristic solutions $u_m(\cdot)$ and $u_M(\cdot)$ are also « viscosity solutions » in the sense of Remark 2.5; this topic is a subject of current research.

3. Stratified problems and solutions.

The theory of stratified sets and mappings initiated by E. Whitney ([25]) and extensively developped R. Thom ([24]), J. Mather ([16]) and other mathematicians working in Differential Topology had been successfully applied also in other areas of Mathematics such as Calculus of Variations ([23]) and Optimal Control Theory ([4], [22], etc.).

The example of Subanalytic sets and functions ([4], [22]-[23])

show that the class of stratified sets and mappings is a significant one from the point of view of applications and its most important feature is the possibility of developping a « calculus » very similar to the one that is valid for differentiable mappings defined on differentiable manifolds.

In view of this necessity the original definition of the so called « Whitney stratified sets and mappings » was weakened in [20] to introduce « weakly stratified sets and mappings » that contain as particular cases the classical differentiable mappings defined on open sets or on differentiable manifolds.

In what follows only the very few properties of the weakly stratified sets and mappings needed in the context of this paper will be presented.

Throughout in the sequel, a nonempty subset $X \subset R^n$ is said to be *weakly C^k-stratified of dimension $m \leqslant n$*, for some $k \in \{1, 2, ..., \infty, \omega\}$ if it has a locally finite partition $\mathcal{S}$ (i.e. any compact subset intersects only a finite number of members of $\mathcal{S}$) into connected differentiable submanifolds of R^n, of class C^k, called *strata*, among which at least one is m-dimensional and the others are of lower dimension; X is said to be *C^k-stratified* if the « stratification » $\mathcal{S}$ has the following additional property: if $S_1, S_2 \in \mathcal{S}$, $S_1 \neq S_2$ and $S_1 \cap \mathrm{Cl}\,(S_2) \neq \emptyset$ then $S_1 \subset \mathrm{Cl}\,(S_2)$ and $\dim\,(S_1) < \dim\,(S_2)$; the subset X is said to be *Whitney C^k-stratified* if it is C^k-stratified and whenever $x_k \to x \in S_1 \in \mathcal{S}$, $x_k \in S_2 \in \mathcal{S}$ for any $k \in N$ one has: $T_x S_1 \subset \lim\limits_{k \to \infty} T_{x_k} S_2$ where the limit is taken in the Grassmann manifold of the suspaces of R^n.

A stratification $\mathcal{S}$ of X is said to be *compatible with a given family* $\mathcal{A}$ of subsets of R^n if every member of $\mathcal{A}$ is either disjoint of X or a union of strata from $\mathcal{S}$.

The tangent space of X at a point $x \in X$ with respect to a stratification $\mathcal{S}$ is defined as follows: $T_x X = T_x S$ if $x \in S \in \mathcal{S}$.

A mapping $f(\cdot): X \subset R^n \to R^m$ is said to be *weakly C^k-stratified* if there exists a weak C^k-stratification $\mathcal{S}_{f,X}$ of X such that for any $S \in \mathcal{S}_{f,X}$ the restriction mapp $f_S(\cdot) = f(\cdot)|S$ is of class C^k; $f(\cdot)$ is said to be a *weakly C^k-stratified submerssion* if there exists a weak C^k-stratification $\mathcal{S}_{f,X}$ of X and a weak C^k-stratification $\mathcal{S}_{f(X)}$ of $f(X) \subset R^m$ such that for any $S \in \mathcal{S}_{f,X}$ one has: $f(S) \in \mathcal{C}_{f(X)}$ and the restriction mapp $f_S(\cdot): S \to f(S)$ is a submerssion of class C^k; C^k-stratified mappings and submerssions are defined in a similar way.

The derivative of $f(\cdot)$ with respect to the stratification $\mathcal{S}_{f,X}$ is defined as follows: $Df(x) = Df_S(x) \in L(T_x X, R^m)$ if $x \in S \in \mathcal{S}_{f,X}$.

We note that at points in open (i.e. n-dimensional) strata the above derivative coincides with the usual (Fréchet) one but at points in lower dimensional strata a C^k-stratified mapping may be even discontinuous.

DEFINITION 3.1. *A function* $u(\cdot): X \subset R^n \to R$ *is said to be a stratified solution of the problem* (1.1)-(1.2) *if it is weakly* C^1*-stratified, satisfies* (1.2) *and for any* $x \in X$ *there exists* $p_x \in R^n$ *such that* (1.4)-(1.5) *are verified.*

REMARK 3.2. Since in case $X \subset R^n$ is open a function $u(\cdot): X \to R$ of class C^1 is a particular case of weakly C^1-stratified function and moreover, the only vector $p_x \in R^n$ satisfying (1.4) is the derivative, $Du(x)$, a classical solution is a particular type of stratified solution; on the other hand, a stratified solution that satisfies (1.3) is obviously, a particular type of a.e. solution in the sense of Definition 2.4.

Stratified solutions as well as the a.e. solutions in Section 2 may exist not only for the classical problems in Definition 2.7 but also for «stratified problems» defined as follows:

DEFINITION 3.3. *The problem* (1.1)-(1.2) *is said to be stratified it its data,* $F(\cdot, \cdot, \cdot)$ *and* $u_0(\cdot)$ *have the following properties:*

(S.1) *The boundary function* $u_0(\cdot): X_0 \to R$ *is a stratified solution of* [1.1)-(1.2) (i.e. it is weakly C^1-stratified and the initial characteristic multifunction $P_0(\cdot)$ in (2.6) has nonempty values at each point $x \in X_0$) *and* $X_0 \subset R^n$ *is of dimension* $m \leqslant n-1$ *as a weakly* C^1*-stratified set*;

