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Spaces of urelements, II

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1. Introduction.

The present paper is concerned with the topological structure of the urelemente-line L. We shall give several characterisations of the continuous T_2-images of compact subsets of L which can be considered as analogies to the Hahn-Mazurkiewicz characterisation of the Peano-continua [The exact analogue would be the theory of the continuous Hausdorff-images of the compact and locally connected subspaces of $\mathbf{R}$]. L is obtained from the set U of all urelements of the Mostowski-permutation-model as the Dedeking-completion of an ordered sum of countably many copies of U. In independence proofs concerning the axiom of choice AC the line L plays a role, similar to that of $\mathbf{R}$ in the Cohen-Halpern-Levy-model.

Our notation will follow [15], [19] in set theory (in particular, the definition of the Mostowski-model MM can be found in [19]) and [14] in topology. Unless stated otherwise, our proofs will be in ZF^0-set theory (i.e.: AC and the axiom of foundation are not used). We recall from [8], that in MM U carries a natural order $<$ which is dense, Dedekind-complete and without endpoints. Hence the order topology $\boldsymbol{U}$ is T_2, locally compact and connected. Moreover, since $\mathbf{P}(U)$ is Dedekind-finite, $\boldsymbol{U}$ is countably compact (in the covering sense).

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1.1. DEFINITION. In the Mostowski-model we set $L = (U \times \mathbb{Z}) \cup \mathbb{Z}$. We order L by $<$ defined through the clauses $(a, n) < (b, n)$, if $a < b$, and $(a, n) < m < (b, m)$ if $n < m$. $\boldsymbol{L}$ is the order topology.

As is easily verified, $<$ is a dense and Dedekind-complete order on L. Therefore $\boldsymbol{A}$ is connected and locally compact (no AC is needed for the proof of the Haar-König criterion which is applied here). From its construction it also follows, that $\boldsymbol{L}$ is σ-compact and not countably compact.

There is some interest to the question, if $\boldsymbol{L}$ is paracompact. Its subspace $\boldsymbol{U}$ is not even metacompact (c.f. [7]), whence $\boldsymbol{L}$ is not hereditarily paracompact. Also, it was shown by van Douwen [12], that one needs some form of AC to prove, that every linearly ordered topological space (LOTS) is normal. In particular, there are non-normal LOTS in MM (but $\boldsymbol{U}$ is normal). These LOTS cannot be paracompact, either, because one does not need AC in the proof of « paracompact $+ T_2 \Rightarrow T_4$ ».

Another question about $\boldsymbol{L}$ concerns the Lindelöf-property. For it is known (see [5]), that the assertion « σ-compact $\Rightarrow$ Lindelöf » is equivalent to the countable choice axiom AC^ω which is false in MM. The same is true (for locally compact T_2-spaces) concerning « σ-compact $\Rightarrow$ weakly Lindelöf » and « weakly Lindelöf $\Rightarrow$ Lindelöf », where a space is weakly Lindelöf, if each open cover has a countable refinement.

