Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 77 (1987), p. 1-13

<http://www.numdam.org/item?id=RSMUP_1987__77__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 77 (1987)

Hilbert Transformation of Beurling Ultradistributions.

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SUMMARY - We give the definitions of the Hilbert transformation on the spaces $\mathcal{D}_{L^s}^{(M_p)}(\mathbb{R})$, $s > 1$, $\mathcal{D}^{(M_p)}(\mathbb{R})$, and on the space $\mathcal{D}_{L^2}^{(M_p)}(\mathbb{R}^q)$.

1. Introduction

We do not give here the historical background of the distributional Hilbert transformation. It was done in [5, 6] and [9] p. 169, so it could be seen there as well as the references.

Following Pandey approach to the Hilbert transformation of Schwartz distributions ([5]), we define the Hilbert transformation of Beurling ultradistributions. In order to do that we first introduce and investigate spaces $\mathcal{D}_{L^s}^{(M_p)}(\mathbf{R}^q)$, $s > 1$, and their duals.

For $s = 2$ we follow the definition of the Hilbert transformation in $\mathcal{D}'_{L^2}(\mathbf{R}^q)$ given by Vladimirov ([9], 10.4) and define the Hilbert transformation in $\mathcal{D}_{L^2}'^{(M_p)}(\mathbf{R}^q)$ by means of the convolution with $\operatorname{Im} \mathcal{K}_C(z)$ where $\mathcal{K}_C(z)$ is the Cauchy kernel. For $s = 2$ and $q = 1$ the given two definitions of the Hilbert transformation in $\mathcal{D}_{L^2}'^{(M_p)}(\mathbf{R})$ are equal.

2. Notions and notation.

We always denote by $q \in \mathbf{N}$ the space dimension and by s a real number greater than 1. The norm in $L^s(\mathbf{R}^q)$ is denoted by $\| \ \|_s$. $\mathcal{D}_{L^s}(\mathbf{R}^q)$

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This material is based on work supported by the U.S.-Yugoslav Joint Fund for Scientific and Technological Cooperation, in cooperation with the NSF under grant (JFP) 544.

is the Schwartz space of all $\varphi \in C^\infty(\mathbb{R}^q)$ such that

$$\|\varphi^{(k)}\|_s < \infty \quad \text{for every } k \in \mathbb{N}_0^q \ (\mathbb{N}_0 = \mathbb{N} \cup \{0\}) \ ([7]) .$$

If $f \in L^s(\mathbb{R})$ $(q = 1)$, then the Hilbert transformation of f, Hf, is defined by

$$(Hf)(x) := P\int_{-\infty}^{\infty} f(t)(t-x)^{-1}\,dt \qquad (P \text{ means principal value})$$

where the integral is being taken in the Cauchy principal value sense. We need the following relations ([8], 132-133):

(1) $$Hf \in L^s(\mathbb{R}) , \qquad H(Hf) = -\pi^2 f ;$$

(2) For every $s > 1$ there is $C_s > 0$ such that

$$\|Hf\|_s \leqslant C_s \|f\|_{s'} , \qquad f \in L^s(\mathbb{R}) .$$

If $\{b_\alpha;\ \alpha \in N_0^q\}$ is a sequence of positive numbers then we put

$$l_1^s(b_\alpha) := \Big\{\{f_\alpha\};\ f_\alpha \in L^s(\mathbb{R}^q),\ \alpha \in \mathbb{N}_0^q,\ \sum_{\alpha \in \mathbb{N}_0^q} b_\alpha \|f_\alpha\|_s < \infty\Big\} ;$$

$$l_\infty^s(b_\alpha) := \Big\{\{f_\alpha\};\ f_\alpha \in L^s(\mathbb{R}^q),\ \alpha \in N_0^q,\ \sup_{\alpha \in \mathbb{N}_0^q} \{b_\alpha \|f_\alpha\|_s\} < \infty\Big\} .$$

Following Komatsu ([2]) we give the definition of the space of Beurling ultradistributions. Let $\{M_p;\ p \in \mathbb{N}_0\}$ be a sequence of positive numbers such that

(3) $$M_p^2 \leqslant M_{p-1} M_{p+1} , \qquad p \in \mathbb{N} ;$$

(4) There are $A > 0$ and $H > 0$ such that

$$M_{p+1} \leqslant A H^p M_p , \qquad p \in \mathbb{N}_0 ;$$

(5) $$\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty .$$

We denote by $K(m)$ the closed ball in $\mathbb{R}^q$ with the center in zero and with the radius $m > 0$. Let $m > 0$ and $n > 0$. We put

$$\mathfrak{D}^{(M_p)}_{m,n}(\mathbf{R}^q) := \{\varphi \in C^\infty(\mathbf{R}^q);\ \operatorname{supp}\varphi \subset K(m)\,,\ \|\varphi\|_{m,n} < \infty\}$$

where

$$\|\varphi\|_{m,n} := \sup_{\alpha \in \mathbf{N}_0^q} \left\{\frac{n^{|\alpha|}\|\varphi^{(\alpha)}(x)\|_\infty}{M_{|\alpha|}}\right\} \qquad (|\alpha| = \alpha_1 + \ldots + \alpha_n)\,.$$

$\mathfrak{D}^{(M_p)}_{m,n}(\mathbf{R}^q)$ is a Banach space under the norm $\|\ \|_{m,n}$ $(m, n > 0)$.