(S.2) *The initial characteristic strip* $A_0 \subset A$ *defined by* (2.7) *is weakly* C^1*-stratified by a stratification* $\mathcal{S}_0$ *that has the following property: for any* $z = (x, p, v) \in A_0$ *and any* $\bar{z} = (\bar{x}, \bar{p}, \bar{v}) \in T_z A_0$ *one has:*

$$(3.1) \qquad \bar{x} \in T_x X_0 , \quad \bar{v} = \langle p, \bar{x} \rangle$$

(S.3) $F(\cdot, \cdot, \cdot): A \to R$ *is locally Lipschitz and weakly* C^1*-stratified by a stratification* $\mathcal{S}_F$ *of* A *such that* $\mathcal{S}_0$ *is compatible with* $\mathcal{S}_F$ *which has the following property: for any* $z = (x, p, v) \in A$ *and any* $\bar{z} = (\bar{x}, \bar{p}, \bar{v}) \in T_z A$ *one has:*

$$(3.2) \qquad (\bar{x}, \bar{p}, \langle p, \bar{x} \rangle) \in T_z A .$$

It is easy to see that a classical problem in the sense of Definition 2.7 is a particular type of stratified problem.

In the case of stratified problems the characteristic vector field defined in (2.4) is replaced by the ***stratified characteristic orientor field*** defined by:

$$(3.3)\quad \zeta_F(x, p, v) = \{(\bar{x}, \bar{p}, \langle p, \bar{x}\rangle) \in T_{(x,p,v)}A;\ DF(x, p, v)\cdot(\bar{y}, \bar{q}, \langle p, \bar{y}\rangle) = = \langle \bar{x}, \bar{q}\rangle - \langle \bar{p}, \bar{y}\rangle \text{ for any } (\bar{y}, \bar{q}, \langle p, \bar{y}\rangle) \in T_{(x,p,v)}A\}$$

which, on open strata of S_F takes the form (2.4) but at points $z \in A$ in lower dimensional strata $\zeta_F(z)$ may be either a singleton or the empty set or a linear manifold of T_zA; in case $\zeta_F(\cdot)$ has empty values on all lower dimensional strata it turns out to be a piecewise continuous vector field.

An absolutely continuous mapping $Z(\cdot) = (X(\cdot), P(\cdot), V(\cdot)): I \subset \subset R \to A$ defined on an interval I that satisfies:

$$(3.4)\qquad Z'(t) \in \zeta_F(Z(t)) \quad \text{a.e. on } I$$

is said to be *a characteristic of the problem* (1.1)-(1.2).

The next result shows that, as in the classical case, the function $F(\cdot, \cdot, \cdot)$ is a first integral of the stratified characteristic orientor field:

PROPOSITION 3.4. *If the problem* (1.1)-(1.2) *is stratified then for any characteristic* $Z(\cdot): I \to A$ *there exists* $C \in R$ *such that*;

$$(3.5)\qquad F(Z(t)) = C \quad \text{for any } t \in I$$

In particular, if there exists $t_0 \in I$ *such that* $Z(t_0) \in A_0$ *then*

$$(3.6)\qquad F(Z(t)) = 0 \quad \text{for any } t \in I$$

PROOF. From Lemma 2.5 in [20] it follows that there exists a subset $J \subset I$ of full measure such that $Z'(t) \in T_{Z(t)}A$ for any $t \in J$; further on, from (3.2)-(3.4) it follows that if $Z(t) = (X(t), P(t), V(t))$ then:

$$(3.7)\qquad V'(t) = \langle P(t), X'(t)\rangle \quad \text{for any } t \in J$$

and therefore

$$DF(X(t), P(t), V(t)) \cdot (\bar{y}, \bar{q}, \langle P(t), \bar{y} \rangle) = \langle X'(t), \bar{q} \rangle - \langle P'(t), \bar{y} \rangle$$

for any $(\bar{y}, \bar{q}, \langle P(t), \bar{y} \rangle) \in T_{Z(t)}A$ hence $DF(Z(t)) \cdot Z'(t) = 0$ for any $t \in J$ which, according to Lemma 2.6 in [20], implies the fact that $t \to F(Z(t))$ is a constant function.

If $Z(t_0) \in A_0$ then from (2.7) and (3.5) it follows (3.6) and the Proposition is proved.

In what follows we denote by $\mathcal{C}$ the set of all characteristics $Z(\cdot) = (X(\cdot), P(\cdot), V(\cdot)) : I(Z(\cdot)) \subset R \to A$ satisfying (2.16) and:

$$0 \in I(Z(\cdot)) , \quad Z(0) \in A_0 \tag{3.8}$$

hence satisfying also (3.6).

We note that in the case of stratified problems, through the same point $z \in A_0$ may pass several characteristics.

Following the general procedure described in Section 2 we define:

$$X = \{X(t); Z(\cdot) = (X(\cdot), P(\cdot), V(\cdot)) \in \mathcal{C}, t \in I(Z(\cdot))\} \tag{3.9}$$

$$Q(x) = \{P(t); Z(\cdot) = (X(\cdot), P(\cdot), V(\cdot)) \in \mathcal{C}, t \in I(Z(\cdot)), X(t) = x\} , \quad x \in X \tag{3.10}$$

$$W(x) = \{V(t); Z(\cdot) \in \mathcal{C}, t \in I(Z(\cdot)), X(t) = x\} , \quad x \in X . \tag{3.11}$$

As in the classical case, from (2.16) and (3.8) it follows: $X_0 \subset X$, $Q(x) = P_0(x)$ and $W(x) = \{u_0(x)\}$ for any $x \in X_0$ hence from Proposition 3.4 and Definition 3.1 follows directly:

PROPOSITION 3.5. *If the problem* (1.1)-(1.2) *is stratified in the sense of Definition* 3.3 *and* $u(\cdot): X \to R$ *is a weakly* C^1*-stratified selection of the multifunction* $W(\cdot)$ *in* (3.11) *such that for any* $x \in X$ *there exists* $Z(\cdot) = (X(\cdot), P(\cdot), V(\cdot)) \in \mathcal{C}$ *and* $t \in I(Z(\cdot))$ *such that* $x = X(t)$ *and* $p_x = P(t) \in Q(x)$ *satisfies* (1.4) *then* $u(\cdot)$ *is a stratified solution of the problem* (1.1)-(1.2) *in the sense of Definition* 3.1.

