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Groups with subnormal subgroups of bounded defect

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Groups with Subnormal Subgroups of Bounded Defect.

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1. The principal object of this note are soluble p-groups with a bound on the defects of their subnormal subgroups. We denote by $\mathfrak{B}_n$ the class of groups in which every subnormal subgroup has defect at most n, and put $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$.

It is well known that soluble groups in $\mathfrak{B}_1$ (more often called T-groups) are metabelian [3], [9]; while there is no bound on the derived length of soluble groups in $\mathfrak{B}_2$, as shown by Robinson in [11]. However in [2] we proved that periodic soluble groups in $\mathfrak{B}_2$ have bounded derived length and that a soluble p-group in $\mathfrak{B}_2$ has derived length at most 4. Here we prove:

THEOREM 1. *For every prime p and every positive integer d, there exists a soluble p-group in $\mathfrak{B}_4$ with derived length exactly d.*

We leave open the question whether soluble p-groups in $\mathfrak{B}_3$ have bounded derived length.

Soluble p-groups in the class $\mathfrak{B}$ are the object of a paper by McDougall [8]. He denotes with $\boldsymbol{Q}$ the class of periodic abelian divisible groups, and with $\boldsymbol{PQ}$ and $\acute{\boldsymbol{P}}\boldsymbol{Q}$ the classes of groups admitting respectively a finite or an ascending series whose factors are $\boldsymbol{Q}$-groups. As a consequence of Theorem 1, we give an answer to a question posed by McDougall in his paper.

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THEOREM 2. *For every prime p, there exists a non soluble p-group which is a $\acute{P}Q$-group (hence hyperabelian) and has a bound on the defects of its subnormal subgroups.*

We will show that the groups constructed in order to prove Theorem 2 actually are in the class $\mathfrak{B}_5$.

Theorem 1 yields also a similar result for SD-groups, that is groups admitting a descending series with abelian factors.

THEOREM 3. *For every prime p, there exists a locally finite p-group in $\mathfrak{B}_4$, which is a SD-group and whose Baer radical is trivial.*

We wish to observe here that Leinen has recently shown in his doctoral dissertation [6], that locally finite p-groups in $\mathfrak{B}_1$ need not to be soluble, thus answering a question which had been open for a certain time (see Robinson [13], Vol. I, pag. 174).

Finite soluble groups in $\mathfrak{B}_2$ have derived length at most 5 ([1], [7]). By contrast Hawkes [5] proves that every finite soluble group is isomorphic to a subgroup of a finite soluble group in $\mathfrak{B}_3$. The groups in $\mathfrak{B}_3$ obtained by Hawkes's procedure have, at least for one prime p, Sylow p-subgroups of «high» derived length; therefore T. Hawkes asked whether there are A-groups in $\mathfrak{B}_n$ of arbitrary derived length, where an A-group is a finite group all of whose Sylow subgroups are abelian. Using the same technique used to prove Theorem 1, we give an answer to this question for $n = 4$.

THEOREM 4. *For every positive integer d, there exists a soluble A-group in $\mathfrak{B}_4$, with derived length exactly d.*

Throughout the paper, $H \operatorname{wr} K$ will always denote the restricted standard wreath product of the group H by the group K, and $\gamma_n(G)$ will denote the n-th term of the lower central series of the group G.

2. The proofs of Theorems 1 and 4 are similar and based upon a construction for which is basic the following extension of a lemma of P. Hall's.

LEMMA 1. *Let H and K be groups, and B denote the base group of $H \operatorname{wr} K$. If $S \leqslant H \operatorname{wr} K$ is subnormal of defect n in BS and the exponent* (1) *of BS/B is greater than n, then $S \geqslant \gamma_{n+1}(B)$.*

(1) By «exponent» of group G, we mean the supremum among the orders of the elements of G.

PROOF. We consider $H \operatorname{wr} K$ as the semidirect product $[B]K$, and proceed by induction on n.