1.2. LEMMA. In the Mostowski-model, L is a paracompact and weakly Lindelöf space which is not Lindelöf.

PROOF. We first prove paracompact and weakly Lindelöf. Let $\mathbf{0} \subseteq \boldsymbol{L}$ be an open cover and let $e \in [U]^{<\omega} = \{e \subseteq U : e \text{ finite}\}$ be a finite support of $\mathbf{0}$:$\mathbf{0} \in \Delta(e) = \{x \in \mathrm{MM} : \mathrm{sym}\, x \supseteq \mathrm{fix}\, e\}$. We will construct a countable and locally finite open refinement $\boldsymbol{V}$. Let $f \in [U \setminus e]^{<\omega}$ be a choice set for the connectedness components of $U \setminus e$ (in $\boldsymbol{U}$). For each $x \in \boldsymbol{L} \cap \Delta(e)$ $(= \mathbb{Z} \cup (e \times \mathbb{Z}))$ we let I_x be the least open interval around x whose endpoints are in $f \times \mathbb{Z}$. Then for each I_x there is a $Q \in \mathbf{0}$ such that $I_x \subseteq 0$ [To this end we take any $0 \in \mathbf{0}$ which contains x and let I be any open interval around x such that $I \cap \Delta(e) = \{x\}$. Then for some $\pi \in \mathrm{fix}\, e$ we have $\pi I = I_x$ and setting $Q = \pi 0 \in \pi \mathbf{0} = \mathbf{0}$ we get $I_x \subseteq Q$]. Similarly, using a choice set g of the components of $U \setminus (e \cup f)$, we can define intervals $I_x \in \Delta(g)$ where $x \in (f \times \mathbb{Z})$, such that $I_x \subseteq Q$, some $Q \in \mathbf{0}$, whence $\boldsymbol{V} = \{I_x : x \in (\boldsymbol{L} \cap \Delta(e)) \cup (f \times \mathbb{Z})\}$ is the desired locally finite countable refinement of $\mathbf{0}$.

We next show, that L is not Lindelöf. We define

$$\boldsymbol{O}_n = \{]-n, n[\setminus E \colon E \in]-n, n[^{<\omega} \subseteq \mathbb{P}(L) \text{ and } |\operatorname{supp} E| = n\}, \qquad n \in \omega,$$

where $|\cdot|$ denotes the cardinality and supp (E) is the least support (c.f. [19]); since $E \to \operatorname{supp} E \in \Delta(\emptyset)$ we have that $O_n \in \Delta(\emptyset)$. Then $\mathbf{0} = \bigcup \{\boldsymbol{O}_n \colon n \in \omega\}$ is an open covering of L in $\Delta(\emptyset)$ which does not have a countable subcovering. Assume on the contrary, that $\{V_n \colon n \in \omega\}$ is such a subcovering and let e be the support of $n \to V_n$. Since $|e| \in \omega \subseteq L$, there is a $V_n \in \mathbf{0}$ containing $|e|$; let $V_n \in \boldsymbol{O}_m$. Then $|e| < m$, because $|e| \in V_n \in \boldsymbol{O}_m$. On the other hand, $V_n =]-m, m[\setminus E$ where $E \in \Delta(e)$ [since $V_n \in \Delta(e)$ and E is defined from V_n via $E =]-m, m[\setminus V_n]$, whence $\operatorname{supp} E \subseteq e$ and $m = |\operatorname{supp} E| \leqslant |e|$, a contradiction. Q.E.D.

Another useful property of $\boldsymbol{L}$ is the fact, that each subspace satisfies the countable antichain condition [this follows from the remark in [7], that there is no subspace of $\boldsymbol{U}$ with an infinite family of pairwise disjoint open sets]. Also, each nowhere dense set in a subspace of L is countable [because each infinite subset of U contains nonempty open intervals]. It follows, that the nowhere dense subsets of a compact subspace X of L are finite [because $X \subseteq [-n, n]$ for some $n \in \omega$, whence X and $\mathbb{P}(X)$ are Dedekind-finite].

In general, the characterisation of a class of spaces which are built up from urelements may lead to rather obscure notions. This is to be expected, because the very existence of these spaces contradicts AC. Interestingly, the class of continuous T_2-images of compact subsets of L can be described by very harmless looking properties. Our main result states, that these are just the Dedekind-finite Lindelöf T_2-spaces which are hereditarily locally Lindelöf and have at most finitely many isolated points. On the other hand, it is wellknown that in MM the hereditarily (weakly) Lindelöf T_2-spaces are well-orderable, while a Dedekind-set (a Dedekind-finite but infinite subset of $\mathbb{R}$) is hereditarily weakly Lindelöf but not Lindelöf (see [7]), when carrying the subspace topology of $\mathbb{R}$ [which is second countable]. We close this section with a proof of half of this theorem.

