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WACŁAW MARZANTOWICZ

ADAM PARUSIŃSKI

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Periodic Solutions Near an Equilibrium of a Differential Equation with a First Integral.

WACŁAW MARZANTOWICZ - ADAM PARUSIŃSKI (*)

SUMMARY - In this paper we present a sufficient condition for the existence of periodic solutions of the autonomous system of ordinary differential equations of the first order near an equilibrium point. This generalise previous local results concerning the existence of periodic solutions near an equilibrium of such a system.

1. Introduction.

We shall consider a system of ordinary differential equations

$$\dot{x} = f(x)\,, \tag{1.1}$$

where $x \in \mathbf{R}^n$ and f is a C^2 map such that $f(0) = 0$. This system can be written in the following form

$$\dot{x} = Ax + \varphi(x)\,,$$

where $A = Df(0)$ is a linear operator on $\mathbf{R}^n$ and $\varphi(x) = o(\|x\|)$. It is assumed that (1.1) has a C^2 first integral $G(x) = G_0 + G_1(x) + \frac{1}{2}G_2(x, x) + \ldots$, with $G_0 = G(0) = DG(0) = G_1 = 0$, and with G_2 non-degenerate. In other words, 0 is a non-degenerate critical point of $G(x)$.

(*) Indirizzo degli AA.: Department of Mathematics, University of Gdańsk, Polonia.

We also assume that A is an isomorphism.

For a discussion of the problem of periodic orbits it is convenient to inject the set of periodic solutions of (1.1) into the space $\mathbf{R}^n \times [0, \infty)$. Observe that the problem of the existence of periodic orbits of (1.1) can be reformulated as the problem of the existence of periodic orbits of the fixed period 1. Indeed, substituting $y(t) = x(t/p)$ into (1.1) we see that periodic solutions of (1.1) correspond to the pairs (y, p) which are solutions of the equation

$$\dot{y} = pAy + p\varphi(y)\,,$$

where y is of period 1. In other words, the set of all periodic solution of (1.1) is the zero set of the map

$$F(x, p) = \dot{x} - pAx - p\varphi(x) \tag{1.2}$$

defined of the space $C^1(S^1, \mathbf{R}^n) \times [0, \infty)$ where $S^1 = \mathbf{R}^1/Z$ is the unit circle. The zero set of F in $C^1(S^1, \mathbf{R}^n) \times [0, \infty)$ is injected into $\mathbf{R}^n \times [0, \infty)$ by the map $(x, p) \to (x(0), p)$.

Note next that $x = 0$ is a periodic solution of an arbitrary period $p > 0$. It follows from the Implicit Function Theorem that the point $(0, p_0)$ is an accumulation point of nontrivial periodic solutions (x, p), $x \neq 0$, only if there is a purely imaginary eigenvalue $\mu = i\beta$, $\beta > 0$, of A such that $p_0 = 2\pi/\beta$.

This paper is devoted to the proof of a theorem which gives a sufficient condition for the existence of a branch of nontrivial periodic solutions at $(0, p_0)$ such that the above necessary condition holds.

2. Main result.

First we introduce some notation.

Let $\mathfrak{Z}$ be the set of zeros of the map $F(x, p)$ injected into $\mathbf{R}^n \times [0, \infty)$, and $\Lambda = \{0\} \times [0, \infty)$ the set of trivial solutions of $F(x, p)$ at the equilibrium point $x = 0$. We denote by $\mathfrak{C}$ the set of critical points of $F(x, p)$ in Λ, or equivalently the set of all points $(0, p_0) \in \Lambda$ such that the equation

$$\dot{x} = p_0 Ax \tag{1.3}$$

has a periodic solution of period 1.

Let $\mathcal{B}$ denote the set $(\mathfrak{Z}\setminus\Lambda) \cup \mathfrak{C}$. The set of all eigenvalues of the operator A will be denoted by $\sigma(A)$. If $\mu \in \sigma(A)$ then we denote by $E_\mu \subset \mathbf{R}^n$ the generalized eigenspace of A corresponding to μ (if μ is not real then E_μ corresponds to both μ and $\bar{\mu}$).