Let $S \lhd BS$ (P. Hall [4], Lemma 4); there exists $1 \neq x \in K$ such that $xB \in SB$, and let $b \in B$ such that $bx \in S$. Let H_z be the coordinate subgroup of B related to $z \in K$, and let $h, g \in H_z$. We have $[h, bx] \in S$ and therefore:

$$S \ni [h, bx, g] = [h^{-1}h^{bx}, g] = [h^{-1}, g]^{h^{bx}}[h^{bx}, g] = [h^{-1}, g]$$

since $h^{bx} \in H_z^{bx} = H_{zx}$ and $[H_{zx}, H_z] = 1$ as $x \neq 1$. Hence $S \geqslant H_z'$. The same is true for any $z \in K$, therefore $S \geqslant B' = \gamma_2(B)$.

Assume now $n > 1$, and let $y \in S$ such that $(yB)^i = y^iB \neq B$ for $i = 1, \ldots, n$, and write $y = bx$ with $b \in B$, $x \in K$ and $x^i \neq 1$ for $i = 1, \ldots, n$. Let $S \lhd S_1 \lhd \lhd BS$, with S_1 of defect $n-1$ in $BS_1 = BS$. Since $\exp(S_1B/B) > n$, we have, by inductive hypothesis, $S_1 \geqslant \gamma_n(B)$. Let H_z be the coordinate subgroup of B corresponding to $z \in K$, and let $f \in H_z$ and $g \in \gamma_n(H_z) \leqslant S_1$. Since $x^i \neq 1$ for $i = 1, \ldots, n$, the groups H_{zx^j} and H_{zx^k} permute if $1 \leqslant j$, $k \leqslant n$ and $j \neq k$. Therefore we have:

$$[f, {}_nx] = f^{(x-1)^n} = f^{(-1)^n}t\,, \qquad \text{where } t \in H_{zx} \times \ldots \times H_{zx^n}$$

and, by induction on n, it is easy to check that:

$$[f, {}_ny] = [f, {}_nbx] = f^{(-1)^n}u\,, \qquad \text{where } u \in H_{zx} \ldots H_{zx^n}\,.$$

Since $g \in S_1$, $S \lhd S_1$ and S has defect n in BS, we now have:

$$S \ni [g, [f, {}_ny]] = [g, f^{(-1)^n}u] = [g, f^{(-1)^n}]^u[g, u] = [g, f^{(-1)^n}]$$

as H_z commutes with $H_{zx} \ldots H_{zx^n}$.

Therefore $S \geqslant [H_z, \gamma_n(H_z)] = \gamma_{n+1}(H_z)$. The same is true for any $z \in K$, hence $S \geqslant \gamma_{n+1}(B)$. □

REMARK. The hypothesis $\exp(SB/B) > n$ in Lemma 1 cannot be relaxed to $|SB/B| > n$. Let, in fact, H be the symmetric group on three objects and K be an elementary abelian group of order 4. It is easy to check that $W = H \operatorname{wr} K$ has a subnormal subgroup S of defect 3, such that $SB = W$ but S does not contain B', where B is again the base group of W.

3. In his paper on soluble p-groups in the class $\mathfrak{B}$ [8], McDougall proves that every soluble p-group in $\mathfrak{B}$ is the extention of a **P*Q***-group (that is a group admitting a finite series whose factors are periodic abelian divisible groups) by a nilpotent group. Since, by a Theorem of Roseblade's [14], nilpotent groups in $\mathfrak{B}_n$ have class bounded by a function of n, in order to prove Theorem 1 we need to construct ***PQ***-groups in $\mathfrak{B}_4$ with arbitrary derived length.

For the properties of ***PQ***-groups that we will need, we refer to the quoted paper by McDougall.

LEMMA 2. *Let H be a p-group in PQ, K a periodic group, B the base group of H wr K, and let S be a subnormal subgroup of BS. Then either $S \leqslant B$ or $S \geqslant B'$.*

PROOF. Assume $S \nleqslant B$. We prove that $S \geqslant B'$ by induction on the defect n of S in BS.