1.3. Lemma. In the Mostowski-model, if X is a T_2-space which is a continuous image of a compact subspace Y of L, then $\mathbb{P}(X)$ is Dedekind-finite and X is compact and hereditarily locally compact.

The set of isolated points of X is finite and every subspace satisfies the countable antichain condition.

PROOF. It follows from the previous remarks, saying $\mathbf{P}(Y)$ is a Dedekind-finite space, every subspace satisfying the countable antichain condition, that the same is true for X, whence the set of isolated points is finite. If $A \subseteq X$, then $A^- \setminus A$ is finite [$f\colon Y \to X$ denotes the continuous onto-mapping, $B = f^{-1}(A)$; then B^- is compact, whence $f(B^-) = f(B)^-$ and $A^- \setminus A = f(B^-) \setminus f(B) \subseteq f(B^- \setminus B)$. As $B^- \setminus B \subseteq Y$ is nowhere dense, it is finite and so is $A^- \setminus A$]. So every subspace A of X is an open subspace of the compact space A^-, whence A is locally compact and X is hereditarily locally compact. Q.E.D.

If AC holds, then a hereditarily locally compact T_2-space has a dense set of isolated points (a consequence of [17]).

2. Main result.

We collect the purely topological arguments needed in the proof of our theorem in the following two lemmas.

2.1. LEMMA. If $\mathbf{R}$ is wellorderable, then the following assertions on a Hausdorff-space X with at most finitely many isolated points are equivalent:

(i) X is Dedekind-finite, Lindelöf and hereditarily locally Lindelöf.

(ii) X is compact and hereditarily locally compact.

(iii) X is compact and every dense set is open.

Moreover, such a space is hereditarily normal and each nowhere dense set is finite.

PROOF. (i) $\Rightarrow$ (ii) follows from [6]: A Dedekind-finite Lindelöf $+$ T_1-space is compact [c.f. the proof of 3.1(i)].

(ii) $\Rightarrow$ (iii): Let D be dense and choose $x \in D$. Since X is regular and D is locally compact, it is regular and there are a regularly open set 0 and a compact set $K \subseteq D$ such that $x \in 0 \cap D \subseteq K$. Since K is closed [because it is compact], $x \in 0 = (0 \cap D)^{-0} \subseteq K \subseteq D$, whence $x \in D^0$ and D is open.

(iii) $\Rightarrow$ « nowhere dense sets are finite »: We show: If $S^0 = \emptyset$, then S is finite. For $S^0 = \emptyset$ implies, that $X \setminus S = S^c$ is dense, whence by (iii) $\boldsymbol{O} = \{\{x\} \cup S^c : x \in S\}$ is an open cover of X. Since X is compact, it has a finite subcover $\boldsymbol{F}$ and $S = (\bigcup \boldsymbol{F}) \cap S$ is finite, too.

(iii) $\Rightarrow T_5$: If Y is a subspace of X, then by the preceding remark the boundaries of subsets of Y are finite. So Y is T_4, because more generally a T_2-space Y in which all boundaries are compact is T_4. For let A, B be closed and disjoint. Comfort [11] has shown, that one does not need AC in order to separate the compact sets $A \setminus A^0$, $B \setminus B^0$ by disjoint open sets P, Q. Then $A^0 \cup (P \setminus B)$ and $B^0 \cup (Q \setminus A)$ are open sets separating A and B.

(iii) $\Rightarrow$ (ii) is now trivial: If $Y \subseteq X$, then $Y^- \setminus Y$ is finite, whence Y is locally compact [since Y is open in the compact space Y^-].