THEOREM 1. *Let $\dot{x} = Ax + \varphi(x)$ be a system of ordinary differential equations as in* (1.1) *with the first integral $G(x) = \frac{1}{2}G_2(x, x) + \ldots$ such that G_2 is a nondegenerate bilinear form on $\mathbf{R}^n$. Suppose that $\mu_1 = i\beta$, $\beta > 0$, is a purely imaginary eigenvalue of A, and G_2, restricted to the space E_μ, for some $\mu_k = k\mu_1 \in \sigma(A)$, $k \in N$, or to the space*

$$E_0 = \bigoplus_{\substack{\mu\in\sigma(A)\\ \mu=k\mu_1, k\in N}} E_N ,$$

has a nonzero signature.

Then there exists a connected set $\mathcal{K} \subset \mathcal{B}$ such that $(0, 2\pi/\beta) \in \mathcal{K}$. Furthermore, there exists a neighbourhood W of $(0, 2\pi j\beta)$ in $\mathbf{R}^n \times [0, \infty)$ such that if $(x, p) \in \mathcal{K} \cap W$, $x \neq 0$, then there exists $k = k(x, p) \in N$ with $ik\beta \in \sigma(A)$ such that the minimal period of x is p/k.

Moreover, $\mathcal{K}$ satisfies one of the following conditions:

either 1°) *$\mathcal{K}$ is unbounded in $\mathbf{R}^n \times [0, \infty)$,*

or 2°) *there exists $(x', p') \in \mathcal{B}$ such that x' is the equilibrium point of* (1.1) *and $(x', p') = (0, p')$ then $p' = k\cdot 2\pi/\beta'$, where $i\beta' \in \sigma(A)$.*

In our proof we follow the approach of D. Schmidt [8] and J. Alexander and J. Yorke [1].

We add to the system (1.1) the perturbation λ grad $G(x)$, with one-dimensional parameter λ, and next apply the Hopf bifurcation theorem to the perturbad system

$$\dot{x} = Ax + \varphi(x) + \lambda \operatorname{grad} G(x) . \tag{1.4}$$

Instead of J. Alexander and J. Yorke's « mod. 2 » version of the Hopf theorem we shall use here a more general version of this theorem proved by S. Chow, J. Mallet-Paret and J. Yorke [2]. For the computation of the bifurcation invariant ([2], [3]) we use its definition as introduced by J. Ize (see [5] for the most general version of the Hopf theorem covering also the case of multidimensional parameter λ).

DEFINITION 1. Let $A(\lambda)$ be a one-dimensional family of linear operators on $\mathbf{R}^n$ with $A(0) = A$ an isomorphism. Assume that $i\beta \in \sigma(A)$, with $\beta > 0$, and that for small $\lambda \neq 0$ there are no eigenvalues of $A(\lambda)$ on the imaginary axis near $ik\beta$ for any $k \in N$. Under this assumption for a given k, we can define a map from a small sphere

$$S_\varrho^1 = \{(\lambda, p) \in \mathbf{R}^2: |\lambda|^2 + |p - p_0|^2 = \varrho^2\}, \quad p_0 = 2\pi/\beta,$$

into $GL(n, \mathbf{C})$ given by

$$(\lambda, p) \to i2\pi k - pA^c(\lambda), \tag{1.5}$$

where $A^c(\lambda)$ is the complexification of $A(\lambda)$.

The class of this map in the first homotopy group $\pi_1(GL(n, \mathbf{C})) = Z$ is, by definition, the *k-th bifurcation invariant* r_k of $A(\lambda)$ at $(0, p_0)$.

We also define $r = \sum_{k=1}^{\infty} r_k$ (r_k is zero for almost all k) which is called the *bifurcation invariant* of $A(\lambda)$ at $(0, p_0)$.

The following version of the Hopf bifurcation theorem is basic for our considerations.

Let us consider a system with one-dimensional parameter:

$$\dot{x} = A(\lambda)x + \varphi(x, \lambda) \tag{1.6}$$

where $A(\lambda) \in GL(n, \mathbf{R})$ as in Definition 1 and $\varphi(x, \lambda) = o(\|x\|)$ uniformly with respect to λ.