In $n = 1$, then the result is a particular case of Lemma 1. Assume $n > 1$ and let $S \lhd S_1 \lhd \lhd SB$ with S_1 of defect $n-1$ in SB. Clearly $S_1 \nleqslant B$, hence, by inductive hypothesis, $S_1 \geqslant B'$. It is clear that B is a ***PQ***-group and, therefore, B/B' is a normal periodic abelian divisible subgroup of SB/B'; since K and, in particular, SB/B' is a periodic group, we have that B/B' normalizes $S_1B'/B' = S_1/B'$ (see Robinson [10], Lemma 2.2). Hence S_1 is normal in SB and so $n = 2$. If SB/B has exponent greater than 2, then by Lemma 1, $S \geqslant \gamma_3(B)$ and, since B is a ***PQ***-group, we have $\gamma_3(B) = B'$ (McDougall [8], Th. 4.8), yielding $S \geqslant B'$. Assume now $\exp(SB/B) = 2$. Let $y \in S \setminus B$, then $y = bx$ with $b \in B$, $x \in K$ and $x^2 = 1$. Let H_z $(z \in K)$ be a coordinate subgroup in B, and $t \in H_z$; since S has defect 2 in SB, we have $[t, {}_2bx] \in S$ and, by expanding the commutaror using the facts that $[H_z, H_{zx}] = 1$ and $x^2 = 1$, we get:

$$S \ni [t^{-1} b^{bx}, bx] = t(t^{-2})^{bx} t^{bb^x} .$$

Let now $f, g, h \in H_z$ and $w = [h, g]$, then $w \in S_1$. Since H/H' is divisible, there exist $t \in H_{zx}$, $u \in H'_{zx}$ such that $ut^2 = h^{xb^{-1}}$. Since $[t, {}_2bx] \in S$, we have:

$$S \ni [w, [t, {}_2bx]] = [w, tt^{bb^x}(t^{-2})^{bx}] = [w, (t^{-2})^{bx}]$$

as $H_{zx^2} = H_z$ commutes with H_{zx}. Hence:

$$(+) \qquad S \ni [w, h^{-1}u^{bx}] = [w, u^{bx}][w, h^{-1}]^{u^{bx}} .$$

Since $u \in B' \leqslant S_1 \leqslant \mathcal{N}(S)$, we have also:

$$S \ni [w, [u, bx]] = [w, u^{-1}u^{bx}] = [w, u^{bx}]$$

and, by comparing with $(+)$, we get: $[w, h^{-1}]^{u^{bx}} \in S$, yielding

$$[w, h^{-1}] = [f, g, h^{-1}] \in S .$$

Hence $S \geqslant \gamma_3(H_z) = H_z'$, and so $S \geqslant B'$. □

Let G be a group; we denote by $D(G) = D_1(G)$ the maximal normal divisible periodic abelian subgroup of G (see e.g. Robinson [10], Lemma 2.2) and we put: $D_n(G)/D_{n-1}(G) = D(G/D_{n-1}(G))$ for any $n > 1$. If $G \in \boldsymbol{PQ}$ then there is a minimal integer n such that $D_n(G) = G$. Such n is called the $\boldsymbol{Q}$-length of G, and it is equal to the derived length of G, according to a result of McDougall ([8], Th. 4.5).

Proof of Theorem 1. By induction on n, we construct a p-group in $\boldsymbol{PQ} \cap \mathfrak{B}_4$ with derived length $2n$ (to get a group of the required type and odd derived length we can take the derived subgroup of one of the groups we will construct). More precisely, we obtain p-groups satisfying the following conditions:

(i) G is a $\boldsymbol{PQ}$-group;

(ii) $C_G(D(G)) = D(G)$;

(iii) if S is subnormal in G, then either $S \leqslant D_2(G)$ or $S \geqslant D_1(G)$;

(iv) G is in $\mathfrak{B}_4$.

Let Q be a divisible abelian p-group and $C \simeq C_{p^\infty}$. Let V be the base group of Q wr C, considering the latter as the semidirect product $[V]C$, let $H = [V, C]C$. Then H satisfies (i)-(iv) and it has derived length 2. Moreover $H' = [V, C] = D_1(H) = C_H(D_1(H))$ (this follows easily from Lemma 2.2 in [10]).

Assume now that we have already constructed a group G_0 which satisfies (i)-(iv) and has derived length $2(n-1)$. Let B be the base group of $W = H \operatorname{wr} G_0$, where again we consider W as the semidirect product $[B]G_0$. Let $D_1 = D_1(G_0) \lhd G_0$. The action of G_0 on B induces an action of G_0 on $\bar{B} = B/B'$; with respect to this action, let $\bar{R} = R/B' = [\bar{B}, D_1]$. Since $D_1 \lhd G_0$, we have $R \lhd W$. We put

$G = RG_0$ and verify that G satisfies (i)-(iv) and has derived length ($=$ $\boldsymbol{Q}$-length) $2n$.