The space $\mathfrak{D}^{(M_p)}_{m}(\mathbf{R}^q)$, $m > 0$, is defined by

$$\mathfrak{D}^{(M_p)}_{m}(\mathbf{R}^q) := \lim_{n \in \mathbf{N}} \operatorname{proj} \mathfrak{D}^{(M_p)}_{m,n}(\mathbf{R}^q)\,,$$

and the space of Beurling ultradifferentiable functions is defined by

$$\mathfrak{D}^{(M_p)}(\mathbf{R}^q) := \lim_{m \in \mathbf{N}} \operatorname{ind} \mathfrak{D}^{(M_p)}_{m}(\mathbf{R}^q)\,.$$

The strong dual of $\mathfrak{D}^{(M_p)}(\mathbf{R}^q)$, is the space of Beurling ultradistributions. This space is deeply analyzed in [2].

For the purpose of the definition of the Hilbert transformation in the space $\mathfrak{D}'^{(M_p)}_{L^s}(\mathbf{R}^q)$ we shall recall some notions from [9].

Let C be a connected open cone in $\mathbf{R}^q$ with vertex at O and let $C^* := \{\xi;\ \langle\xi, x\rangle = \xi_1 x_1 + \ldots + \xi_q x_q \geqslant 0,\ x \in C\}$. Then the Cauchy kernel of a tubular region $T^C = \{x + iy,\ x \in \mathbf{R}^q,\ y \in C\}$; denoted by $\mathcal{K}_C(z)$, is defined by

$$\mathcal{K}_C(z) := \int_{C^*} \exp\left(i\langle z, \xi\rangle\right) d\xi\,, \qquad z = x + iy \in T^C\,,$$

and $\mathcal{K}_C(x)$ is defined by

$$\mathcal{K}_C(x) = \lim_{\substack{y \to 0\\ y \in C}} \mathcal{K}_C(x + iy) \quad \text{in } \mathcal{H}_s \quad \text{for } s < -\frac{q}{2}$$

(see [9] p. 161). Since $\mathfrak{D}'_{L^2}(\mathbf{R}^q) = \bigcup_{s=0}^{\infty} \mathcal{H}_{-s}$ ([9], p. 156) this implies that $\mathcal{K}_C(x) \in \mathfrak{D}'_{L^2}(\mathbf{R}^q)$ and that

$$\operatorname{Im}\mathcal{K}_C(x)\,, \qquad \operatorname{Re}\mathcal{K}_C(x) \in \mathfrak{D}'_{L^2}(\mathbf{R}^q)\,.$$

We also need the following formulae ([9], p. 162):

$$-\operatorname{Im}\mathcal{K}_C * \operatorname{Im}\mathcal{K}_C = \operatorname{Re}\mathcal{K}_C * \operatorname{Re}\mathcal{K}_C = \tfrac{1}{2}(2\pi)^q \operatorname{Re}\mathcal{K}_C\,, \tag{6}$$

$$\operatorname{Im}\mathcal{K}_C * \operatorname{Re}\mathcal{K}_C = \tfrac{1}{2}(2\pi)^q \operatorname{Im}\mathcal{K}_C\,. \tag{7}$$

3. Space $\mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q)$.

Let $h > 0$. We denote by $\mathcal{D}_{L^s,h}^{(M_p)}(\mathbb{R}^q)$ the space of all $\varphi \in C^\infty(\mathbb{R}^q)$ for which

$$\gamma_{s,h}(\varphi) := \sum_{\alpha\in\mathbf{N}_0^q} \frac{h^{|\alpha|}\|\varphi^{(\alpha)}\|_s}{M_{|\alpha|}} < \infty\,.$$

Since $\mathcal{D}_{L^s,h}^{(M_p)}(\mathbb{R}^q)$ is a subspace of complete space $\mathcal{D}_{L^s}(\mathbb{R}^q)$, one can easily prove that $\mathcal{D}_{L^s,h}^{(M_p)}(\mathbb{R}^q)$ is a Banach space under the norm $\gamma_{s,h}$.