As before, we can inject the set of periodic solutions of (1.6) into $\mathbf{R}^n \times [0, \infty) \times \mathbf{R}$. The Hopf bifurcation theorem in the version of S. Chow, J. Mallet-Paret, J. Yorke [2] and J. Ize [5] reads as follows.

(1.7) THE HOPF BIFURCATION THEOREM. If for some $k \in N$ the invariant r_k (or r) of the problem (1.6) is different from zero then there exists a connected set $\mathcal{K} \subset \mathbf{R}^n \times [0, \infty) \times \mathbf{R}$ of periodic solutions of (1.6) such that $(0, 2\pi/\beta, 0) \in \mathcal{K}$. The only trivial periodic solutions that can be in $\mathcal{K}$ are of the form $(0, 2\pi/\beta \cdot k, 0)$ where $i\beta \in \sigma(A)$, $\beta > 0$ and $k \in N$. Furthermore, there exists a neighbourhood W of $(0, 2\pi/\beta, 0)$ in $\mathbf{R}^n \times [0, \infty) \times \mathbf{R}$ such that if $(x, p, \lambda) \in \mathcal{K} \cap W$, $x \neq 0$, then there exists $k = k(x, p, \lambda) \in N$ with $ik\beta \in \sigma(A)$ such that the minimal period of x is p/k. Moreover, $\mathcal{K}$ satisfies one of the following conditions:

either 1°) $\mathcal{K}$ is unbounded in $\mathbf{R}^n \times [0, \infty) \times \mathbf{R}$, or 2°) there exists a solution (x', p', λ') such that x' is the equilibrium point of (1.6) and if $(x', p', \lambda') = (0, p', \lambda')$ then $p' = k\cdot 2\pi/\beta'$ where $i\beta' \in \sigma(A)$.

We also need some further notations. Let $G_2\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be a belinear form. For every scalar product $\langle\,,\rangle$ in $\mathbf{R}^n$ we define a linear map $S\colon \mathbf{R}^n \to \mathbf{R}^n$ by

$$G_2(v, w) = \langle v, Sw\rangle \tag{1.8}$$

for every $v, w \in \mathbf{R}^n$.

Assume that some scalar product $\langle\,,\rangle$ is chosen in $\mathbf{R}^n$ then for a given linear operator A we denote by A^* the operator adjoint to A.

We shall use the following lemma.

LEMMA 1. *Let $A \in GL(n, \mathbf{R})$ be the linear part of the right-hand side of equation* (1.1). *Then for every scalar product $\langle\,,\rangle$ we have*

$$A^*S + SA = 0$$

where S is the operator corresponding to the quadratic part $G_2 = DG(0)$ of the first integral $G(x)$ of (1.1).

PROOF. Having chosen a scalar product, we can write $G(x) = \frac{1}{2}\langle x, Sx\rangle +$ terms of the higher order in x.

Differentiating $G(x)$ along an integral curve $x(t)$ of (1.1) we obtain

$$0 = \frac{d}{dt}\,G(x(t))|_0 = \frac{1}{2}\,(\langle x(0), SAx(0)\rangle + \langle Ax(0), Sx(0)\rangle) + \\ + o0(\|x\|^2) = \frac{1}{2}\,\langle x(0), (SA + AS)x(0)\rangle + o0(\|x\|^2)\,.$$

Using the fact that in some small convex neighbourhood of 0, for every point v there is an integral curve $x(t)$ of (1.1) such that $x(0) = v$, and that the operator $SA + A^*S$ is self-adjoint, we get the lemma (see [6] for more details).

We are now in a position to formulate an assertion which enables us to compute the bifurcation invariants r_k and r of the system (1.4).

We begin with the following.

DEFINITION 2. Let $A\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear operator. We say that a scalar product $\langle\,,\rangle$ is *associated with A* if there exists an orthonormal basis $e_1, e_2, \ldots, e_n$ of $\mathbf{R}^n$ in which A has the canonical Jordan form.

