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# A note on the converse of the Clifford's theorem and some consequences

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# A Note on the Converse of the Clifford's Theorem and Some Consequences.

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Throughout this paper « group » means « finite group ». All characters considered are $\mathbb{C}$-characters. We use the standard notation from the books of *B.* Huppert [1] and I. M. Isaacs [2].

It is well known that the class of $p$-decomposable groups, where $p$ is a prime, is a saturated Fitting formation. The purpose of this paper is to prove the following

THEOREM A. *Let $G$ be a group. Let $P$ be a Sylow $p$-subgroup of $G$. Then $G = P \times K$ if and only if $\forall \theta \in \mathrm{Irr}_C(G)$ $\theta_P = n\chi$, where $\chi \in \mathrm{Irr}_C(P)$ and $n$ is some integer.*

Before proving Theorem *A*, we show that the converse of the Clifford's Theorem is true. Exactly, we give the following result

THEOREM B. *Let $G$ be a group and $N \leqslant G$ such that $\forall \theta \in \mathrm{Irr}_C(G)$ $\theta_N = n(\chi^{g_1} + \ldots + \chi^{g_k})$, where $g_i \in N_G(N)$, $\chi \in \mathrm{Irr}_C(N)$ and $n$ is some integer. Then $N \trianglelefteq G$.*

PROOF OF THEOREM B. Let $1_N$ be the principal character of $N$. If $x \in G$, then

$$(1_N)^G(x) = 1/|N| \sum_{g \in G} (1_N)^0 (g^{-1}xg) = 1/|N| \sum_{g^{-1} x g \in N} 1_N(g^{-1}xg) = |T(x)|/|N|$$

where $T(x) = \{g \in G : x \in N^{g^{-1}}\}$.

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If $\chi$ is an irreducible character of $G$, then

$$\chi_N = n_\chi(\eta_\chi^{g_1} + \dots + \eta_\chi^{g_{k_\chi}}), \qquad g_j \in N_G(N), \qquad \eta_\chi \in \mathrm{Irr}_C(N) .$$

By Frobenius reciprocity:

$$\big((1_N)^G, \chi\big)_G = (1_N, \chi_N)_N = \big(1_N, n_\chi(\eta_\chi^{g_1} + \dots + \eta_\chi^{g_{k_\chi}})\big)_N .$$

Suppose that there is an integer $s$ satisfying $\eta_\chi^{g_s} = 1_N$.
Then $\eta_\chi^{g_i} = 1_N$, $\forall i$, and so, we get

$$\big((1_N)^G, \chi\big)_G = n_\chi k_\chi \quad \text{and} \quad \chi_N = n_\chi k_\chi 1_N .$$

Otherwise, $\big((1_N)^G, \chi\big)_G = 0$. Then, we have,

$$(1_N)^G = \sum_{\chi_N = n_\chi k_\chi 1_N} n_\chi k_\chi \chi .$$

In particular:

$$|G|/|N| = (1_N)^G(1) = \sum_{\chi_N = n_\chi k_\chi 1_N} n_\chi k_\chi \chi(1) = \sum_{\chi_N = n_\chi k_\chi 1_N} (n_\chi k_\chi)^2 .$$

Let $\tau = (1_N)^G$. Then

$$\tau_N = \sum_{\chi_N = n_\chi k_\chi 1_N} n_\chi k_\chi \chi_N = \Big(\sum_{\chi_N = n_\chi k_\chi 1_N} (n_\chi k_\chi)^2\Big) 1_N = \big(|G|/|N|\big) 1_N .$$

Inducing on both sides:

$$(\tau_N)^G = \big(|G|/|N|\big)(1_N)^G.$$

If $x \in G$,

$$(\tau_N)^G(x) = 1/|N| \sum_{g \in G} (\tau_N)^0 (g^{-1}xg) = 1/|N| \sum_{g^{1-}xg \in N} \tau(g^{-1}xg) =$$
$$= \big(\tau(x)/|N|\big) |T(x)| = |T(x)|^2/|N|^2 .$$

On the other hand, $\big(|G|/|N|\big)(1_N)^G(x) = \big(|G|/|N|^2\big)|T(x)|$. Then, we have $\big(|G|/|N|^2\big)|T(x)| = |T(x)|^2/|N|^2$.

If $x \in N$, since $|T(x)| \geqslant |N| > 0$, the last equality gives us $|G| = |T(x)|$. Hence, $x$ belongs to all conjugates of $N$. Consequently, $N \trianglelefteq G$.

Proof of Theorem A: By Theorem B, $P \trianglelefteq G$. It is enough to see that $G$ is $p$-nilpotent. We use induction on $|G|$ and we can assume that $p \mid |G|$. If $N$ is a normal subgroup of $G$, it is not difficult to see that $G/N$ satisfies the induction hypothesis.

Assume that there exists $\theta \in \mathrm{Irr}_C(G)$ such that $\ker \theta = 1$.

By hypothesis, $\theta_P = n\chi$, $\chi \in \mathrm{Irr}_C(P)$.

Since $P > 1$, it follows that $Z(P) > 1$. Let $x \in Z(P)$. $\chi$ is irreducible and faithful, then $|\chi(x)| = \chi(1)$. Thus,

$$|\theta(x)| = n|\chi(x)| = n\chi(1) = \theta(1) ,$$

which implies that $x \in Z(G)$.

Then, $Z(G) > 1$ and $G/Z(G)$ is $p$-nilpotent.

From this it follows that there exits $M \leqslant G$ $p$-nilpotent such that $G = MZ(G)$. But then, $M \trianglelefteq G$ and $G$ is $p$-nilpotent.

So, we can suppose that $\ker \theta > 1$, $\forall \theta \in \mathrm{Irr}_C(G)$.

Now, by the induction hypothesis $G/\ker \theta$ is $p$-nilpotent.

Hence, $G \simeq G/ \bigcap_{\theta \in \mathrm{Irr}^a(G)} \ker \theta$ is $p$-nilpotent.

Clearly, the converse holds.

## REFERENCES

[1] B. Huppert, *Endliche Gruppen* I, Springer-Verlag, 1967.
[2] I. Martin Isaacs, *Character Theory of Finite Groups*, Academic Press, 1976.

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