(ii) $\Rightarrow$ (i) is based on a lemma from [8]: If $\mathbb{R}$ is wellorderable and X is an antianticompact T_2-space (*i.e.*: each infinite $Y \subseteq X$ has an infinite compact subset), then X is Dedekind-finite. To show, that X is antianticompact, let $Y \subseteq X$ be infinite. As was shown before, $Y \setminus Y^0$ is finite and so Y^0 is infinite, too. If every compact subset of Y^0 is finite, then Y^0 is discrete, since Y^0 is a locally compact T_2-space. It follows, that Y^0 is an infinite set of isolated points of X, a contradiction to our assumption. Q.E.D.

In 2.1(iii) one cannot weaken « compact » to « Lindelöf ». For as was shown by Sharma [23] (in response to a question in Bankston [2]), $ZF^0 + AC +$ the continuum hypothesis implies, that there is a Lindelöf T_2-space X without isolated points (not Dedekind-finite of course) in which every dense set is open. The property, that dense sets are open was introduced by Hewitt in his doctoral thesis [18] who called it MI. It is motivated by some classical results in real analysis: A family S of sets is reducible, if there is a set D such that $D \cap 0 \neq \emptyset$ and $O \setminus D \neq \emptyset$ for all $0 \neq \emptyset$ in S (*i.e.* D and its complement are dense). This notion can be traced back to Bernstein [3], who used it (without naming it so) in his construction of a Bernstein set. Hewitt's paper was the first systematic study of reducible topologies which was later extended *e.g.* by Katetov [20], Ceder [10], Aull [1] and Elkin [13] (as for precedessors, the Lusin and Sierpinski paper [22] should be mentioned). MI is a strong negation of reducibility. [In 2.1 « irreducible » is too weak: a finite product of intervals of U is irreducible]. For example, every maximal T_2-space (i.e.: the topology is maximal with respect to having no isolated points) is MI ([18]). In view of the importance of Hewitt's [18] ideas and concepts for the characterisation of the continuous T_2-images X of compact subsets of L we decide to call the class of these spaces X ***Hewitt-spaces***. In contrast to the theory of the Peano-continua, a Hewitt-space X satisfies $\operatorname{ind}(X) \leqslant 1$ [1.3 + 2.1]. Also, Hewitt spaces are not closed under finite products, but they are closed-hereditary.

2.2. LEMMA. A Dedekind-finite Suslin-line which is not compact has only two T_2-compactifications, namely the one-point compactification and the order-compactification (which may coincide).

PROOF. X is a Souslin-line, if X is a linearly ordered and connected topological space which satisfies the countable antichain condition but is not separable (automatically true, if X is Dedekind-finite); let $<$ be the order of X [it is dense and Dedekind-complete]. Since X is locally compact, the one point compactification $a(X)$ exists and the order compactification oX reduces to the addition of an upper and/or lower bound; we assume «and» [i.e. $X =]\leftarrow, \rightarrow[$, the other cases being analogous to this one]. Let $k: X \to kX$ be any T_2-compactification, where k is the embedding; we assume $kx = x$ for all $x \in X$. Since kX is regular, the regularly closed sets form local bases. We first prove the following remark: If P is a regularly closed neighbourhood of $p \in kX \setminus X$, then $P \cap X$ contains an unbounded interval ($]\leftarrow, x[$ or $]x, \rightarrow[$); since there are only two types of unbounded intervals, T_2 implies: $|kX \setminus X| \leqslant 2$. Because X is locally connected and satisfies the countable antichain condition, in X $(P \cap X)^0$ is the union of a countable family $\boldsymbol{P}$ of pairwise disjoint open intervals, the components. As X is Dedekind-finite, the countable set of their endpoints is finite, whence also $\boldsymbol{P}$ is finite. One of the intervals in $\boldsymbol{P}$ is unbounded, for otherwise $(P \cap X)^0$ is relatively compact in X, which is impossible [because P is regularly closed, $P = (P \cap X)^-$ and $P \cap X$ is regularly closed in X, whence in X $(P \cap X)^{0-} = P \cap X$ is compact, yielding $P = P \cap X$, while $p \in P \setminus X$]. We now assume, that $kX \setminus X = \{p, q\}$, $p \neq q$, and show, that $k = o$. [If $kX \setminus X = \{p\}$ we similarily will obtain $k = a$]. We set $P_x = \{p\} \cup]\leftarrow, x[$ and $Q_x = \{q\} \cup [x, \rightarrow[$, $x \in X$, and fix two disjoint regularly closed neighbourhoods P and Q of p and q. It follows from the above observation that—modulo an obvious change in notation—we have $P_x \subseteq P$ and $Q_y \subseteq Q$ for some x, y. If $x' < x$, then $kX \setminus P_{x'} = Q \cup [x', y]$ is closed in kX, whence $P_{x'}$ and $Q_{y'}$ are open neighbourhoods of p and q, where $x > x'$ and $y < y'$. If $P' \subseteq P$ is any regularly closed neighbourhood of p, then since $P' \cap Q_y = \emptyset$ the unbounded intervals in $P' \cap X$ have to be unbounded to the left, whence $P_{x'} \subseteq P'$ for some $x' \in X$. Hence $\{P_{x'}^{\bullet}: x' < x\}$ and $\{Q_{y'}^{\circ}: y < y'\}$ are neighbourhoodbases of p and q, proving $k = o$. Q.E.D.