As in the case of the characteristic a.e. solutions in Section 2 one may prove, under suitable hypotheses, that the marginal functions

$u_m(\cdot)$ and $u_M(\cdot)$ defined in (2.23) and (2.24), respectively, are stratified solutions. In the case of stratified problems additional difficulties arise in connection with the existence and other properties of the characteristics in the set $\mathcal{C}$ considered above.

We shal consider the case in which the set $\mathcal{C}$ of the characteristics may be organized as a family of « piecewise smooth flow »—direct generalization of the situation met in the case of classical problems:

DEFINITION 3.6. *Let $\tilde{A} \subset R^{2n+1}$ and $\tilde{B} \subset R \times \tilde{A}$ be open subset, let $Z(\cdot\,;\,\cdot)\colon \tilde{B} \to \tilde{A}$ be of class C^1 and let $S \in \mathcal{S}_F$ be a stratum such that $S \subset \tilde{A}$.*

The mapping $Z(\cdot\,;\,\cdot)$ is said to be a smooth characteristic flow over the stratum S if there exists a vector field $\zeta_S(\cdot)\colon \tilde{A} \to R^{2n+1}$ of class C^1 such that:

$$\zeta_S(z) \in \zeta_F(z) \cap T_z S \quad \text{for any } z \in S \tag{3.12}$$

and such that $Z(\cdot\,;\,\cdot)$ is the flow of $\zeta_S(\cdot)$.

A mapping $Z(\cdot\,;\,\cdot) = (X(\cdot\,;\,\cdot), P(\cdot\,;\,\cdot), V(\cdot\,;\,\cdot))\colon B \subset R \times A_0 \to A$ is said to be a piecewise smooth characteristic flow of the stratified problem (1.1)-(1.2) *if there exist the « switching functions »*

$$t_k^-(\cdot)\colon A_0 \to [-\infty, 0]\,, \qquad k = 0, 1, \ldots, k_0\,,$$

$$t_j^+(\cdot)\colon A_0 \to [0, \infty]\,, \qquad j = 0, 1, \ldots, j_0$$

with the following properties:

(P.S.1) *For any k, j, the functions $t_k^-(\cdot), t_j^+(\cdot)$ are weakly C^1-stratified by $\mathcal{S}_0$ and for any $z \in A_0$ satisfy:*

$$t_{k-1}^-(z) \leqslant t_k^-(z) \leqslant t_0^-(z) = 0 = t_0^+(z) \leqslant t_j^+(z) \leqslant t_{j+1}^+(z) \tag{3.13}$$

and for any stratum $S \in \mathcal{S}_0$ and any k, j, either the inequality is strict for any $z \in S$ or $t_{k+1}^-(z) = t_k^-(z)$ (respectively, $t_{j+1}^+(z) = t_j^+(z)$) *for any $z \in S$;*

(P.S.2) *For any $S \in \mathcal{S}_0$ and any $k = 0, 1, \ldots, k_0 - 1$ $j = 0, 1, \ldots$ $\ldots, j_0 - 1$ there exist the strata S_k^-, σ_k^-, S_j^+, σ_j^+ and the smooth characteristic flows: $Z_k^-(\cdot\,;\,\cdot)\colon B_k^- \subset R \times A_k^- \to A_k^-$ over the stratum S_k^- and $Z_j^+(\cdot\,;\,\cdot)\colon B_j^+ \subset R \times A_j^+ \to A_j^+$ over the stratum S^+ such that for any $z \in S$ the follow-*

ing relations hold:

$$Z(t_k^-(z);\,z) = \tilde{Z}_k^-(z) \in \sigma_k^- \subset A_k^- \cap A_{k-1}^- \tag{3.14}$$

$$Z(t;\,z) = Z_k^-(t - t_k^-(z);\,\tilde{Z}_k^-(z)) \in S_k^- \quad \text{for any } t \in (t_{k+1}^-(z), t_k^-(z)) \tag{3.15}$$

$$Z(t_j^+(z);\,z) = \tilde{Z}_j^+(z) \in \sigma_j^+ \subset A_j^+ \cap A_{j-1}^+ \tag{3.16}$$

$$Z(t;\,z) = Z_j^+(t - t_j^+(z);\,\tilde{Z}_j^+(z)) \in S_j^+ \quad \text{for any } t \in (t_j^+(z), t_{j-1}^+(z)) \tag{3.17}$$

(P.S.3) *The first component,* $X(\cdot\,;\,\cdot)$ *of* $Z(\cdot\,;\,\cdot) = (X(\cdot\,;\,\cdot), P(\cdot\,;\,\cdot), V(\cdot\,;\,\cdot))$ *has property* (2.16) *where for any* $z \in A_0$ *the interval* $I(z)$ *is defined by:*

$$I(z) = (t_{k_0}^-(z), t_{j_0}^+(z))\,, \quad z \in A_0 \tag{3.18}$$

Obviously, a piecewise smooth characteristic flow has many of the properties of the characteristic flow in the classical case.