It follows from the definition that for every two distinct eigenvalues μ_1, μ_2 of A the generalized eigenspaces E_{μ_1}, E_{μ_2} are orthogonal in such a scalar product.

PROPOSITION 1. *Suppose we have $A \in GL(n, \mathbf{R})$ with $ik\beta \in \sigma(A)$ for some $\beta > 0$, $k \in N$. Let E_k be the generalized eivenspace corresponding to $p_k = ik\beta$ and E'_k the direct sum of all remaining generalized eigenspaces of A. Suppose next that in some scalar product associated with A we have a selfadjoint operator $S\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $A^*S + SA = 0$. Then:*

a) S preserves E_k and thus E'_k,

b) If $S|_{E_k}$ is an isomorphism then the map $(\lambda, p) \to i2\pi k - pA^c - p\lambda S^c$ maps $S^1_\varrho = \{(x, p)\colon |\lambda|^2 + |p - p_0|^2 = \varrho^2\}$, $p_0 = 2\pi/\beta$ into $GL(n, \mathbf{C})$ for sufficiently small ϱ.

Moreover the class of this map in $\pi_1(GL(n, \mathbf{C})) = Z$ is equal to $\frac{1}{2}\operatorname{Sign}\langle x, Sx\rangle|_{E_k}$.

We will prove Proposition 1 in the next section.

We now return to the proof of Theorem 1.

PROOF OF THEOREM 1. Using the notations we have introduced, we consider the perturbed system (1.4) with one-dimensional parameter λ:

$$\dot{x} = Ax + \varphi(x) + \lambda \operatorname{grad} G(x)$$

where the gradient of the first integral $G(x)$ is taken with respect to a scalar product associated with A.

First observe that if $\lambda \neq 0$, then (1.4) has no periodic solutions in some small neighbourhood of 0([8]). Indeed, taking the derivative

$$\frac{d\dot{G}(x(t))}{dt}\Big|_0 = \lambda \|\operatorname{grad} G(x)\|^2$$

we see that the function $G(x)$ is strictly monotonic along every trajectory $x(t)$ of (1.4) if $\lambda \neq 0$ because 0 is nondegenerate critical point of $G(x)$.

On the other hand we show that for the system (1.4) the Hopf bifurcation occurs at the point $(0, 2\pi/\beta, 0)$.

Since grad $G(x) = Sx + r(x)$ with $r(x) = o(x)$, it follows that (1.4) can be written in the form

$$\dot{x} = (A + \lambda S)x + \varphi(x, \lambda)$$

where $\varphi(x, \lambda)$ is a C^1 map and $\varphi(x, \lambda) = o(\|x\|)$ uniformuly with respect to λ. In other words, we have to study the Hopf bifurcation problem with the linear part

$$A(\lambda) = A + \lambda S \tag{1.9}$$

From Lemma 1 we know that $A^*S + SA = 0$ and we can apply Proposition 1 to the computation of the bifurcation invariant r_k and r. By Definition 1, for small $\varrho \neq 0$, the integer r_k is equal to the class of the map

$$(\lambda, p) \to i2\pi k - pA^c - p\lambda S^c$$

from S^1_ϱ into $GL(n, \mathbf{C})$. It follows from Proposition 1 that this class is equal to $\frac{1}{2}$ Sign $G_{2|E_k}$.

Consequently $r =$ Sign $G_{2|E_0}$, because $r = \sum_{k=1}^{\infty} r_k$ and the signature is additive. It now follows from the Hopf bifurcation theorem (see (1.7)) that there exists a branch $\mathcal{K} \subset \mathbf{R}^n \times [0, \infty) \times \mathbf{R}$ of nontrivial periodic solutions of (1.4) with the described properties at the point $(0, 2\pi/\beta, 0)$. But we have proved that it consists of points of the form $(x, p, 0) \in \mathbf{R}^n \times [0, \infty) \times \{0\}$. This shows $\mathcal{K}$ that is the desired branch of nontrivial periodic solutions of (1.1) satisfying all conditions of Theorem 1. The proof is complete.