The space $\mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q)$ is defined by

$$\mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q) := \lim_{n\in\mathbf{N}} \operatorname{proj} \mathcal{D}_{L^s,n}^{(M_p)}(\mathbb{R}^q)\,.$$

Using [1] p. 46, one can easily prove:

THEOREM 1. *The space $\mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q)$ is an (FG)-space (Gelfand space)*

(For the definition of (FG)-spaces see [1]).

Condition (4) implies that the mappings $\mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q) \to \mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q)$ defined by $\varphi \to \varphi^{(k)}$, $k \in \mathbf{N}_0^q$, are continuous.

Obviously, $\mathcal{D}^{(M_p)}(\mathbb{R}^q) \subset \mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q)$. Condition (5) implies that $\mathcal{D}^{(M_p)}(\mathbb{R}^q)$ is non-trivial (see [2], Theorem 4.2). $\mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q)$ is larger than $\mathcal{D}^{(M_p)}(\mathbb{R}^q)$; for example if $s = 2$, $q = 1$ $M_p = p^{\alpha p}$ $p \in \mathbf{N}_0$, $\alpha > 1$, then Hermite functions

$$h_n(x) = (-1)^n \pi^{-\frac{1}{4}}(2^n n!)^{-\frac{1}{2}} \exp\left(\frac{x^2}{2}\right)\left(\exp(-x^2)\right)^{(n)}, \qquad n \in \mathbf{N}_0\,,$$

belong to $\mathcal{D}_{L^s}^{(p^{\alpha p})}(\mathbb{R})$ but do not belong to $\mathcal{D}^{(p^{\alpha p})}(\mathbb{R})$.

THEOREM 2. *$\mathcal{D}^{(M_p)}(\mathbb{R}^q)$ is a dense subspace of $\mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q)$.*

PROOF. There exists $\eta \in \mathcal{D}^{(M_p)}(\mathbb{R}^q)$ such that

$$\eta \geqslant 0 , \quad \eta \equiv 1 \text{ in the ball } K(\tfrac{1}{2}) ; \quad \eta \equiv 0 \text{ in } \mathbb{R}^q \setminus K(1)$$

(see [2], Lemma 4.1).

We put $\eta_k(t) := \eta(t/k)$, $k \in \mathbb{N}$. $(t/k = (t_1/k, \ldots, t_q/k).)$

We are going to prove that for arbitrary $\varphi \in \mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q)$, $\eta_k\varphi \in \mathcal{D}^{(M_p)}(\mathbb{R}^q)$, $k \in \mathbb{N}$, and $\eta_k\varphi \to \varphi$ in $\mathcal{D}_{L^s}^{(M_p)}(\mathbb{R}^q)$ as $k \to \infty$.

Let m, $h \in \mathbb{N}$. Since (3) implies

$$M_\beta M_{\alpha-\beta} \leqslant M_0 M_\alpha , \quad \beta, \alpha \in \mathbb{N}_0 , \quad \beta \leqslant \alpha ,$$

we obtain $(k \in \mathbb{N})$

$$\|\eta_k\varphi\|_{m,h} \leqslant \sup_{\substack{\alpha\in\mathbb{N}_0^q \\ x\in K(m)}} \left\{ \frac{h^{|\alpha|}}{M_{|\alpha|}} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \frac{1}{k^{|\beta|}} \left| \eta^{(\beta)}\left(\frac{x}{k}\right) \right| |\varphi^{(\alpha-\beta)}(x)| \right\} \leqslant \tag{8}$$

$$\leqslant M_0 \sup_{\substack{\alpha\in\mathbb{N}_0^q \\ x\in K(m)}} \left\{ \frac{1}{2^{|\alpha|}} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \frac{(2h)^{|\beta|} |\eta^{(\beta)}(x/k)|}{M_{|\beta|}} \, \frac{(2h)^{|\alpha-\beta|} |\varphi^{(\alpha-\beta)}(x)|}{M_{|\alpha-\beta|}} \right\} \leqslant$$

$$\leqslant M_0 \sup_{\substack{\beta\in\mathbb{N}_0^q \\ x\in K(m)}} \left\{ \frac{(2h)^{|\beta|} |\eta^{(\beta)}(x/k)|}{M_{|\beta|}} \right\} \sup_{\substack{\alpha\in\mathbb{N}_0^q \\ x\in K(m)}} \left\{ \frac{(2h)^{|\alpha|} |\varphi^{(\alpha)}(x)|}{M_{|\alpha|}} \right\} .$$

Using Sobolev's lemma (see for example [4], p. 197) for a suitable C_m and $r > q/s$, $r \in \mathbb{N}$, we obtain

$$\sup_{\substack{\alpha\in\mathbb{N}_0^q \\ x\in K(m)}} \left\{ \frac{(2h)^{|\alpha|} |\varphi^{(\alpha)}(x)|}{M_{|\alpha|}} \right\} \leqslant C_m \sup_{\alpha\in\mathbb{N}_0^q} \left\{ \frac{(2h)^\alpha}{M_{|\alpha|}} \sum_{i=0}^{r} \left(\int_{-m}^{m} |\varphi^{(\alpha+i)}(x)|^s dx \right)^{1/s} \right\} .$$