In particular, a piecewise smooth characteristic flow verifies relation (2.22) and has the properties in the following analog of Lemma 2.8:

LEMMA 3.7. *If* $Z(\cdot\,;\,\cdot) = (X(\cdot\,;\,\cdot)\cdot P(\cdot\,;\,\cdot), V(\cdot\,;\,\cdot)) : B \subset R \times A_0 \to A$ *is a piecewise smooth characteristic flow in the sense of Definition* 3.6 *then for any* $z \in A_0$, $\bar{t} \in R$ *and* $\bar{z} \in T_z A_0$ *then function* $(r(\cdot\,;\,z, \bar{t}, \bar{z})$ *defined in* (2.26) *on the set* $J(z) = I(z) \setminus \{t_k^-(z), t_j^+(z);\, k = 0, 1, \dots, k_0 - 1, j = 0, 1, \dots, j_0 - 1\}$ *and by:*

$$r(t_k^-(z);\,z, \bar{t}, \bar{z}) = D\tilde{V}_k^-(z)\cdot\bar{z} - \langle \tilde{P}_k^-(z), D\tilde{X}_k^-(z)\cdot\bar{z}\rangle\,, \qquad k = 0, 1, \dots, k_0 - 1 \tag{3.19}$$

$$r(t_j^+(z);\,z, \bar{t}, \bar{z}) = D\tilde{V}_j^+(z)\cdot\bar{z} - \langle \tilde{P}_j^+(z), D\tilde{X}_j^+(z)\cdot\bar{z}\rangle\,, \qquad j = 0, 1, \dots, j_0 - 1 \tag{3.20}$$

on the set $I(z) \setminus J(z)$ *verifies relation* (2.30).

PROOF. Since according to (3.13)-(3.15), for any $t \in [t_1^-(z), 0]$ one has: $Z(t;\,z) = Z_0^-(t;\,z)$, on this interval relation (2.30) follows from Lemma 2.8 applied to the smooth characteristic flow $Z_0^-(\cdot\,;\,\cdot)$.

Assuming that (2.30) is satisfied on the interval $[t_k^-(z), 0]$, we note that from (3.15) it follows that for any $t \in (t_{k+1}^-(z), t_k^-(z))$ one has:

$$DZ(t;\, z)\cdot(\bar{t}, \bar{z}) = D_1 Z_k^-(t - t_k^-(z);\, \tilde{Z}_k^-(z))\cdot(\bar{t} - Dt_k^-(z)\cdot\bar{z}) + \\ + D_2 Z_k^-(t - t_k^-(z);\, \tilde{Z}_k^-(z))\cdot D\tilde{Z}_k^-(z))\cdot\bar{z}$$

hence using (3.7) it follows:

$$r(t;\, z, \bar{t}, \bar{z}) = D_2 V(t;\, z)\cdot\bar{z} - \langle P(t;\, z), D_2 X(t;\, z)\cdot\bar{z}\rangle = \\ = D_2 \tilde{V}_k^-(t - t_k^-(z);\, \tilde{Z}_k^-(z))\, D\tilde{Z}_k^-(z)\cdot\bar{z} - \\ - \langle P(t - t_k^-(z));\, \tilde{Z}_k^-(z)), D_2 X_k^-(t - t_k^-(z);\, \tilde{Z}_k^-(z))\cdot D\tilde{Z}_k^-(z)\cdot\bar{z}\rangle$$

and therefore one has: $r(t;\, z, \bar{t}, \bar{z}) \to r(t_k^-(z);\, z, \bar{t}, \bar{z})$ given by (3.19) as $t \to t_k^-(z)$, $t < t_k^-(z)$.

From Lemma 2.8 applied to the smooth characteristic flow $Z_k^-(\cdot\,;\,\cdot)$ it follows that on the interval $[t_{k+1}^-(z), t_k^-(z)]$ the function $r(\cdot\,;\, z, \bar{t}, \bar{z})$ is the solution of a linear differential equation and since (2.30) is assumed to be satisfied on the interval $[t_k^-(z), 0]$, one has: $r(t_k^-(z);\, z, \bar{t}, \bar{z}) = 0$ hence (2.30) is satisfied on the interval $[t_{k+1}^-(z), 0]$; by induction it follows that (2.30) is satisfied on the interval $[t_{k_0}^-(z), 0]$; on the interval $[0, t_{j_0}^+(z)]$ the proof is entirely similar.

The main result of this paper is the following generalization of Theorem 2.11:

THEOREM 3.8. *Let the problem* (1.1)-(1.2) *be stratified in the sense of Definition* 3.3, *let* $Z_i(\cdot\,;\,\cdot) = (X_i(\cdot\,;\,\cdot), P_i(\cdot\,;\,\cdot), V_i(\cdot\,;\,\cdot))\colon B_i \subset R \times \times A_0 \to A$, $i \in \mathfrak{J}$, *be piecewise smooth characteristic flows, let* X, $Q(\cdot)$ *and* $W(\cdot)$ *be defined as in* (3.9)-(3.11), *respectively, where* $\mathcal{C} = \{Z_i(\cdot\,;\, z);\ i \in \mathfrak{J}, z \in A_0\}$, *and let* $u_m(\cdot)\colon X \to R$ *be defined by* (2.23).

If for every $i \in \mathfrak{J}$ *the first component,* $X_i(\cdot, \cdot)\colon B_i \to X$ *of* $Z_i(\cdot, \cdot)$ *is a weakly* C^1*-stratified submerssion by* $(\mathcal{S}_{B_i}, \mathcal{S}_{i,X})$ *and* $u_m(\cdot)$ *is weakly* C^1*-stratified by* $\mathcal{S}_m$ *such that* $\mathcal{S}_m$ *is compatible with* $\mathcal{S}_{i,X}$ *for every* $i \in \mathfrak{J}$, *then* $u_m(\cdot)$ *is a stratified solution of the problem* (1.1)-(1.2) *in the sense of Definition* 3.1.