2. Proof of Proposition 1.

Let A be a linear operator of $\mathbf{R}^n$ as in Proposition 1. First note that the generalized eigenspaces of A^* are then the same as those of A. By definition of E_k there exists $m \in N$ such that

$$E_k = \ker(A^2 + k^2\beta^2)^m . \tag{2.1}$$

By (2.1) $E_k = \ker(A^2 + k^2\beta^2)^m = \ker((A^*)^2 + k^2\beta^2)^m$ and since $A^*S +$

$+SA = 0$, we have $(A^*)^2 S = SA^2$ and consequently

$$((A^*)^2 + k^2\beta^2)^l S = S(A^2 + k^2\beta^2)^l$$

for every $l \in N$. Taking an element $v \in E_k$ we have

$$((A^*)^2 + k^2\beta^2)^m Sv = S(A^2 + k^2\beta^2)^m v = 0$$

which shows that $Sv \in E_k$ and consequently $S(E'_k) \subset E'_k$ because S is selfadjoint and E'_k is the orthogonal complement of E_k. This proves part (*a*) of the statement of Proposition 1.

The proof of part (*b*) is divided into a sequence of lemmas.

LEMMA 2. *Let A and S be linear operators on $\mathbb{R}^n$, S selfadjoint. Then the function $\langle x, Sx\rangle$ is a first integral of the system*

$$\dot{x} = Ax$$

if and only if $A^*S + SA = 0$.

PROOF. Observe that for every integral curve $x(t)$ of the system $\dot{x} = Ax$ we have

$$\frac{d}{dt}\langle x(t), Sx(t)\rangle = \langle x(t), (A^*S + SA)x(t)\rangle ,$$

so that $\langle x, Sx\rangle$ is a first integral if $A^*S + SA = 0$. The necessity follows from Lemma 1.

LEMMA 3. *Let A and S be as in Proposition* 1. *Suppose also that $S|_{E_k}$ is an isomorphism. Then for every $\lambda \neq 0$ the map $A + \lambda S$ has no purely imaginary eigenvalues near $ik\beta$.*

If S is an isomorphism then $A + \lambda S$ has no purely imaginary eigenvalues at all if $\lambda \neq 0$.

PROOF. First observe that $A + \lambda S$ has purely imaginary eigenvalue if and only if the system

$$\dot{x} = (A + \lambda S)x$$

has a periodic solution. On the other hand, for every such solution

$x(t)$ we have

$$\frac{d}{dt}\langle x(t), Sx(t)\rangle = \lambda\|Sx\|^2, \tag{2.2}$$

thus for $\lambda \neq 0$ the map $A + \lambda S$ has no purely imaginary eigenvalue if S is isomorphism.

In the case $\ker S \neq 0$ we see that A maps $\ker S$ into $\ker S$ because $SA = -A^*S$. From 2.2 we deduce that a periodic solution of the system $\dot{x} = (A + \lambda S)x$ has to belong to $\ker S$ and $A + \lambda S|_{\ker S} = A|_{\ker S}$. Since $S|_{E_k}$ is an isomorphism, we see that every purely imaginary eigenvalue of $A + \lambda S$ is different from $ik\beta$, $-ik\beta$ if $\lambda \neq 0$.

2.3. COROLLARY. *Under the assumption of Proposition* 1, *the map* $i2\pi k - pA^c - p\lambda S^c$ *is a linear isomorphism for* (λ, p) *such that* $\lambda^2 + (p - p_0)^2 = \varrho^2$, $p_0 = 2\pi/\beta$ *and* ϱ *sufficiently small.*

For the above ϱ we denote by $L_k(\lambda, p)$ the map from S^1_ϱ into $GL(n, \mathbf{C})$ given by

$$L_k(\lambda, p) = i2\pi k - pA^c - p\lambda S^c.$$

The restriction $L_k(\lambda, p)|_{E_k\otimes\mathbf{C}}$ gives a map from S^1_ϱ into $GL(E_k \otimes \mathbf{C})$ which is denoted by $\tilde{L}_k(\lambda, p)$.