(i) Since H is a metabelian $\boldsymbol{PQ}$-group, such is B; therefore $\bar{B}$ is divisible and so $\bar{R}$ is divisible. Hence R is a $\boldsymbol{PQ}$-group of derived length 2. Since the class $\boldsymbol{PQ}$ is closed under extensions, we get $G \in \boldsymbol{PQ}$.

(ii) Let $K = D_1(G)$; then $K \cap R = D_1(R)$ and, by the structure of H, we have $B' = D_1(B)$, in particular $B' = D_1(R) = K \cap R$. It is $C_G(B') = G \cap C_W(B')$ and, since K is abelian, $K \leqslant C_G(B')$. Let $C = C_W(B')$, then $C \lhd W$ and so

$$[C, B] \leqslant C \cap B = B' , \qquad \text{since } C_H(H') = H' .$$

But $W/B' \simeq (H/H') \operatorname{wr} G_0$ and, since C centralizes B/B', we have $C \leqslant B$, that is $C = B'$. In particular $C_G(B') = B'$, yielding $K = B'$ and $C_G(K) = K$, thus proving (ii).

We observe now that $D_2(G) = R$. Clearly, in fact, $D_2(G) \geqslant R$. Assume $D_2(G) \neq R$. Then $(D_2(G) \cap G_0)B'/B'$ is infinite, and therefore, its centralizer in the base group B/B' of the restricted wreath product W/B' is trivial (see Robinson [13], Lemma 6.28), contradicting the fact that $D_2(G)$ centralizes $R/B' \leqslant B/B'$. Hence $D_2(G) = R$. This proves that the $\boldsymbol{Q}$-length, and so the derived length, of G is is equal to the $\boldsymbol{Q}$-length of G_0 plus two, that is $2(n-1) + 2 = 2n$.

(iii) Let S be a subnormal subgroup of G and assume that $S \nleqslant D_2(G) = R$. We use «$\sim$» to denote subgroups modulo R. We have $\tilde{S} = SR/R \neq 1$ and $\tilde{S} \lhd\lhd \tilde{G}$, hence $\tilde{S}$ is normalized by $\widetilde{D_1(G_0)}$. If $\tilde{S} \cap \widetilde{D_1(G_0)} = 1$, then $\tilde{S}$ centralizes $\widetilde{D_1(G_0)}$ and, since G_0 satisfies (ii), we get $\tilde{S} = 1$, contradicting $S \nleqslant R$. Hence $L/R = \tilde{S} \cap \widetilde{D_1(G_0)} \neq 1$; let $T = L \cap S$, then $T \lhd\lhd G$ and $TR = (L \cap S)R = L \cap SR = L$, and so $T \nleqslant R$. It is therefore sufficient to prove that $T \geqslant D_1(G) = B'$. Looking at W, we have: $T \lhd\lhd L \trianglelefteq RD_1(G_0) \trianglelefteq BD_1(G_0) \trianglelefteq W$, yielding $T \lhd\lhd W$ and $T \nleqslant B$, since $T \nleqslant R$. Now, by applying Lemma 2, we get $T \geqslant B' = D_1(G)$, as we wanted.

(iv) Let S be a subnormal subgroup of G.

Now, R is in $\mathfrak{B}_2$, since it is a $\boldsymbol{PQ}$-group of length 2 (McDougall [8], Th. 4.7); hence if $S \leqslant D_2(G) = R$, then S has defect at most 2 in R and, since R is normal in G, it has defect at most 3 in G. Otherwise $S \geqslant D_1(G) = B'$. Since R/B' is a normal abelian divisible subgroup of G/B', it normalizes every subnormal subgroup, in particular: $S/B' \trianglelefteq SR/B'$ and so $S \triangleleft SR \triangleleft\triangleleft G$. Since G satisfies (iii) we have two cases:

a) $SR \geqslant D_1(G_0)R$, in this case, working modulo B', we get:

$$\bar{R} = [\overline{D_1(G_0)R}, \bar{R}] \leqslant [\overline{SR}, \bar{R}] = [\bar{S}, \bar{R}],$$

and, since $\bar{S}$ is subnormal in $\bar{G}$, $\bar{S} \geqslant \bar{R}$, yielding $S \geqslant R$ as $S \geqslant B'$. It follows from the property (iv) of G_0, that S has defect at most 4 in G.

b) $SR \leqslant D_2(G_0)R'$, then we have $S \trianglelefteq SR \triangleleft\triangleleft D_2(G_0)R \trianglelefteq G$, with SR of defect at most 2 in $D_2(G_0)R$; hence S has defect at most 4 in G.