PROOF. We consider $x \in X_m$, $i \in \mathfrak{J}$, $(t, z) \in B_i$ such that $u_m(x) = V_i(t;\, z)$, $x = X_i(t;\, z)$; according to Lemma 2.8 in [18] (an analog of Lemma 2.9 in Section 2 for stratified marginal functions), for any $\bar{x} \in T_x X_m$ one has: $Du_m(x)\cdot\bar{x} = DV_i(t;\, z)\cdot(\bar{t}, \bar{z})$ for any $(\bar{t}, \bar{z}) \in T_{(t,z)} B_i$ for which $DX_i(t;\, z)\cdot(\bar{t}, \bar{z}) = \bar{x}$. From Lemma 3.7 it follows that

$$DV_i(t;\, z)\cdot(\bar{t}, \bar{z}) = \langle P_i(t;\, z), DX_i(t;\, z)\cdot(\bar{t}, \bar{z})\rangle$$

hence $Du_m(x)\cdot\bar{x} = \langle P_i(t; z), \bar{x}\rangle$ for any $\bar{x} \in T_x X_m$ and therefore, since $P_i(t; z) \in Q(x)$, from Proposition 3.5 it follows that $u_m(\cdot)$ is a stratified solution and the theorem is proved.

REMARK 3.9. In the case of stratified problems, the choice of the marginal functions $u_m(\cdot)$ and $u_M(\cdot)$, defined by (2.23) and (2.24) respectively, as (remarkable) stratified solutions is additionally justified by the results in [22] and [23] according to which $u_m(\cdot)$ and $u_M(\cdot)$ are subanalytic (hence C^ω-stratified) provided the piecewise smooth characteristici flows, $Z_i(\cdot\,;\,\cdot)$ are subanalytic and their first components are proper mappings on Cl (B_i).

REMARK 3.10. As some examples from Optimal Control Theory show, a characteristic orientor field may have not only piecewise smooth characteristics generated by the piecewise smooth characteristic flows in Definition 3.6 but also *regular characteristics* (i.e. that have a countable number of switching points) or even more general absolutely continuous characteristics. To cover such cases, the piecewise smooth characteristic flow in Definition 3.6 may be further generalized to « regular » characteristic flow ([20]) defined by countable many switching function $t_k^-(\cdot), t_j^+(\cdot)$ and such that Lemma 3.7 and therefore Theorem 3.8 remain valid.

We note that the a.e. solutions in Definition 2.4 may be considered also for stratified problems not only for the classical ones in Section 2.

Moreover, the general procedure described in Section 2 may be used not only for stratified (and classical) problems but also for other types of problems for which a corresponding generalization of the characteristic vector field may be introduced.

For example, for Hamilton-Jacobi equations of the form:

$$H(x, Du(x)) = 0\,, \qquad u(\cdot)|X_0 = u_0(\cdot) \tag{3.21}$$

defined by locally Lipschitz Hamiltonians $H(\cdot,\cdot)\colon A \subset R^n \times R^n \to R$ the characteristic vector field (2.4) is generalized in [6] by the following « Hamiltonian inclusion »:

$$(p', -x') \in \partial H(x, p) \tag{3.22}$$

where $\partial H(x, p)$ denotes Clarke's generalized gradient of $H(\cdot,\cdot)$ at the point $(x, p) \in A$.

It is conceivable that such generalizations of the characteristic vector field may be found for even more general problems using suitable generalized derivatives (e.g. [17]).

4. Examples.

EXAMPLE 4.1. We consider the following « Cauchy problem » ([10]):

$$D_1 u(t, x) + H\big(t, x, D_2 u(t, x)\big) = 0\,, \qquad (t, x) \in [T_0, T] \times R^n \tag{4.1}$$

$$u(T, x) = u_0(x)\,, \qquad x \in R^n \tag{4.2}$$

defined by the functions $H(\cdot, \cdot, \cdot)\colon (T_0 - r, T + r) \times R^n \times R^n \to R$, $r > 0$, and $u_0(\cdot)\colon R^n \to R$ which are assumed to be of class C^2.

Obviously this is a problem of the form (1.1)-(1.2) for which the initial characteristic multifunction $P_0(\cdot)$ defined in (2.6) is given by: $P_0(T, x) = \big(-H(T, x, Du_0(x)), Du_0(x)\big)$ for $(T, x) \in X_0 = \{T\} \times R^n$ hence it is single-valued and of class C^1 and therefore (4.1)-(4.2) is a classical problem in the sense of Definition 2.1.

It is easy to see that the characteristic flow is completely defined by the flow of the (non-autonomous) Hamiltonian vector field $\xi_H(t, x, p) = \big(D_3 H(t, x, p), -D_2 H(t, x, p)\big)$.

Since the initial characteristic strip A_0 in (2.7) is given in this case by: $A_0 = \Big\{\Big((T, x), \big(-H(T, x, Du_0(x)), Du_0(x)\big), u_0(x)\Big);\ x \in R^n\Big\}$, it follows that if for any $a \in R^n$ the mapping $\big(X(\cdot\,; a), P(\cdot\,; a)\big)\colon I(a) \subset R \to R^n \times R^n$ denotes the unique non-continuable solution of the initial value problem:

$$\begin{cases} x' = D_3 H(t, x, p)\,, & x(T) = a \\ p' = -D_2 H(t, x, p)\,, & p(T) = Du_0(a) \end{cases} \tag{4.3}$$

and if $V(\cdot\,; a)$ is defined by:

$$V(t; a) = \tag{4.4}$$

$$= u_0(a) + \int_T^t \big[\langle P(s; a), D_1 X(s; a)\rangle - H\big(s, X(s; a), P(s; a)\big)\big]\, ds$$

then the set $X \subset [T_0, T] \times R^n$ and the « minimal characteristic solution » $u_m(\cdot, \cdot)$ defined by (2.17) and (2.23), respectively, are given by:

$$X = \{(t, X(t; a)); a \in R^n, t \in I(a) \cap [T_0; T]\} \tag{4.5}$$

$$u_m(t, x) = \min \{V(t; a); a \in R^n, t \in I(a) \cap [T_0, T], X(t; a) = x\} \tag{4.6}$$