Since the map $\det\colon GL(n, \mathbf{C}) \to \mathbf{C}^*$ induces an isomorphism of the fundamental groups, it is sufficient to study the map

$$\det L_k(\lambda, p), \quad (\text{or } \det \tilde{L}_k(\lambda, p))$$

from S^1_ϱ into $\mathbf{C}^*$.

Let μ be an eigenvalue of A. Recall that μ is said to have the algebraic multiplicity equal to the geometrie multiplicity if there is no nilpotent part in the factors corresponding to μ in the Jordan decomposition.

LEMMA 4. *Suppose that* $\mu_k = ik\beta$, $\beta \in \mathbf{R}$, $k \in N$ *is an eigenvalue of* A *with the algebraic multiplicity equal to the geometric multiplicity. Then the element represented by* $\tilde{L}_k(\lambda, p)$ *in* $\pi_1(GL(E_k \otimes \mathbf{C}))$ *is equal to* $\frac{1}{2}\,\mathrm{Sign}\,\langle x, Sx\rangle|_{E_k}$.

PROOF. By assumptions, in some orthonormal basis $A|_{E_k}$ is represented by the matrix with 2×2 blocks

$$\begin{bmatrix} 0 & k\beta \\ -k\beta & 0 \end{bmatrix}$$

on the diagonal only. This means that $A^*|_{E_k} = -A|_{E_k}$ and consequently $SA = AS$ on E_k, because $S(E_k) \subset E_k$. This shows every eigenspace of $S|_{E_k}$ admists a complex structure induced by $A|_{E_k}$, in particular every eigenvalue of $S|_{E_k}$ is double. Taking the complexification we see that $A^c|_{E_k\otimes\mathbf{C}}$ and $S^c|_{E_k\otimes\mathbf{C}}$ can be written as

$$A = \begin{bmatrix} ik\beta & 0 \\ 0 & -ik\beta \end{bmatrix}, \qquad S = \begin{bmatrix} a_1 & & & & & 0 \\ & a_2 & & & & \\ & & \ddots & & & \\ & & & a_1 & & \\ 0 & & & & a_2 & \\ & & & & & \ddots \end{bmatrix}$$

where $E_k \otimes \mathbf{C} = F_1 \oplus F_2$, $F_2 = \bar{F}_1$ and $S|_{F_1}$ has the same eigenvalues as $S|_{F_2}$. Finally, in this basis, the map $\tilde{L}_k(\lambda, p)$ is represented by the matrix

$$\begin{bmatrix} i2\pi k\left(1-\frac{p}{p_0}\right)-p\lambda a_1 & & & & \\ & i2\pi k\left(1-\frac{p}{p_0}\right)-p\lambda a_2 & & 0 & \\ & & \ddots & & \\ & & & i2\pi k\left(1+\frac{p}{p_0}\right)-p\lambda a_1 & \\ & 0 & & & i2\pi k\left(1+\frac{p}{p_0}\right)-p\lambda a_2 \\ & & & & & \ddots \end{bmatrix}. \tag{2.4}$$

Since $\det(A \oplus B) = \det(A)\cdot\det(B)$, this shows that the class of $\tilde{L}_k(p, \lambda)|_{E_k\otimes\mathbf{C}}$ is the sum of the images of the classes represented by the maps

$$(\lambda, p) \to i2\pi k\left(1-\frac{p}{p_0}\right)-p\lambda a_j\,, \qquad a_j \neq 0\,, \tag{2.5}$$

from S^1_ϱ into $\mathbf{C}^*$, because the terms in the second factor give zero in $\pi_1(C^*)$. It is clear that the loop (2.5) gives in $\pi_1(C^*) = Z$ the element equal to sign a_j and consequently the class of (2.4) is equal to $\frac{1}{2}$ Sign $S|_{E_k}$. The proof is complete.

LEMMA 5. *Let A and S be as in Proposition* 1. *Assume that $\sigma(A) = \{i\beta, -i\beta\}$, $\beta \in \mathbf{R}$, $\beta > 0$. Let next $A = \tilde{A} + N$ be the canonicat decomposition of A into the semisimple and the nilpotent part.*

Then: 1°) $\tilde{A}S = S\tilde{A}$,

2°) $N^*S + SN = 0$.