Now, using (4) we obtain that for suitable $C > 0$ and $t \in \mathbb{N}$

$$\sup_{\substack{\alpha\in\mathbb{N}_0^q \\ x\in K(m)}} \left\{ \frac{(2h)^{|\alpha|} |\varphi^{(\alpha)}(x)|}{M_{|\alpha|}} \right\} \leqslant C \gamma_{s,t}(\varphi) . \tag{9}$$

From (8) and (9) it follows

$$\|\eta_k\varphi\|_{m,h} \leqslant C \|\eta\|_{1,2h} \gamma_{s,t}(\varphi) , \quad k \in \mathbb{N} . \tag{10}$$

To prove that $\gamma_{s,h}(\varphi-\eta_k\varphi)\to 0$, $k\to\infty$, for every $h\in\mathbf{N}$, we first notice that for any $t\in\mathbf{N}$

$$\sup_{\alpha\in\mathbf{N}_0^q}\left\{\frac{t^{|\alpha|}}{M_{|\alpha|}}\left(\int_{|x|>R}|\varphi^{(\alpha)}(x)|^s dx\right)^{1/s}\right\}\to 0 \quad \text{as } R\to\infty. \tag{11}$$

We have

$$\gamma_{s,h}(\varphi-\eta_k\varphi)\leqslant \sup_{\alpha\in\mathbf{N}_0^q}\left\{\frac{h^{|\alpha|}}{M_{|\alpha|}}\left(\int_{k/2<|x|<k}|(\eta_k\varphi)^{|\alpha|}(x)|^s dx\right)^{1/s}+\right. \\ \left. +\sup_{\alpha\in\mathbf{N}_0^q}\left\{\frac{h^{|\alpha|}}{M_{|\alpha|}}\left(\int_{|x|>k}|\varphi^{(\alpha)}(x)|^s dx\right)^{1/s}\right\}. \tag{12}$$

In the similar way as in the first part of the proof one can prove that the first member on the right side of (12) is smaller than

$$C\|\eta\|_{1,2h}\sup_{\alpha\in\mathbf{N}_0^q}\left\{\frac{t^{|\alpha|}}{M_{|\alpha|}}\left(\int_{k/2<|x|<k}|\varphi^{(\alpha)}(x)|^s dx\right)^{1/s}\right\}$$

where C and t are from (10). Now by (11) we obtain that

$$\gamma_{s,h}(\varphi-\eta_k\varphi)\to 0 \quad \text{as } k\to\infty .$$

If $\varphi\in\mathcal{D}_{L^s}(\mathbf{R}^q)$, η is as in Theorem 2 and $C=\int\eta(x)\,dx$, one can easily prove that $\varphi(x)*(k^q\eta(kx))/C$ is a sequence from $\mathcal{D}_{L^s}^{(M_p)}(\mathbf{R}^q)$ which converges to φ in $\mathcal{D}_{L^s}(\mathbf{R}^q)$. Thus, we obtain

THEOREM 3. *$\mathcal{D}_{L^s}^{(M_p)}(\mathbf{R}^q)$ is a dense subspace of $\mathcal{D}_{L^s}(\mathbf{R}^q)$.*

4. Space $\mathcal{D}_{L^s}^{\prime(M_p)}(\mathbf{R}^q)$.

We denote by $\mathcal{D}_{L^s}^{\prime(M_p)}(\mathbf{R}^q)$ the strong dual of $\mathcal{D}_{L^s}^{(M_p)}(\mathbf{R}^q)$.
If $f\in\mathcal{D}_{L^s}^{\prime(M_p)}(\mathbf{R}^q)$ and $k\in\mathbf{N}^q$ we put

$$\langle f^{(k)}(x),\varphi(x)\rangle := (-1)^{|k|}\langle f(x),\varphi^{(k)}(x)\rangle , \quad \varphi\in\mathcal{D}_{L^s}^{(M_p)}(\mathbf{R}^q) .$$

Obviously

$$f \to f^{(k)}$$

is a continuous mapping from $\mathfrak{D}'^{(M_p)}_{L^s}(\mathbb{R}^q)$ into $\mathfrak{D}'^{(M_p)}_{L^s}(\mathbb{R}^q)$, $k \in \mathbb{N}^q$.