From Theorem 2.11 it follows that whenever the subset $X_m \subset X$ on which the minimum in (4.6) exists and satisfies condition (1.3) and $u_m(\cdot, \cdot)$ is a.e. differentiable on Int (X_m), $u_m(\cdot, \cdot)$ is an a.e. solution of the problem (4.1)-(4.2). Using results of the type of statements (iii) and (iv) in Lemma 2.9 concerning regularity properties of the marginal functions of the form (2.32) one may obtain corresponding results concerning the minimal characteristic solution $u_m(\cdot, \cdot)$ in (4.6) in terms of the first component, $X(\cdot, \cdot)$, of the solution of (4.3). For instance, as an immediate corollary of Lemma 2.9 we get:

PROPOSITION 4.2. *Let us assume that the initial value problem* (4.3) *has the following properties: for any* $(t, x) \in [T_0, T] \times R^n$ *there exists* $a \in R^n$ *such that the solution* $(X(\cdot\,; a), P(\cdot\,; a))$ *of* (4.3) *is defined on the interval* $[t, T]$ *and satisfies* : $X(t; a) = x$ *and for any* $s \in [t, T]$ *one has:* $\lim_{\|a\| \to \infty} \|X(s; a)\| = \infty$ ($X(s; \cdot)$ is proper).

Then $X_m = [T_0, T] \times R^n$ *and* $u_m(\cdot, \cdot): X_m \to R$ *defined by* (4.6) *is an* (global) *a.e. solution of the problem* (4.1)-(4.2) *that is lower semicontinuous and a.e. locally Lipschits.*

Using different types of results in the theory of Ordinary Differential Equations one may obtain a large variety of properties of $H(\cdot, \cdot, \cdot)$ and $u_0(\cdot)$ that imply the hypotheses in Proposition 4.2.

On the other hand, as in many other problems, very fruitful turns out to be the so called « variational method » in obtaining interesting results concerning the properties of the minimal characteristic solution $u_m(\cdot, \cdot)$ defined in (4.6):

THEOREM 4.3. *Let us assume that the data* $(H\cdot, \cdot, \cdot)$ *and* $u_0(\cdot)$ *of the problem* [4.1)-(4.2) *are of class* C^2 *and have the following additional properties:*

(i) *For any* $(t, x, p) \in [T_0, T] \times R^n \times R^n$ *the second derivative of* $H(t, x, \cdot)$ *is negative definite and* $H(t, x, p) \to -\infty$ *as* $\|p\| \to \infty$.

(ii) *There exist* $C_1, C_2 \in R$ *and* $\varrho(\cdot)\colon [0, \infty) \to R$ *such that* $\varrho(s)/s \to \infty$ *as* $s \to \infty$, $\varrho(s) \geqslant C_1$ *for any* $s \in [0, \infty)$ *and for any* $(t, x, p) \in [T_0, T] \times R^n \times R^n$ *one has*:

$$(4.7) \qquad u_0(x) \geqslant C_2 , \qquad H(t, x, p) - \langle p, D_3H(t, x, p)\rangle \geqslant \varrho(\|D_3H(t, x, p)\|)$$

Then $X_m = [T_0, T] \times R^n$ *and the function* $u_m(\cdot, \cdot)\colon X_m \to R$ *in* (4.6) *is a* (global) *locally Lipschitz a.e. solution of the problem* (4.1)-(4.2) *that coincides with the « variational solution » in* [10].

PROOF. From hypothesis (i) in the theorem it follows that the « Lagrangean » defined by:

$$(4.8) \qquad L(t, x, y) = \max\{H(t, x, p) - \langle y, p\rangle ; p \in R^n\} , \qquad (t, x, y) \in [T_0, T] \times R^n \times R^n$$

is of class C^1 and satisfies: if $q(t, x, y) = -D_3L(t, x, y)$ then:

$$(4.9) \qquad D_3H(t, x, q(t, x, y)) = y , \qquad D_2H(t, x, q(t, x, y)) = D_2L(t, x, y)$$
$$L(t, x, y) = H(t, x, q(t, x, y)) - \langle y, q(t, x, y)\rangle$$
$$(\forall)(t, x, y) \in [T_0, T] \times R^n \times R^n$$

and therefore from (4.7) it follows that $L(t, x, y) \geqslant \varrho(\|y\|)$ for any (t, x, y).

From the existence theorem for Bolza problems in Calculus of Variations (e.g. [5], Theorem 11.4 (i) and Section 11.5) it follows now that for any $[t, x) \in [T_0, T] \times R^n$ there exists an absolutely continuous mapping $\bar{x}(\cdot)\colon [t, T] \to R^n$ that minimizes the functional $C(x(\cdot)) = u_0(x(T)) + \int_t^T L(s, x(s), x'(s))\, ds$ over all absolutely continuous mappings $x(\cdot)\colon [t, T] \to R^n$ that satisfy: $x(t) = x$. Further on, the necessary optimality conditions in Calculus of Variations (or, equivalently, Pontryagin's Maximum Principle, e.g. [5], Section 5.1) imply the existence of an absolutely continuous mapping $\tilde{p}(\cdot)\colon [t, T] \to R^n$ such that $(\bar{x}(\cdot), \tilde{p}(\cdot))$ is a solution of the Hamiltonian system (4.3) and satisfy the « transversality condition »: $\tilde{p}(T) = Du_0(\bar{x}(T))$, hence if we denote $a = \bar{x}(T) \in R^n$ then $(\bar{x}(\cdot), \tilde{p}(\cdot)) = (X(\cdot\,; a), P(\cdot\,; a))$ is the solution of the initial value problem (4.3).

On the other hand, from (4.9) it follows that for any $a' \in R^n$ for which the solution $\big((X(\cdot\,; a'), P(\cdot\,; a')\big)$ of (4.3) is defined on $[t, T]$ and satisfies: $X(t; a') = x$ one has: $P(s; a') = q\big(s, X(s; a'), D_1X(s; a')\big)$ for any $s \in [t, T]$ and therefore from (4.4) and (4.9) it follows: $V(t; a') = C\big(X(\cdot\,; a')\big)$.

