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The Dirichlet Problem in Half-Space for Elliptic Equations with Unbounded Coefficients.

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Introduction.

Let $R_n^+ = \{x; x \in R_n, x_n > 0\}$, for a point $x \in R_n^+$ we write $x = (x', x_n)$, where $x' \in R_{n-1}$. In this paper we consider the Dirichlet problem

$$(1) \qquad Lu + \lambda u = \sum_{i,j=1}^{n} D_i(a_{ij}(x) D_j u) + \sum_{i=1}^{n} b_i(x) D_i u + (c(x) + \lambda) u = f(x) \quad \text{in } R_n^+,$$

$$(2) \qquad u(x', 0) = \varphi(x'),$$

where $\varphi \in L^2_\Psi(R_{n-1})$ (for the definition of $L^2_\Psi(R_{n-1})$ see Section 2).

In recent years the Dirichlet problem with L^2-boundary data has been studied by several authors (see [2], [3], [5], [6], [8], [9] and the references given there). In particular, the author has investigated this problem in R_n^+(see [3]) and established the existence theorem in the case $L^2_\Psi(_{n-1}) = L^2(R_{n-1})$. The aim of this work is to weaken the assumptions from [3] on the coefficients b_i and c. More specifically, in [3] it is assumed that

$$b_i \in L^\infty(R_n^+) \cap L^n(R_n^+) \quad \text{and} \quad c \in L^n(R_n^+) + L^\infty(R_n^+)$$

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$(i = 1, \ldots, n)$. Here we assume that for every $\delta > 0$

$$b_i \in L^n(R_{n-1} \times (\delta, \infty)) \cap L^\infty(R_{n-1} \times (\delta, \infty)) \quad (i = 1, \ldots, n)$$

and

$$c \in L^n(R_{n-1} \times (\delta, \infty)) + L^\infty(R_{n-1} \times (\delta, \infty))$$

$(i = 1, \ldots, n)$ and moreover we allow b_i and c grow to ∞ in certain way as $x_n \to 0$, namely

$$b_i(x) = 0\,(x_n^{-\beta_1}) \quad (i = 1, \ldots, n) \qquad \text{and} \qquad c(x) = 0\,(x_n^{-\beta_2})$$

for x_n close to 0, where $0 < \beta_1 < 1$ and $0 < \beta_2 < 2$.

The paper is organized as follows. We derive basic properties of traces of solutions in $W^{1,2}_{\text{loc}}(R_n^+)$ od (1) in Sections 1 anf 2. The results of Sections 1 and 2 justify the formulation of the Dirichlet problem adopted in this work. In Section 3 we examine solutions u_δ of (1), (2) in $R_{n-1} \times (\delta, \infty)$ with boundary condition $u_\delta(x', \delta) = 0$ on R_{n-1}. In particular we establish an energy estimate for u_δ, which is used to show that the limit of u_δ, as $\delta \to 0$, exists and solves the Dirichlet problem (1), (2) with $\varphi \equiv 0$ on R_{n-1}. By a standard argument we use this result to solve the problem (1), (2) with $\varphi \not\equiv 0$ and $\varphi \in L^2_\Psi(R_{n-1})$ (see Section 4). We note the that methods of proofs here are not new and have appeared in [3]. Bibliographic information not given in this paper is available in [2] and [3]. Finally we point out that similar problem in the case of the Dirichlet problem in bounded domains has been considered by the author in [4].

1. Assumptions and preliminaries.

Let Ψ be a positive function in $C^1(R_{n-1})$ such that

$$(\Psi) \qquad |D\Psi(x')| \leqslant N\Psi(x') \quad \text{in } R_{n-1}$$

for certain constant N.

We put

$$L^2_\Psi(R_n^+) = \left\{u;\ L^2_{\text{loc}}(R_n^+),\ \int_{R_n^+} u(x)^2 \Psi(x')^2\,dx < \infty\right\},$$

$$L^2_\Psi(R_{n-1}) = \left\{u;\ u \in L^2_{\text{loc}}(R_{n-1}),\ \int_{R_{n-1}} u(x')^2 \Psi(x')^2 dx' < \infty\right\},$$

and

$$W^{1,2}_\Psi(R_n^+) = \left\{u;\ u \in W^{1,2}_{\text{loc}}(R_n^+),\ \int_T^\infty \mathrm{d}x_n \int_{R_{n-1}} u(x)^2 \Psi(x')^2 \mathrm{d}x' < \infty \quad \text{for all } T > 0\right\},$$

where $W^{1,2}_{\text{loc}}(R_n^+)$ denotes a Sobolev space, i.e., $W^{1,2}_{\text{loc}}(R_n^+)$ is the space of all functions in $L^2_{\text{loc}}(R_n^+)$ with first order weak derivatives also in $L^2_{\text{loc}}(R_n^+)$.