PROOF. By our assumptions, in some orthonormal basis we can write A as a sum of matrices A_l of the form $A_l = \tilde{A}_l + N_l$, where

$$\tilde{A}_l = \begin{bmatrix} 0 & \beta & & & \\ -\beta & 0 & & & \\ & & 0 & \beta & \\ & & -\beta & 0 & \\ & & & & \ddots \end{bmatrix} \qquad N_l = \begin{bmatrix} 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ & & 0 & 0 & 1 & 0 & \\ & & 0 & 0 & 0 & 1 & \\ & & & \ddots & & \ddots & \end{bmatrix}$$

and $\tilde{A} = \bigoplus_l \tilde{A}_l$, $N = \bigoplus_l N_l$.

First observe that 2° follows from 1° because $(\tilde{A})^* = -\tilde{A}$. To prove 1° we extend the scalar product on $\mathbb{R}^n \otimes \mathbb{C}$, and accordingly extend the operation $*$. We have $(S^c)^* = S^c$. In the orthonormal basis $f_j = \frac{1}{2}(e_{2j-1} + ie_{2j})$, $f_{n+j} = \frac{1}{2}(e_{2j-1} - ie_{2j})$ the operator A^c is represented by a sum of matrices of the form

$$\begin{bmatrix} i\beta & 1 & & & & 0 \\ & i\beta & 1 & & & \\ & & \ddots & \ddots & & \\ & & & -i\beta & 1 & \\ 0 & & & & -i\beta & 1 \\ & & & & & \ddots \end{bmatrix}$$

and the matrix of $\tilde{A}^c$ is a sum of diagonal parts of these matrices.

From this it follows that $\mathbb{R}^n \otimes \mathbb{C}$ splits into $F_1 \oplus F(F_2 = \overline{F}_1)$, and moreover

$$F_1 = \ker\big((A^c + i\beta)^m\big) = \ker\Big(\big((A^c)^* - i\beta\big)^m\Big)$$

for some $m \in N$. Since $i\beta S^c + S^c(-i\beta) = 0$ and $(A^c)^* S^c + S^c A^c = 0$, we obtain

$$\big((A^c)^* + i\beta\big)^m S^c = S^c(A^c - i\beta)^m = 0\,. \tag{2.6}$$

From (2.6) it follows that $S^c(F_1) \subset F_1$ and $S^c(F_2) \subset F_2$, so that $\tilde{A}^c S^c = S^c \tilde{A}^c$ and consequently $\tilde{A}S = S\tilde{A}$, which is our claim.

We can now conclude the proof of part (*b*) of Proposition 1. First observe that the element represented by $L_k(\lambda, p)$ in $\pi_1(GL(n, \mathbf{C}))$ depends only on the restriction $\tilde{L}_k(\lambda, p)$. Moreover, the following diagram commutes:

$$\begin{tikzcd} \pi_1(GL(n, \mathbf{C})) \arrow[rr, "i_*"] \arrow[dr, "\det_*"'] & & \pi_1(GL(E_k \otimes \mathbf{C})) \arrow[dl, "\det_*"] \\ & \pi_1(\mathbf{C}^*) & \end{tikzcd}$$

In fact, from part (*a*) we know that S^c preserves $E_k \otimes \mathbf{C}$ and $E_k' \otimes \mathbf{C}$ thus $[\det L_k(\lambda, p)] = [\det \tilde{L}_k(\lambda, p)] \cdot [\det L_k(\lambda, p)|_{E_k' \otimes \mathbf{C}}]$. The second sumand on the right is zero in $\pi_1(GL(E_k' \otimes \mathbf{C}))$ because $A + \lambda S|_{E_k}$ has no purely imaginary eigenvalue near ik (Lemma 3). It is therefore sufficient to study $\tilde{L}_k(\lambda, p)$. After this reduction we can form the family

$$A_t = \tilde{A} + tN$$

where $A = \tilde{A} + N$ is the canonical decomposition of $A|_{E_k}$ as in Lemma 5 and $t \in [0, 1]$.