We have thus proved that for any $(t, x) \in [T_0, T] \times R^n$ there exists $a \in R^n$ such that $\big(X(\cdot\,; a), P(\cdot\,; a)\big)$ is defined on $I(a) \supset [t, T]$ and satisfies: $X(t; a) = x$ and $V(t; a) = C\big(X(\cdot\,; a)\big) \leqslant C\big(X(\cdot\,; a')\big)$ for any $a' \in R^n$ for which $\big(X(\cdot\,; a'), P(\cdot\,; a')\big)$ is defined on $I(a') \supset [t, T]$ and satisfies: $X(t; a') = x$; therefore $X_m = [T_0, T] \times R^n$ and the function $u_m(\cdot, \cdot)$ in (4.6) is given by: $u_m(t, x) = \min \big\{C(x(\cdot)); x(t) = x, x(\cdot)\colon [t, T] \to R^n$ is absolutely continuous$\big\}$ which means that it is the « variational solution » of the problem (4.1)-(4.2) as defined in [10]. The fact that $u_m(\cdot, \cdot)$ is locally Lipschitz follows in the same way as in the proof of Theorem 1 in [10]; from Rademacher's theorem and from Theorem 2.11 (or Proposition 4.2) it follows that $u_m(\cdot, \cdot)$ is a Lipschitzian a.e. solution in the sense of Definition 2.4 for the problem (4.1)-(4.2).

EXAMPLE 4.4. We consider next the following problem:

$$D_1u(x_1, x_2) + \big(D_2u(x_1, x_2)\big)^2 = 0\,, \qquad u(0, x_2) = u_0(0, x_2) = (x_2)^2 \tag{4.10}$$

which is discussed in several places in [2] and is of the same type as the problem (4.1)-(4.2) where $[T_0, T]$ is replaced by R.

Standard computations show that the characteristic flow is given by:

$$Z_F(t; z) = \Big(\big(t, (4t+1)s\big), (-4s^2, 2s), (4t+1)s^2\Big)$$

for any $z = \big((0, s), (-4s^2, 2s), s^2\big) \in A_0$, $s \in R$, $t \in I(z) = R$ whose first component, $X(t; z) = \big(t, (4t+1)s\big)$ is invertible on the subset $\big(R \setminus \{-1/4\}\big) \times R$ and therefore, using (2.11) we obtain the unique classical solution:

$$u^*(x) = (x_2)^2/(4x_1 + 1) \quad \text{for} \quad x = (x_1, x_2) \in X^* = R^2 \setminus \{(-1/4, x_2);\, x_2 \in R\} \tag{4.11}$$

for which the set X^* is the maximal domain of definition that contains $X_0 = \{0\} \times R$.

However, the marginal characteristic solutions defined in (2.23) and (2.24) given in this case by:

$$(4.12) \qquad u_m(x) = u_M(x) = \begin{cases} u^*(x) & \text{if } x \in X^* \\ 0 & \text{if } x \in R^2 \setminus X^* \end{cases}$$

are globally defined and coincide; the function $u_m(\cdot)$ above is, obviously, an a.e. solution as well as a stratified solution and is discontinuous at the points in $R^2 \setminus X^*$ hence it cannot be a viscosity solution in the sense defined in Remark 2.5.

In fact any function $u(\cdot, \cdot)$: $R^2 \to R$ that is an extension of the classical solution $u^*(\cdot, \cdot)$ in (4.11) is an a.e. solution in the sense of Definition 2.4 but the « characteristic solution in (4.12) has the following remarkable property: for any $x_0 \in X_m = R^2$ there exists a characteristic $Z(\cdot) = (X(\cdot), P(\cdot))$, $V(\cdot)$: $[0, T] \to A$ such that: $X(0) \in X_0$, $X(T) = x_0$, $u_m(X(t)) = V(t)$ for any $t \in [0, T]$ and (in the case) $Du_m(X(t)) = P(t)$ a.e. on $[0, T]$ (we allow $T \in R$ to be also negative). Such properties may be important in some applications in Calculus of Variations, Optimal Control Theory, Theoretical Mechanics, etc.

EXAMPLE 4.5. Let us consider the optimal control problem of minimizing for any $(t_0, x_0) \in E_0 = \{(t, x) \in R^2;\, x < t\}$ the functional $C(u(\cdot); t_0, x_0) = \int_{t_0}^{t_F} (u(t))^2 dt$ over the set $\mathcal{U}(t_0, x_0)$ of all measurable bounded « admissible controls » $u(\cdot)$: $[t_0, t_F] \to R$ for which the absolutely continuous solution, $x(\cdot)$, of the initial value problem: $x' = u(t)$, $x(t_0) = x_0$ is defined on $[t_0, t_F]$ and satisfies the restrictions: $x(t) < t$ for any $t \in [t_0, t_F)$ and $x(t_F) = t_F$.

It is easy to see that the partial differential equation of Dynamic Programming (known also as Bellman-Isaacs equation) associated to this problem (e.g. [5], [13], [18], [20], etc.) is the following boundary value problem of the form (1.1)-(1.2):

$$(4.13) \qquad D_1 W(t, x) - (1/4)(D_2 W(t, x))^2 = 0\,,$$

$$(t, x) \in E_0\,, \qquad W(t, t) = 0\,, \qquad t \in R$$

Standard computations similar to the ones in the previous examples show that the marginal characteristic solutions (2.23) and (2.24) of this problem are given by:

$$(4.14)\qquad W_m(t,x)=0\,,\qquad W_M(t,x)=4(t-x)$$
$$(\forall)(t,x)\in E=\{(t,x)\in R^2;\ x\leqslant t\}\,.$$

These functions are, both, classical, hence viscosity solutions of the problem (4.13), but only $W_M(\cdot,\cdot)$ coincides with the so called « value function » of the optimal control problem defined by: $W(t,x)=\min\{C(u(\cdot);t,x);\ u(\cdot)\in\mathcal{U}(t,x)\}$ for $(t,x)\in E_0$.