This completes the proof of Theorem 1. □

REMARK. In the notation used in the previous proof, $G > B'G_0 \simeq H' \operatorname{wr} G_0$ and H' is an arbitrary divisible abelian p-group; it is therefore easy to verify that Theorem 1 can be restated in the following way: every finite p-group is isomorphic to a subgroup of a p-group in $\mathfrak{B}_4$, which is a $\boldsymbol{PQ}$-group (in particular it is soluble).

4. Before giving the proofs of Theorems 2 and 3, we state in a lemma a well known fact concerning the classes $\mathfrak{B}_n$ (see e.g. Robinson [12], Ex. 1, pag. 127).

LEMMA 3. *For each positive integer n, $\mathfrak{B}_n$ is a local class.*

That is, $G \in \mathfrak{B}_n$ if every finite subset of G is contained in a subgroup of G, which is a $\mathfrak{B}_n$-group.

Throughout this section G_n will denote the group constructed as in the proof of Theorem 1, of derived length $2n$, for any fixed prime p.

PROOF OF THEOREM 2. For every prime p, we may take the group $G = \operatorname{Dir}_{n \in \mathbf{N}} G_n$, and apply the following result:

PROPOSITION. *If, for a fixed n, $H_\lambda \in \mathfrak{B}_n$ for every $\lambda \in \Lambda$, then $\operatorname{Dir}_{\lambda \in \Lambda} H \in \mathfrak{B}_{\sigma(n)}$, where $\sigma(n)$ is a function depending only on n.*

PROOF. Let S be a subnormal subgroup of $H = \operatorname{Dir}_{\lambda \in \Lambda} H$, of defect say d, and let Π_λ be the projection on H_λ for each $\lambda \in \Lambda$. Then: $\gamma_{d+1}(S\Pi_\lambda) \leqslant [H_\lambda, {}_d S\Pi_\lambda] = [H_\lambda, {}_d S] \leqslant S$, hence $S\Pi_\lambda/(H_\lambda \cap S)$ is nilpotent, for every $\lambda \in \Lambda$. Now $S\Pi_\lambda$ is subnormal in H_λ, which is a $\mathfrak{B}_n$-group; therefore all the subgroups of $S\Pi_\lambda/(H_\lambda \cap S)$ have defect at most n. By Roseblade's Theorem [14], there exists a function $\eta(n)$ such that the class of $S\Pi_\lambda/(H_\lambda \cap S)$ is at most $\eta(n)$, for every $\lambda \in \Lambda$. Let

$$K = \langle S\Pi_\lambda \colon \lambda \in \Lambda \rangle \quad \text{and} \quad T = \langle S \cap H_\lambda \colon \lambda \in \Lambda \rangle$$

Then $T \leqslant S \leqslant K$ and K is subnormal in H of defect at most n. But K/T is nilpotent of class at most $\eta(n)$, hence S has defect at most $\eta(n)$ in K, and defect at most $\eta(n) + n$ in H. Therefore $H \in \mathfrak{B}_{\sigma(n)}$, where $\sigma(n) = \eta(n) + n$. □

(For this property, see also Smith [15], Th. 4.16.)