Observe that $A_t^* S + SA_t = 0$ and $\sigma(A_t) = \{ik\beta, -ik\beta\}$. This fact together with Corollary 2.3 guarantee that

$$\tilde{L}_k(\lambda, p, t) = i2\pi k - pA_t^c - p\lambda S^c|_{E_k \otimes \mathbf{C}}$$

belongs to $GL(E_k \otimes \mathbf{C})$ for every $t \in [0, 1]$.

Finally $\tilde{L}_k(\lambda, p) = \tilde{L}_k(\lambda, p, 1)$ and $\tilde{L}_k(\lambda, p, 0)$ give the same in $\pi_1(GL(E_k \otimes \mathbf{C}))$, to $\frac{1}{2}\operatorname{Sign}\langle Sx, x\rangle|_{E_k}$ by Lemma 4. The proof Proposition 1 is complete.

As we have said, J. Alexander and J. Yorke used the same approach to the special case of the Hamiltonian system

$$\dot{x} = I \operatorname{grad} H(x) = IH_2 x + I\tilde{H}(x) \tag{2.7}$$

where I is the symplectic matrix, $H\colon \mathbf{R}^{2n} \to \mathbf{R}$ a Hamiltonian function and $H_2 = (\partial^2 H(0)/\partial x_i\, \partial x_j)$ the Hessian of H at the equilibrium point. They proved that if $\frac{1}{2}\dim_{\mathbf{R}} E_0$ is odd then the conclusion of Theorem 1 holds [1]. But then $r = \frac{1}{2}$ Sign $H_2|_{E_0}$ is an odd integer and this result can be deduced from Theorem 1.

Our theorem also generalizes earlier result of J. Moser ([7] Th. 3). We have the following corollary of Theorem 1.

2.8. COROLLARY. *Assume that in Theorem* 1 *the form* G_2 *restricted to* E_0 *is positive define.*

Then for sufficiently small $\varepsilon > 0$ *every surface* $G(x) = \varepsilon^2$ *contains at least one periodic solution whose period is close to* $2\pi/\beta$.

PROOF. By connectivity of $\mathcal{K}$ it is sufficient to show that G is positive on $\mathcal{K}\setminus\{(0, 2\pi/\beta)\}$ locally near $(0, 2\pi/\beta)$, where G is extended on $\mathbb{R}^n \times [0, \infty)$ by the formula $G(x, p) = G(x)$.

First observe that from our assumption on G_2 it follows that there exists a constant $c > 0$ such that G is positive on the set

$$P_c = \{(v_1, v_2) \in E_0 \oplus E_0^{\perp} = \mathbb{R}^n\colon \|v_2\| < c\|v_1\|\}\,.$$

Next, using the Liapunov-Schmidt procedure we can, locally near $(0, p_0)$, embed the zero set of the map F defined in 1.2 in a graph of C^2 map

$$u\colon \ker D_x F(0, p_0)\cdot[0, \infty) \to X_2$$

where $\ker D_x F(0, p_0)$ is a finite dimensional subspace of $C^1(S^1, \mathbb{R}^n)$ isomorphic to a subspace of E_0 by the map $x \to x(0)$, and X_2 is the complement of $\ker D_x F(0, p_0)$ in $C^1(S^1, \mathbb{R}^n)$.

Furthermore $u(0, p) = 0$ for every p and $D_{x_1} u(0, p_0) = 0$, which shows that $\|u(x_1, p)\|/\|x_1\| \to 0$ if $(x_1, p) \to (0, p_0)$ and $x \neq 0$. The last means that, for a given $c > 0$, $x(0)$ belongs to $P_c \cup \{0\}$ if $(x, p) \in \mathcal{K}$ and (x, p) is sufficiently close to $(0, p_0)$.

From it follows that G is positive on $\mathcal{K}\setminus\{(0, 2\pi/\beta)\}$ which ends the proof.

Opposite to the case of the Hamiltonian system ([7], [4]) Theorem 1 does not give any information on the number of different periodic solutions on integral surface near the equilibrium. We must add that *N.* Dancer recently has shown a theorem similar to Theorem 1 for the case of the Hamiltonian system (2.7). He has used the periodic point index theory, introduced by him [3].

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