EXAMPLE 4.6. *The time-optimal control problem* of reaching the point $x_F=(0,1)\in R^2$ in minimal time, along the trajectories of the system:

$$x_1'=x_2\,,\qquad x_2'=u(t)\in[-1,1]\,,\qquad x(0)=x_0\in R^2\setminus\{x_F\}\,,\qquad x(t_F)=x_F\,,$$

$u(\cdot)\colon[0,t_F]\to[-1,1]$ being measurable, leads to the following « stratified » boundary value problem:

$$(4.15)\qquad x_2D_1W(x_1,x_2)-|D_2W(x_1,x_2)|+1=0\,,$$
$$x=(x_1,x_2)\in R^2\setminus\{x_F\}\,,\qquad W(x_F)=0$$

which is the equation of Dynamic Programming for the considered optimal control problem.

It is easy to see that the data of the problem: $A=R^2\times R^2\times R$, $X_0=\{x_F\}=\{(0,1)\}$, $F(x,p,v)=x_2p_1-|p_2|+1$, $W_0(x_F)=0$ and the initial characteristic strip given by: $A_0=\{((0,1),(|s|-1,s),0);\ s\in R\}$ have the properties in Definition 3.3 that makes the problem (4.15) a stratified one.

The « natural » stratification $\{A_1,A_2,A_3\}$, $A_1=\{z=(x,p,v)\in A;\ p_2>0\}$, $A_2=\{z=(x,p,v)\in A;\ p_2<0\}$, $A_3=\{z=(x,p,v)\in A;\ p_2=0\}$ of the function $F(\cdot,\cdot,\cdot)$ may be further refined to the C^ω-stratification $S_F=\{A_{0i},A_{i0},A_{03},A_{31},A_{32};\ i=1,2\}$ where:

$$A_{0i}=A_i\cap A_0\,,\quad i=1,2,3\,,\qquad A_{i0}=A_i\setminus A_{0i}\,,\quad i=1,2\,,$$
$$A_{31}=\{(x,p,v)\in A_3;\ p_1\neq 0\}\setminus A_{03}\,,\qquad A_{32}=\{(x,p,v)\in A_3;\ p_1=0\}$$

such that $\mathcal{S}_0 = \{A_{0i};\ i = 1, 2, 3\}$ is compatible with $\mathcal{S}_F$ and the stratified characteristic orientor field in (3.3) is given by:

$$(4.16)\qquad \zeta_F(z) = \begin{cases} \zeta_1(z) = \big((x_2, -1),\ (0, -p_1),\ x_2p_1 - p_2\big) & \text{if } z \in A_1 = A_{10} \cup A_{01} \\ \zeta_2(z) = \big((x_2, 1),\ (0, -p_1),\ x_2p_1 + p_2\big) & \text{if } z \in A_2 = A_{20} \cup A_{02} \\ \big\{\big((x_2, r),\ (0, -p_1),\ x_2p_1 + rp_2;\ r \in R\big\} & \text{if } z \in A_{32} \\ \emptyset & \text{if } z \in A_{31} \cup A_{03}\,. \end{cases}$$

An easy computation shows that through each point $z = \big((0, 1), (|s| - 1, s), 0\big) \in A_0$ there passes a unique characteristic $Z_F(\cdot\,; z)\colon I(z) = R \to A$ and the mapping $Z_F(\cdot\,; \cdot)$ may be described in an obvious way as a piecewise analytic stratified flow in the sense of Definition 3.6 which is defined by the (analytic) flows, $Z_i(\cdot\,; \cdot)$, of the vector fields $\zeta_i(\cdot)$ in (4.16), $i = 1, 2$, and by the switching functions:

$$t_1^-(z) = \begin{cases} s/(|s| - 1) & \text{if } s \in (-\infty, -1) \cup (0, 1) \\ -\infty & \text{otherwise} \end{cases}$$

$$t_1^+(z) = \begin{cases} s/(|s| - 1) & \text{if } s \in (-1, 0) \cup (1, \infty) \\ \infty & \text{otherwise}\,. \end{cases}$$

Obviously, the stratifications $\mathcal{S}_0$ and $\mathcal{S}_F$ may be further refined so that $t_1^-(\cdot)$ and $t_1^+(\cdot)$ become C^ω-stratified by $\mathcal{S}_0$ and satisfy the other properties in Definition 3.6.

Writing explicitely the components $X(\cdot\,; \cdot)$ and $V(\cdot\,; \cdot)$ of the characteristic flow, $Z_F(\cdot\,; \cdot)$, one may see that the set X defined in (2.17) is the whole plane, R^2, and for any $x \in X \setminus \{x_F\}$ the set $W(x)$ defined by (2.19) consists of two distinct points: $W_1(x) = -t$ if $X(t; x) = x$ and $t < 0$ and $W_2(x) = -t$ if $X(t; x) = x$ and $t > 0$. It is easy to see that the two marginal functions defined in (2.23) and (2.24) given in this case by: $W_m(x) = W_2(x)$ and $W_M(x) = W_1(x)$ for any $x \in X_m = X_m = X = R^2$ are C^ω-stratified solutions (and also a.e. solutions in the sense of Definition 2.4) of the problem (4.15), the maximal characteristic solution, $W_M(\cdot) = W_1(\cdot)$ which is discontinuous at some points including $x_F = (0, 1)$ being the minimal-time function of the considered time-optimal control problem ([9]); the minimal characteristic solution, $W_m(\cdot) = W_2(\cdot)$ (which is also discontinuous at some points) has the following interpretation: for any $x \in R^2 \setminus \{x_F\}$, $-W_m(x)$ is the minimal time in which the point $x_F = (0, 1)$ can be steered to x along the trajectories of the differential system defining the optimal control problem.

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