Throughout this article we make the foollwing assumptions about the operator L:

(A) L is uniformly elliptic in R_n^+, i.e., there exists a positve constant γ such that

$$\gamma|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(x)\, \xi_i\, \xi_j$$

for all $x \in R_n^+$ and $\xi \in R_n$ and moreover $a_{ij} \in L^\infty(R_n^+)$ $(i, j, = 1, \ldots, n)$.

(B) (i) There exist positive constant K and $0 < \alpha < 1$ such that

$$|a_{nn}(x', x_n) - a_{nn}(x', \bar{x}_n)| \leqslant K|x_n - \bar{x}_n|^\alpha$$

for all $x' \in R_{n-1}$ and all $x_n, \bar{x}_n \in (0, \infty)$.

(ii) The coefficients a_{in} $(i = 1, \ldots, n-1)$ have partial derivatives $D_i\, a_{in}$ satisfying the inequality

$$|D_i\, a_{in}(x)| \leqslant K_1 x_n^{-\beta} \qquad (i = 1, \ldots, n-1)$$

for all $x' \in R_{n-1}$ and $x_n \in (0, b]$, where K_1, b and β are positive constants with $0 \leqslant \beta < 1$ and moreover

$$D_i\, a_{in} \in L^\infty(R_{n-1} \times [b, \infty)) \qquad (i = 1, \ldots, n-1)\,.$$

(iii) $b_i \in L^\infty(R_{n-1} \times (\delta, \infty))$ $(i = 1, \ldots, n)$ for all $\delta > 0$ and

$$|b_i(x)| \leqslant B x_n^{-\beta_1} \quad \text{in } R_{n-1} \times (0, b)$$

where B and β_1 are positive constants with $0 \leqslant \beta_1 < 1$.

$$\text{(iv)} \quad c \in L^\infty(R_{n-1} \times (\delta, \infty)) + L^n(R_{n-1} \times (\delta, \infty)) \quad \text{for all } \delta > 0$$

and

$$|c(x)| \leqslant B x_n^{-\beta_2} \quad \text{in } R_{n-1} \times (0, b),$$

where C and β_2 are positive constants with $0 \leqslant \beta_2 < 2$.

$$(C) \qquad f \in L^2_{\text{loc}}(R_n^+) \text{ and } \int_{R_n^+} f(x)^2 \Psi(x')^2 [\min(1, x_n)]^\Theta \, dx < \infty,$$

where $2 \leqslant \Theta < 3$.

In the sequel we need the following elementary lemmas

LEMMA 1. *If* $u \in L^2_\Psi(R_n^+)$ *and*

$$\sup_{0 < x_n \leqslant T} \int_{R_{n-1}} u(x', x_n)^2 \Psi(x')^2 dx' < \infty$$

for certain $T > 0$, *then*

$$\int_{R_n^+} u(x)^2 \Psi(x')^2 [\min(1, x_n)]^{-\mu} dx < \infty$$

for every $0 \leqslant \mu < 1$.

LEMMA 2. *Let* $u \in W^{1,2}_\Psi(R_n^+)$ *and* $\int_{R_n^+} |D_n u(x)|^2 \min(1, x_n) \Psi(x')^2 dx < \infty$, *then for every* T *and* $0 \leqslant \mu < 1$

$$\int_\delta^T dx_n \int_{R_{n-1}} u(x)^2 \Psi(x')^2 (x_n - \delta)^{-\mu} dx' \leqslant \frac{2(T-\delta)^{1-\mu}}{1-\mu} \int_{R_{n-1}} u(x', T)^2 \Psi(x')^2 dx' +$$

$$+ \frac{4T^{1-\mu}}{(1-\mu)^2} \int_\delta^T dx_n \int_{R_{n-1}} |D_n u(x)|^2 (x_n - \delta) \Psi(x')^2 dx \qquad \textit{for all } \delta \in \left[0, \frac{T}{2}\right].$$

The proof is identical to that of Lemma 2 in [3].