2.3. THEOREM. In the Mostowski-model the following assertions are equivalent for a Hausdorff-space X.

(i) X is a continuous image of a compact subset of L (a Hewitt space).

(ii) X is obtained from a compact subset of L by a finite number of identification of finite sets.

(iii) X is Lindelöf, hereditarily locally Lindelöf, Dedekind-finite and the set of isolated points is finite.

Moreover, every Hewitt space is hereditarily locally connected and it is not hereditarily metacompact (unless it is finite).

PROOF. (ii) means, that there is a compact $Y \subseteq L$ and a mapping $f: Y \to X$ which is a composite of homeomorphisms and quotient maps which identify finite sets with points. The « moreover » statements are consequences of (ii): Since every subset M of L is a locally finite union of possibly degenerate intervals, it is locally connected. As every subspace Z of X is the quotient of some $M \subseteq L$, Z is locally connected in view of the ZF^0-result of Whyburn [25]. If X is infinite, then X is obtained from an infinite $Y \subseteq L$ by the collapse of some finite sets E_i, $i \in n$. In L $\left(Y \setminus \bigcup_{i \in n} E_i\right)^0 \neq \emptyset$, and so there is an open interval $I \subseteq Y$ which is embedded homeomorphically into X. Since a subset of U is metacompact iff it is compact ([7]) we obtain a non-metacompact subset of X in this way.

(ii) $\Rightarrow$ (i) is trivial and (i) $\Rightarrow$ (iii) is 1.3.

(iii) $\Rightarrow$ (ii): We recall from the proof of 2.1, last part, that X is antianticompact. Let $e \subseteq U$ be a support of X and $\mathbf{X}$. We first show, that $|\text{supp}(x) \setminus e| \leqslant 1$ for all $x \in X$. Let $x \notin \Delta(e)$. If we write $\text{supp}(x) \setminus e = \text{im}(\boldsymbol{a})$ for some strictly increasing $\boldsymbol{a} \in (U \setminus e)^n$, $n = |\text{supp}(x) \setminus e|$, then $\text{orb}_e\, x = \{\pi x: \pi \in \text{fix}\, e\}$ can be identified with the set $\text{orb}_e\, \boldsymbol{a} \subseteq (U \setminus e)^n$, using a mapping $f_0: \text{orb}_e\, a \to \text{orb}_e\, x: = 0$ in $\Delta(e)$. In [8] it was shown, that the antianticompactness of X