We now show that $G = \operatorname{Dir}_{n \in \mathbb{N}} G_n$ actually is in $\mathfrak{B}_5$. We first observe that the groups G_n, besides of (i)-(iv) in the proof of Theorem 1, satisfy the following property:

$$(+) \qquad \text{if } S \lhd \lhd G_n \quad \text{and} \quad S \nleqslant D_4(G_n)\,, \qquad \text{then } S \geqslant D_2(G_n)\,;$$

in fact, (by repeating an argomentation used in the proof of Theorem 1) if $S \nleqslant D_4(G_n)$ then $SD_2(G_n) \geqslant D_3(G_n)$ by (iii), since $G_n/D_2(G_n) \simeq \simeq G_{n-1}$. Hence

$$D_2(G_n) = [D_3(G_n), D_2(G_n)] \leqslant [SD_2(G_n), D_2(G_n)] \leqslant [S, D_2(G_n)]$$

since $S \geqslant D_1(G_n)$ by (iii). Then, S subnormal yields $S \geqslant D_2(G_n)$.

Since $\mathfrak{B}_5$ is a local property, in order to show that G is in $\mathfrak{B}_5$, we need only to show that, for a finite number of integers $n_1, \ldots, n_r$, the group $\operatorname{Dir}_{i=1,\ldots,r} G_{n_i}$ is in $\mathfrak{B}_5$. We make induction on Σn_i. The case $\Sigma n_i = 1$ is trivial, hence let $\Sigma n_i > 1$ and $H = \operatorname{Dir}_{i=1,\ldots,r} G_{n_i}$, and let T be a subnormal subgroup of H. If $T \leqslant D_4(H)$ then, by McDougall ([8], Th. 4.7), T has defect at most 4 in $D_4(H)$ and so it has defect at most 5 in H. Otherwise, for some index $i \in \{1, \ldots, r\}$ which we may assume to be 1, the projection $T\Pi_1$ of T in G_{n_1} is not contained in $D_4(G_{n_1})$, hence by (+), $T\Pi_1 \geqslant D_2(G_{n_1})$. Now, for some integer d:

$$T \geqslant [G_{n_1}, {}_dT] = [G_{n_1}, {}_dT\Pi_1] \geqslant [G_{n_1}, {}_dD_2(G_{n_1})] = D_2(G_{n_1})$$

(observe that, since $T\Pi_1$ is not contained in $D_4(G_{n_1})$, it is $n_1 > 2$, and so $[G_{n_1}, D_2(G_{n_1})] = D_2(G_{n_1})$). Hence $T \geqslant K = D_2(G_{n_1}) \trianglelefteq H$. Now $H/K \cong G_{n_1}/K \times \ldots \times G_{n_r} \cong G_{n_1-1} \times G_{n_2} \times \ldots \times G_{n_r}$, and by inductive hypothesis, we have that $T = TK$ has defect at most 5 in H, concluding the proof that H (and therefore G) is in $\mathfrak{B}_5$.

PROOF OF THEOREM 3. Let H be the group of PQ-length 2, as defined in the first part of the proof of Theorem 1. Then G_{n+1} can be viewed as a subgroup of $H \operatorname{wr} G_n$, containing a subgroup isomorphic to G_n. We have therefore the sequence of inclusions:

$$G_1 \to G_2 \to G_3 \to \ldots .$$

Let G be the direct limit of it. Then $G = \bigcup_{n \in \mathbf{N}} \bar{G}_n$, where $\bar{G}_n \cong G_n$ and $\bar{G}_n \leqslant \bar{G}_{n+1}$ for every $n \in \mathbf{N}$. Clearly G is a locally finite p-group and, since $\mathfrak{B}_4$ is a local class, G is in $\mathfrak{B}_4$. It is also easy to check that G is a SD-group.

We have only to show now that the Baer radical of G is trivial. Let $\langle x \rangle$ be a cyclic subnormal subgroup of G; then $x \in \bar{G}_k$ for some $k \in \mathbf{N}$. Now $\bar{G}_{k+1} = R\bar{G}_k$ (where R is the same as in the proof of Theorem 1), with $R \cap \bar{G}_k = 1$, and the subnormal subgroups of $\bar{G}_{k+1}$ which are not contained in R contain $D_1(\bar{G}_{k+1})$. In particular $\langle x \rangle$ cannot contain $D_1(\bar{G}_{k+1})$, therefore $x \in R \cap \bar{G}_k = 1$, proving that the Baer radical of G is trivial. □

5. Since the proof of Theorem 4 is similar to the proof of Theorem 1, we give only a sketch of it, leaving the details to the reader. We first need an analogous of Lemma 2 for finite groups.