If $f \in L^t(\mathbb{R}^q)$, where $t = s/(s-1)$, then by

$$\varphi \to \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx\,, \qquad \varphi \in \mathfrak{D}^{(M_p)}_{L^s}(\mathbb{R}^q)\,,$$

an element from $\mathfrak{D}'^{(M_p)}_{L^s}(\mathbb{R}^q)$ is defined. Clearly, different functions from $L^t(\mathbb{R}^q)$ define different elements in $\mathfrak{D}'^{(M_p)}(\mathbb{R}^q)$.

Let us denote by $\overline{\mathfrak{D}}^{(M_p)}_{L^s,n}(\mathbb{R}^q)$, $n \in \mathbb{N}$, the completion of $\mathfrak{D}^{(M_p)}_{L^s}(\mathbb{R}^q)$ under the norm $\gamma_{s,n}$. From ([1], p. 47, 2.2 Satz) we directly obtain

THEOREM 4. $\mathfrak{D}^{(M_p)}_{L^s}(\mathbb{R}^q) = \lim\limits_{n\in\mathbb{N}} \operatorname{proj} \overline{\mathfrak{D}}^{(M_p)}_{L^s,n}(\mathbb{R}^q)$ *is a strict* (FG)*-space.*

This implies (see [1], p. 59, 3.1, Satz) that

$$\mathfrak{D}'^{(M_p)}_{L^s}(\mathbb{R}^q) = \bigcup_{n=1} \overline{\mathfrak{D}}'^{(M_p)}_{L^s,n}(\mathbb{R}^q) \tag{13}$$

in the set- theoretical sense.

THEOREM 5. *If* $f \in \mathfrak{D}'^{(M_p)}_{L^s}(\mathbb{R}^q)$ *then for some* $n \in \mathbb{N}$ *there is an element* $\{f_\alpha\}$ *from* $l^t_\infty(M_\alpha/n^\alpha)$, $t = s/(s-1)$, *such that*

$$f = \sum_{\alpha\in\mathbb{N}_0^q} f_\alpha^{(\alpha)} \tag{14}$$

in the sense of weak topology in $\mathfrak{D}'^{(M_p)}_{L^s}(\mathbb{R}^q)$. *Conversely, if for some* $n \in \mathbb{N}$ $\{f_\alpha\} \in l^t_\infty(M_\alpha/n^\alpha)$, *then there is an element* f *from* $\mathfrak{D}^{(M_p)}_{L^s}(\mathbb{R}^q)$ *such that* $\sum_{\alpha\in\mathbb{N}_0^q} f_\alpha^{(\alpha)}$ *converges in the weak sense to* f.

PROOF. We use in the proof the theory of Köthe spaces ([3], § 28.8, p. 359).

If $f \in \mathfrak{D}'^{(M_p)}_{L^s}(\mathbb{R}^q)$, Theorem 4 implies that for some n the extension of f on $\overline{\mathfrak{D}}^{(M_p)}_{L^s,n}(\mathbb{R}^q)$, denoted again by f, belongs to $\overline{\mathfrak{D}}'^{(M_p)}_{L^s,n}(\mathbb{R}^q)$. The space $\overline{\mathfrak{D}}^{(M_p)}_{L^s,n}(\mathbb{R}^q)$ is isometrically isomorphic to a closed subspace of $l^s_1\,(n^\alpha/M_\alpha)$

under the mapping

$$\mathcal{D}_{L^s,n}^{(M_p)}(\mathbb{R}^q) \in \varphi \to \{(-1)^{|\alpha|}\varphi^{(\alpha)};\ \alpha \in \mathbf{N}_0^q\} \in l_1^s\left(\frac{n^\alpha}{M_\alpha}\right).$$

We denote this subspace of $l_1^s(n^\alpha/M_\alpha)$ by Λ and define a continuous linear functional f_1 on Λ by

$$\langle f_1, \{(-1)^{|\alpha|}\varphi^{(\alpha)}\}\rangle := \langle f, \varphi\rangle . \tag{15}$$

Again, by Hahn-Banach theorem we extend f_1 on $l_1^s(n^\alpha/M_\alpha)$ to be linear and continuous (with the same dual norm). We denote this extension by F. Since the continuous dual of $l_1^s(n^\alpha/M_\alpha)$ is the space $l_\infty^t(M_\alpha/n^\alpha)$ we obtain that there exists $\{f_\alpha\} \in l_\infty^t(M_\alpha/n^\alpha)$ such that

$$\langle F, \{\psi_\alpha\}\rangle = \sum_{\alpha\in\mathbf{N}_0^q} \int_{\mathbf{R}^q} f_\alpha(x)\psi_\alpha(x)\,dx\,, \qquad \{\psi_\alpha\} \in l_1^s\left(\frac{n^\alpha}{M_\alpha}\right).$$

This implies (see (15)) that

$$\langle f, \varphi\rangle = \sum_{\alpha\in\mathbf{N}_0^q} (-1)^{|\alpha|} \int_{\mathbf{R}^q} f_\alpha(x)\varphi^{(\alpha)}(x)\,dx\,.$$

Now it is easy to see that (14) holds.