In this paper we use the notion of a weak solution of (1) involving the Sobolev spaces $W^{\text{loc}}_{1,2}(R_n^+)$ and $W^{1,2}(R_n^+)$, A function $u \in W^{1,2}_{\text{loc}}(R_n^+)$ is said to be a weak solution of (1), if it satisfies

$$(3)\qquad \int_{R_n^+}\Big[\sum_{1,j=1}^{u} a_{ij}(x)\,D_i u\, D_j v + \sum_{i=1}^{n} b_i(x)\,D_i u\cdot v + \big(c(x)+\lambda\big)\,uv\Big]\,dx = \int_{R_n^+} f(x)\,v\,dx$$

for every $v \in W^{1,2}(R_n^+)$ with compact support in R_n^+.

Lemma 3. *Let $u \in W^{1,2}_{\Psi}(R_n^+)$ be a weak solution of(1) in R_n^+. Then for every $r > 0$*

$$(4)\qquad \int_{2r}^{\infty}\int_{R_{n-1}} |Du(x)|^2\Psi(x')^2\,dx \leqslant M\Big[(|\lambda|+1)\int_{r}^{\infty}\int_{R_{n-1}} u(x)^2\Psi(x')^2\,dx + \int_{r}^{\infty}\int_{R_{n-1}} f(x)^2\Psi(x')^2\,dx\Big],$$

where a positive constant M depends on the norms of the coefficients in $R_{n-1}\times[r,\infty)$, γ and n.

Proof. Let $v = u\,\Phi^2$, where $\Phi \in C_0^1(R_n^+)$. Using v as a test function in (2) we obtain

$$\int_{R_n^+}\sum_{i,j=1}^{n} a_{ij} D_i u\, D_j u\,\Phi^2 dx + 2\int_{R_n^+}\sum_{i,j=1}^{n} a_{ij}\, D_i u\cdot u D_j\Phi\cdot\Phi\,dx + \int_{R_n^+}\sum_{i=1}^{n} b_i\, D_i u\cdot u\Phi^2\,dx + \int_{R_n^+}(c+\lambda)\,u^2\,\Phi^2\,dx = \int_{R_n^+} f u^2\,\Phi^2\,dx\,.$$

It follows from the ellipticity of L and the inequalities of Young and Sobolev that

$$\int_{R_n^+}|Du|^2\,\Phi^2\,dx \leqslant M_1\int_{R_n^+} u^2(\Phi^2+|D\Phi|^2)\,dx + |\lambda|\int_{R_n^+} u^2\,\Phi^2\,dx + \int_{R_n^+} f^2\,\Phi^2\,dx\,,$$

where a positive constant M_1 depends on the norms of the coefficients on supp Φ, n and γ. Here we have used the fact that $c = c_1 + c_2$, with $c_1 \in L^n(\operatorname{supp} \Phi)$ and $c_2 \in L^\infty(\operatorname{supp} \Phi)$ and applied the Sobolev inequality to the term $\int_{R_n^+} c_1 u^2 \Phi^2 \, dx$. To complete the proof we put $\Phi(x) = \Psi(x') \Phi_\nu(x)$, where $\{\Phi_\nu\}$ is an increasing sequence of non-negative functions in $C_0^1(R_n^+)$ with the gradient bounded independently of ν and converging to a non-negative function R_n^+ equal to 1 for $x_n \geqslant 2r$ and vanishing for $x_n < r$.

2. Traces of solutions in $W_\Psi^{1,2}(R_n^+)$.

All constants in the following theorems will be denoted by C_i. The statement « C_i depends on the structure of the operator L » means that C_i depends on $n, \gamma, \beta, \beta_1, \beta_2, \alpha, B, C, b, K, K_1$ and Θ and the norms of the coefficients in the appropriate spaces.

THEOREM 1. *Let* $u \in W_\Psi^{1,2}(R_n^+)$ *be a weak solution of* (1) *in* R_n^+. *Then the following conditions are equivalent*:

(I) *there exists* $T > 0$ *such that*

$$\sup_{0<x_n<T} \int_{R_{n-1}} u(x', x_n)^2 \Psi(x')^2 \, dx' < \infty ,$$

(II)
$$\int_{R_n^+} |Du(x)|^2 \Psi(x')^2 \min(1, x_n) \, dx < \infty .$$

PROOF. The proof is similar to that of Theorem 1 in [3] and therefore we only give an outline.