LEMMA 4. *Let H and K be finite groups, with $\gamma_3(H) = H'$ and $(|H/H'|, |K|) = 1$, and let B be the base group of $H \operatorname{wr} K$. If $S \leqslant H \operatorname{wr} K$ is subnormal in SB, then either $S \leqslant B$ or $S \geqslant B'$.*

PROOF. Omitted. □

PROOF OF THEOREM 4. For a given finite group G let $F_i(G)$ be the i-th term of the Fitting series of G.

By induction on n, one constructs a finite soluble group, with derived length $2n$, satisfying the following conditions:

(i*) $F_i(G)/F_{i-1}(G)$ is an elementary abelian p_i-group, where the p_i's are given primes with $p_i \neq p_{i-1}$ for $i = 1, 2, \ldots, 2n$. In particular if the p_i's are all distinct, G is an A-group;

(ii*) if S is a subnormal subgroup of G, then either $S \leqslant F_2(G)$ or $S \geqslant F_1(G)$;

(iii*) G is in $\mathfrak{B}_4$.

Let $p_1, \ldots, p_{2n}$ be given primes with $p_i \neq p_{i-1}$ for $i = 2, \ldots, 2n$. Suppose we have already constructed a group G_0 of derived length $2(n-1)$, satisfying (i*)-(iii*) with involved primes $p_3, \ldots, p_{2n}$. Let H be a group such that $F_1(H) = H' = \gamma_3(H)$ is an elementary abelian p_1-group and H/H' is an elementary abelian p_2-group. Let B be the base group of $H \operatorname{wr} G_0$: we take $G = B'[B, F_1(G_0)]\, G_0$ where G_0 is viewed in a natural way as a subgroup of $H \operatorname{wr} G_0$. Then G satisfies (i*)-(iii*) and has derived length $2n$. □

Remark. With a similar construction it is possible to embed any finite soluble group of derived length n in a finite soluble group in $\mathfrak{B}_4$, of derived length $2n$.

REFERENCES

[1] C. Casolo, *Gruppi finiti risolubili in cui tutti i sottogruppi subnormali hanno difetto al più* **2**, Rend. Sem. Mat. Univ. Padova, **74** (1984), pp. 257-271.

[2] C. Casolo, *Periodic soluble groups in which every subnormal subgroup has defect at most two*, Arch. Math., **46** (1986), pp. 1-7.

[3] W. Gaschütz: *Gruppen in denen das Normalteilersein transitiv ist*, J. Reine Angew. Math., **198** (1957), pp. 87-92.

[4] P. Hall, *Wreath powers and characteristically simple groups*, Proc. Cambridge Phil. Soc., **58** (1962), pp. 170-184.

[5] T. O. Hawkes, *Groups whose subnormal subgroups have bounded defect*, Arch. Math., **43** (1984), pp. 289-294.

[6] F. Leinen, *Existenziell abgeschlossene $L\chi$-Gruppen*, Dissertation, Albert-Ludwigs Univ., Friburg i.Br., 1984.

[7] D. J. McCaughan - S. E. Stonehewer, *Finite soluble groups whose subnormal subgroups have defect at most two*, Arch. Math., **35** (1980), pp. 56-60.

[8] D. McDougall, *The subnormal structure of some classes of soluble groups*, J. Austral. Math. Soc., **13** (1972), pp. 365-377.

[9] D. J. S. Robinson, *On groups in which normality is a transitive relation*, Proc. Cambridge Phil. Soc., **60** (1964), pp. 21-38.

[10] D. J. S. Robinson, *On the theory of subnormal subgroups*, Math. Zeit., **89** (1965), pp. 30-51.

[11] D. J. S. Robinson, *Wreath products and indices of subnormality*, Proc. London Math. Soc., (3) **17** (1967), pp. 257-270.

[12] D. J. S. Robinson, *Infinite soluble and nilpotent groups*, London, Q.M.C. Math. Notes (1968).

[13] D. J. S. Robinson, *Finiteness conditions and generalised soluble groups*, Springer, Berlin-Heidelberg-New York, 1972.

[14] J. E. Roseblade, *On groups in which every subgroup is subnormal*, J. Algebra, **2** (1965), pp. 402-412.

[15] H. Smith, *Groups with the subnormal join property*, Can. J. Math., **37** (1985), pp. 1-16.

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