The converse part of theorem easily follows.

5. Hilbert transformation.

Now, we are ready to define the Hilbert transformation in the spaces $\mathcal{D}_{L^s}^{\prime(M_p)}(\mathbb{R})$ and $\mathcal{D}^{\prime(M_p)}(\mathbb{R})$.

It was proved in [6], Theorem 1 that for $\varphi \in \mathcal{D}_{L^s}(\mathbb{R})$

$$H(\varphi^{(k)}) = (H\varphi)^{(k)}, \qquad k \in \mathbf{N}_0\,. \tag{16}$$

Thus, using (1) and (2) we obtain

THEOREM 6. *The mapping*

$$\varphi \to H\varphi = P\int_{-\infty}^{\infty} \varphi(t)(t-x)^{-1}\,dt\,, \qquad \varphi \in \mathfrak{D}_{L^s}^{(M_p)}(\mathbb{R})$$

is a linear homeomorphism of $\mathfrak{D}_{L^s}^{(M_p)}(\mathbb{R})$ *onto* $\mathfrak{D}_{L^s}^{(M_p)}(\mathbb{R})$.

This theorem enable us to define the H-transform on $\mathfrak{D}_{L^s}^{\prime(M_p)}(\mathbb{R})$ as an adjoined mapping to the mapping

$$H\colon \mathfrak{D}_{L^s}^{(M_p)}(\mathbb{R}) \to \mathfrak{D}_{L^s}^{(M_p)}(\mathbb{R})\,.$$

Thus

$$\langle \tilde{H}f, \varphi\rangle := -\langle f, H\varphi\rangle\,, \qquad \varphi \in \mathfrak{D}_{L^s}^{(M_p)}(\mathbb{R})\,.$$

Theorem 6 directly implies

Theorem 7. $\tilde{H}$ *is a linear homeomorphism of* $\mathfrak{D}_{L^s}^{\prime(M_p)}(\mathbb{R})$ *onto itself.*

Let $t = s/(s-1)$. If $f \in L^t(\mathbb{R}) \subset \mathfrak{D}_{L^s}^{\prime(M_p)}(\mathbb{R})$ then ([8], p. 132) implies

$$\int_{-\infty}^{\infty} (Hf)(x)\varphi(x)\,dx = -\int_{-\infty}^{\infty} f(x)(H\varphi)(x)\,dx\,, \qquad \varphi \in \mathfrak{D}_{L^s}^{(M_p)}(\mathbb{R})\,.$$

This allow us to use instead of $\tilde{H}$ the notation H for the Hilbert transformation of elements from $\mathfrak{D}_{L^s}^{\prime(M_p)}(\mathbb{R})$.

We denote by $H(\mathfrak{D}^{(M_p)}(\mathbb{R}))$ the image of $\mathfrak{D}^{(M_p)}(\mathbb{R})$ under the mapping H. Since $\mathfrak{D}^{(M_p)}(\mathbb{R})$ is a dense subspace of $\mathfrak{D}_{L^s}^{(M_p)}(\mathbb{R})$, Theorem 6 implies

THEOREM 8. (i) $H(\mathfrak{D}^{(M_p)}(\mathbb{R}))$ *is a dense subspace of* $\mathfrak{D}_{L^s}^{(M_p)}(\mathbb{R})$.

(ii) $H\big(H(\mathfrak{D}^{(M_p)}(\mathbb{R}))\big) = \mathfrak{D}^{(M_p)}(\mathbb{R})\,.$

We transport the topology from $\mathfrak{D}^{(M_p)}(\mathbb{R}^q)$ to the space $H(\mathfrak{D}^{(M_p)}(\mathbb{R}))$. The Hilbert transformation on $\mathfrak{D}'^{(M_p)}(\mathbb{R})$ denoted again by H is a mapping from $\mathfrak{D}'^{(M_p)}(\mathbb{R})$ into $\big(H(\mathfrak{D}^{(M_p)}(\mathbb{R}))\big)'$ defined by

$$\langle Hf, \varphi\rangle = -\langle f, H(\varphi)\rangle\,, \qquad \varphi \in H(\mathfrak{D}^{(M_p)}(\mathbb{R}))\,.$$

We supply $\big(H(\mathfrak{D}^{(M_p)}(\mathbb{R}))\big)'$ by the strong dual topology.

THEOREM 9. *The Hilbert transformation is a homeomorphism of* $\mathcal{D}'^{(M_p)}(\mathbb{R})$ *onto* $\big(H(\mathcal{D}^{(M_p)}(\mathbb{R}))\big)'$.