implies, that the relative topology $X|\mathrm{orb}_e x$ is homeomorphic under f_0 with the topology on $\mathrm{orb}_e\, \boldsymbol{a}$ which is generated by the product topology $\boldsymbol{U}^n$. Hence if $n \geqslant 2$, then there are some nontrivial intervals $I_i \subseteq U$, $i \in n$, such that $\prod_{i \in n} I_i \subseteq \mathrm{orb}\, \boldsymbol{a}$. But this is impossible, since then by 2.1 every nowhere dense set—like $\{\boldsymbol{a}|(n-1)\} \times I_{n-1}$—should be finite. Therefore for $x \in X \setminus \Delta(e)$, $\mathrm{orb}_e x = 0$ is homeomorphic to the open interval $I_0 = \mathrm{orb}_e\, \boldsymbol{a} \subseteq U$ whose endpoints are in $e \cup \{\leftarrow, \rightarrow\}$ and f_0 is the homeomorphism. We next show, that $\boldsymbol{O} = \{\mathrm{orb}_e x \colon x \in \in X \setminus \Delta(e)\}$ is finite. If on the contrary, $\boldsymbol{O}$ is infinite, then by the pigeon-hole principle there is some component I of $U \setminus e$ such that $\boldsymbol{Q} = \{o \in \boldsymbol{O} \colon \mathrm{dom}\, f_0 = I\}$ is infinite. Therefore, for any $u \in I$, the set $\{f_0(u) \colon o \in \boldsymbol{Q}\} \subseteq X$ is infinite and wellorderable, contradicting the Dedekind-finiteness of X. We now define a compact set $Y \subseteq L$ and a quotient map $f\colon Y \to X$. To this end we enumerate $\boldsymbol{O} = \{o_i \colon i \in n\}$ and $X \cap \Delta(e) = \{x_i \colon i \in m\}$; corresponding to o_i we have a component I_i of $U \setminus e$ and a homeomorphism $f_i \colon I_i \to o_i$. We set $Y = \{-i-\, -1 \colon i \in m\} \cup \{(I_i \times \{2i\})^- \colon i \in n\}$. Since by 2.1 $f_i \colon I_i \to o_1 -$ is a T_2-compactification we conclude from 2.2, that $f_\iota = a$ or $f_i = 0$. Since $(I_i \times \{2i\})^- = oI_i$ and because aI_i is obtained from oI_i by collapsing the finite set of endpoints, we obtain onto mappings $g_i \colon (I_i \times \{2i\})^- \to o_i^-$ which collapse only finite sets and an onto mapping $g \colon \{-i-1 \colon i \in \in m\} \subseteq L \to X \cap \Delta(e)$. Then $f = g \cup \bigcup \{g_i \colon i \in n\}$ is the required mapping $Y \to X$. Q.E.D.

We finally observe, that in the Mostowski-model not every compact and connected linearly ordered space X with $\mathbf{P}(X)$ Dedekind-finite is a Hewitt-space: If X is $o(U \cdot 2)$, $U \cdot 2$ the lexicographic product, then X is reducible.

3. Related results.

The notion of reducibility appears in several independence results. (i) Let us say, a family S of sets is σ-reducible, if it is a countable union of reducible sets. Then a nonprincipal ultrafilter on ω is σ-reducible, iff it is not a Ramsey-point, the existence of Ramsey points being independent of *ZFC* (classical results due to Rudin, Booth and Kunen). (ii) If *AC* holds, then some nonempty $S \subseteq [\omega]^\omega$ is not σ-reducible. But in the model of Blass [4], each nonempty set $S \subseteq [\omega]^\omega$ is σ-reducible [which follows from a Fefermantype argument, c.f. [15]].

Also, combining Easton-forcing with the Blass-construction, we obtain a model such that every nonempty family S of infinite sets is σ-reducible, provided that $\cup S$ is wellorderable. (iii) It follows from topological results due to Malhyn, that the existence of nonprincipal ultrafilters is equivalent to the existence of maximal T_2-spaces (in particular, the dense sets form an ultrafilter, if X is maximal T_2).