Relation (16) implies

$$H(f^{(k)}) = (Hf)^{(k)}, \qquad f \in \mathcal{D}'^{(M_p)}(\mathbb{R}) .$$

6. Hilbert transformation in $\mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$.

In order to extend the definition of the Hilbert transformation from the space $\mathcal{D}'_{L^2}(\mathbb{R}^q)$ (see [9], p. 168) to the space $\mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$, we first define the convolution in $\mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$.

If $f \in \mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$ and $\varphi \in \mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$ then we put

$$(f * \varphi)(x) := \langle f(t), \varphi(x-t)\rangle .$$

Using Theorem 5 one can easily prove that

$$(f * \varphi) \in C^\infty(\mathbb{R}^q) \qquad \text{i.e.} \qquad (f * \varphi)^{(k)}(x) = \sum_{\alpha \in \mathbf{N}_0^q} (-1)^{|\alpha|} \int_{\mathbf{R}^q} f_\alpha(t)\varphi^{(k+\alpha)}(x-t)\, dt ,$$

where f is of the form (14).

Let $g \in \mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$ have the following properties:

For every $\varphi \in \mathcal{D}_{L^2}^{(M_p)}(\mathbb{R}^q)$, $g * \varphi \in \mathcal{D}_{L^2}^{(M_p)}(\mathbb{R}^q)$;

$\varphi \to g * \varphi$ is a continuous mapping from $\mathcal{D}_{L^2}^{(M_p)}(\mathbb{R}^q)$ into $\mathcal{D}_{L^2}^{(M_p)}(\mathbb{R}^q)$.

Then, g is called the convolutor. The space of all convolutors is denoted by $O'_c(\mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q))$.

If $f \in \mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$ and $g \in O'_c(\mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q))$ we define $f * g$ as an element from $\mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$ by

$$\langle f * g, \varphi\rangle := \langle f, \check{g} * \varphi\rangle , \qquad \varphi \in \mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q) \qquad (\check{f}(x) = f(-x)) .$$

From Theorem 3 it follows that $\mathcal{D}'_{L^2}(\mathbb{R}) \subset \mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$

THEOREM 10. *If* $f \in \mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)$, g, $h \in O'_c(\mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q)) \cap \mathcal{D}'_{L^2}(\mathbb{R}^q)$ *and* $g * h \in O'_c(\mathcal{D}_{L^2}'^{(M_p)}(\mathbb{R}^q))$, *then*

$$(f * g) * h = f * (g * h) = f * (h * g) .$$

PROOF. We notice that if $g, h \in \mathcal{D}'_{L^2}(\mathbb{R}^q)$ then the convolution $g * h$ is defined in [9] p. 154, to be an element from $\mathcal{S}'(\mathbb{R}^q)$ ba means of the Fourier transformation. If $g, h \in \mathcal{D}'_{L^2}(\mathbb{R}^q)$, $g \in O'_c(\mathcal{D}'^{(M_p)}_{L^2}(\mathbb{R}^q))$ then the restriction of the convolution $g * h$ defined in ([9], (1.7), p. 154) on the test functions from $\mathcal{D}^{(M_p)}(\mathbb{R}^q)$ is equal to the restriction of the convolution $g * h$, defined in this section, on $\mathcal{D}^{(M_p)}(\mathbb{R}^q)$. By the associativity of the convolution of elements from $\mathcal{D}'_{L^2}(\mathbb{R}^q)$ we obtain

$$(g * h) * \varphi = g * (h * \varphi), \quad \varphi \in \mathcal{D}^{(M_p)}(\mathbb{R}^q). \tag{17}$$

Since $\mathcal{D}^{(M_p)}(\mathbb{R}^q)$ is a dense subspace of $\mathcal{D}^{(M_p)}_{L^2}(\mathbb{R}^q)$ and the boath sides of (17) exist (g, h and $g * h$ are convolutors) when $\varphi \in \mathcal{D}^{(M_p)}_{L^2}(\mathbb{R}^q)$, by the definition of a convolutor we obtain

$$(g * h) * \varphi = g * (h * \varphi), \quad \varphi \in \mathcal{D}^{(M_p)}_{L^2}(\mathbb{R}^q).$$

Thus we have

$$\begin{aligned}\langle (f * g) * h, \varphi \rangle &= \langle f * g, \check{h} * \varphi \rangle = \langle f, \check{g} * (\check{h} * \varphi) \rangle = \\ &= \langle f, (g \check{*} h) * \varphi \rangle = \langle f * (g * h), \varphi \rangle\end{aligned}$$

and the proof of the first equality in Theorem 10 is complete. The second equality can be proved in a similar way.

THEOREM 11. $\operatorname{Im} \mathcal{K}_C(x)$, $\operatorname{Re} \mathcal{K}_C(x) \in \mathcal{D}'_{L^2}(\mathbb{R}^q) \cap O'_c(\mathcal{D}'^{(M_p)}_{L^2}(\mathbb{R}^q))$.

PROOF. It was pointed out in Section 2 that

$$\operatorname{Im} \mathcal{K}_C(x), \quad \operatorname{Re} \mathcal{K}_C(x) \in \mathcal{D}'_{L^2}(\mathbb{R}^q).$$

If $\varphi \in \mathcal{D}^{(M_p)}_{L^2}(\mathbb{R}^q)$ then by ([9], (2.8), p. 161) we obtain

$$\|\operatorname{Im} \mathcal{K}_C * \varphi\|_2^2 = (2\pi)^q 2^{-1} \|(\theta_{C^*} - \theta_{-C^*})\check{\hat{\varphi}}\|_2^2 \leqslant (2\pi)^q \|\hat{\varphi}\|_2^2 = (2\pi)^{2q} \|\varphi\|_2^2$$

where $\theta_{\pm C^*}$ is the characteristic function of $\pm C^*$. This inequality implies that $\operatorname{Im} \mathcal{K}_C(x) \in O'_c(\mathcal{D}'^{(M_p)}_{L^2}(\mathbb{R}^q))$. Similarly, we prove that $\operatorname{Re} \mathcal{K}_C(x) \in$ $\in O'_c(\mathcal{D}'^{(M_p)}_{L^2}(\mathbb{R}^q))$.

For a given $f \in \mathcal{D}'^{(M_p)}_{L^2}(\mathbb{R}^q)$ we define

$$f_1 := f * \operatorname{Im} \mathcal{K}_C. \tag{18}$$

$f_1 \in \mathcal{D}_{L^2}^{\prime(M_p)}(\mathbb{R}^q)$. Using formulae given in (6), (7), Theorems 10 and 11, one can easily prove

THEOREM 12. *If $f \in \mathcal{D}_{L^2}^{\prime(M_p)}(\mathbb{R}^q)$ and f_1 is defined by* (18) *then the following two conditions are equivalent*:

$$a) \quad f = -\frac{4}{(2\pi)^{2q}} f_1 * \operatorname{Im} \mathcal{K}_C ;$$

$$b) \quad f = \frac{2}{(2\pi)^q} f * \operatorname{Re} \mathcal{K}_C .$$

As in [9], p. 169, we say that ultradistributions f and f_1 from $\mathcal{D}_{L^2}^{\prime(M_p)}(\mathbb{R}^q)$, where f_1 is defined by (18), form a pair of Hilbert transformations if they satisfy relation (a) from Theorem 12. In this case we say that f_1 is the Hilbert transformation of f and write $f_1 = Hf$.

If $q = 1$ then $C = (0, \infty)$ and

$$\operatorname{Re} \mathcal{K}_C(x) = \pi\delta(x) , \qquad \operatorname{Im} \mathcal{K}_C(x) = \frac{1}{x} \ \text{([9], p. 169)} .$$

In this case for every $f \in \mathcal{D}_{L^2}^{\prime(M_p)}(\mathbb{R})$, f and f_1 (defined by (18)) form a pair of Hilbert transformations.

Obviously, for $q = 1$, $s = 2$ the Hilbert transformation on $\mathcal{D}_{L^2}^{\prime(M_p)}(\mathbb{R})$ defined in this section is equal to the Hilbert trasnformation defined in Section 5.

We do not give in this paper any application. We only remark that in [5] and [6] some assertions are obtained in a way that they are proved for a finite series of the form

$$\sum_{\alpha=0}^{k} f_\alpha^{(\alpha)} , \qquad f_\alpha \in L^t \qquad (t = s/(s-1)) .$$

The same assertions also hold if this series is observed as an element of the space $\mathcal{D}_{L^2}^{\prime(M_p)}(\mathbb{R})$. Now we have possibilities to give the generalized assertions for the infinite series of the form

$$\sum_{\alpha=0}^{\infty} f_\alpha^{(\alpha)} , \qquad \{f_\alpha ;\ \alpha \in \mathbb{N}_0\} \in l_\infty^t \left(\frac{M_\alpha}{n^\alpha}\right) , \qquad \text{for some } n \in \mathbb{N} .$$

For example, one can prove that for given $x \in \mathbb{R}$, $\eta > 0$ and

$z = x + iy$ with $y \neq 0$,

$$(t - x)/((t - x)^2 + \eta^2)\,, \qquad 1/(t - z) \in \mathcal{D}'^{(M_p)}_{L^s}(\mathbb{R})\,.$$

Thus, as in [6] (for elements from $\mathcal{D}'_{L^s}(\mathbb{R})$) one can study the approximate Hilbert transformation and the analytic representation of elements from $\mathcal{D}'^{(M_p)}_{L^s}(\mathbb{R})$.

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Manoscritto pervenuto in redazione il 28 giugno 1985.