One of the ingredients for the proof of 2.3 was the observation, that Dedekind-finite Lindelöf T_1-spaces are compact. There are even models, where several notions of compactness coincide; 3.1(ii) answers one of the problems listed in [21].

3.1. LEMMA. (i) In the Cohen-Halpern-Levy model, if X is Lindelöf and T_1, then X is compact.

(ii) In Gitik's model, if X is countably compact, then X is compact.

PROOF. In both cases, we shall actually prove a more general statement than asserted.

(i) If there is a Dedekind-set, then every Lindelöf + T_1-space is compact.

As the first step we observe, that ω with the discrete topology is not Lindelöf. For, since there is an almost disjoint family $\boldsymbol{A}$ of infinite subsets of ω such that $\boldsymbol{A}$ is equipotent with $\mathbf{R}$ [a well-known ZF^0 result of Sierpinski for which Buddenhagen [9] has an elegant topological proof], our assumption implies, that there is an infinite, Dedekind-finite almost disjoint family $\boldsymbol{B} \subseteq [\omega]^\omega$ such that $\cup \boldsymbol{B} = \omega$. If ω were Lindelöf, then there would exist a finite subcover $\boldsymbol{F}$ of $\boldsymbol{B}$, which is impossible, since for $B \in \boldsymbol{B} \setminus \boldsymbol{F}$ $B \cap \bigcup \boldsymbol{F} \in [\omega]^{<\omega}$ while B is infinite. We next show, that every Lindelöf + T_1-space which is not compact contains a closed copy of ω. Since it is not countably compact, there is a countable open cover $\langle O_n : n \in \omega \rangle$ such that $O_n \subsetneq O_{n+1}$. Via T_1 we define an auxiliary cover $\boldsymbol{O}$: $\boldsymbol{O} = \{O_{n+1} \setminus \{x\} : x \in O_{n+1} \setminus O_n$ and $n \in \omega\}$. Then a countable subcover of $\boldsymbol{O}$ defines an infinite sequence x_k, $x_k \in O_{\bar{n}+1} \setminus O_{\bar{n}}$ for some $\bar{n}(k)$, and by a routine verification we see, that $\{x_k : k \in \omega\}$ is closed an discrete, hence Lindelöf which is impossible. As there exists a Dedekind-set in the Cohen-model, the result follows.

(ii) Specker's axiom, that every set has a grad implies, that every countably compact space is compact.

If we set C_0 = class of countable sets, C'_α = class of countable unions of sets in $\bigcup\{C_\beta: \beta < \alpha\{$ and $C_\alpha = C'_\alpha \setminus \bigcup\{C_\beta: \beta < \alpha\}$, then Specker's axiom [24] says, that $V = \bigcup\{C_\alpha: \alpha \in On\}$. In [16] Gitik has constructed a model for this axiom under the assumption, that there are arbitrarily large strongly compact cardinals. We assume, that $(X, \boldsymbol{X})$ is countably compact but not compact. Let α be the least ordinal, such that there is an open cover $\boldsymbol{O} \in C_\alpha$ without a finite subcover; $\alpha > 0$. $\boldsymbol{O}$ is a countable union of families $\boldsymbol{O}_n \in \bigcup\{C_\beta: \beta < \alpha\}$ and by countably compactness there is a $F \in [\omega]^{<\omega}$ such that $\{\bigcup \boldsymbol{O}_n: n \in F\}$ covers X. But grad $(\bigcup\{\boldsymbol{O}_n: n \in F\}) < \alpha$, whence by the minimality of α there is a finite subcover of $\bigcup\{\boldsymbol{O}_n: n \in F\}$ and a fortiori of $\boldsymbol{O}$, a contradiction. Q.E.D.

We finally note, that Specker's axiom is equivalent to the following assertion: AT_2-space is discrete, if countable intersections of open sets are open.

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