Let $0 < 3\delta_0 < 1$. We may assume that $3\delta_0 \leqslant b$. We define a non-negative function $\eta \in C^2([0, \infty))$ such that $\eta(x_n) = x_n$ for $x_n \leqslant 2\delta_0$, $\eta(x_n) = 1$ for $x_n \geqslant 3\delta_0$ and $\eta(x_n) \geqslant \delta$ for all $x_n \geqslant \delta$ and $0 < \delta < 3\delta_0$. We may also assume that $\lambda = 0$.

Let

$$v(x) = \begin{cases} u(x)\big(\eta(x_n) - \delta\big)\Phi(x')^2 & \text{for } x_n > \delta, \\ 0 & \text{elsewhere,} \end{cases}$$

where Φ is a non-negative function in $C_0^1(R_{n-1})$. Since for every $\delta < x_n$,

$v(\cdot, x_n)$ has a compact support in R_{n-1}, it follows from Lemma 3 that v is an admissible test function. Applying the assumption (A) we obtain from (3)

$$(5)\qquad \gamma\int_\delta^\infty\int_{R_{n-1}}|Du|^2(\eta-\delta)\Phi^2\,dx\leqslant\int_\delta^\infty\int_{R_{n-1}}fu(\eta-\delta)\Phi^2\,dx-$$

$$-\int_\delta^\infty\int_{R_{n-1}}\sum_{i=1}^{n-1}a_{in}D_iu\cdot uD_n\eta\Phi^2\,dx-\int_\delta^\infty\int_{R_{n-1}}a_{nn}D_nu\cdot uD_n\eta\Phi^2\,dx-$$

$$-2\int_\delta^\infty\int_{R_{n-1}}\sum_{i,j=1}^{n}a_{ij}D_iu\cdot u(\eta-\delta)\Phi D_j\Phi\,dx-$$

$$-\int_\delta^\infty\int_{R_{n-1}}\sum_{i=1}^{n}b_iD_iu\cdot u(\eta-\delta)\Phi^2\,dx-\int_\delta^\infty\int_{R_{n-1}}cu^2(\eta-\delta)\Phi^2\,dx\,.$$

All integrals on the right side can be estimated in the same way as in [3] except the integrals involving coefficients c and b_i. We only derive an estimate of the last integral. Let us denote this integral by J.

To estimate J we use decomposition $c=c_1+c_2$ on $R_{n-1}\times[\delta_0,\infty)$, with $c_1\in L^\infty(R_{n-1}\times[\delta_0,\infty))$ and $c_2\in L^n(R_{n-1}\times[\delta_0,\infty))$ and by the assumption (Biv) we get

$$|J|\leqslant C\int_\delta^{\delta_0}\int_{R_{n-1}}u^2x_n^{-\beta_2}(\eta-\delta)\Phi^2\,dx+\int_\delta^\infty\int_{R_{n-1}}|c_1|u^2(\eta-\delta)\Phi^2\,dx+$$

$$+\int_\delta^\infty\int_{R_{n-1}}|c_2|u^2(\eta-\delta)\Phi^2\,dx\,.$$

To evaluate the last integral we set

$$\bar{c}_2(x)=\begin{cases}c_2(x) & \text{on } R_{n-1}\times[\delta_0,\infty),\\ 0 & \text{elsewhere,}\end{cases}$$

then $\bar{c}_2 \in L^n(R_n^+)$. By Hölder's inequality we have

$$\int_\delta^\infty \int_{R_{n-1}} |\bar{c}_2| u^2(\eta-\delta)\Phi^2 dx \leqslant \|\bar{c}_2\|_{L^\infty} \Big[\int_\delta^\infty \int_{R_{n-1}} u^2\Phi^2 dx\Big]^{1/2} \cdot$$
$$\cdot \Big\{\int_\delta^\infty \int_{R_{n-1}} [u(\eta-\delta)\Phi]^{2^*} dx\Big\}^{1/2^*},$$

where $1/2^* = 1/2 - 1/n$. Now by Sobolev's inequality

$$\Big\{\int_\delta^\infty \int_{R_{n-1}} |u(\eta-\delta)\Phi|^{2^*} dx\Big\}^{1/2^*} \leqslant S\Big[\int_\delta^\infty \int_{R_{n-1}} |Du|^2(\eta-\delta)\Phi^2 dx +$$
$$+ \int_\delta^\infty \int_{R_{n-1}} u^2|D\eta|^2\Phi^2 dx + \int_\delta^\infty \int_{R_{n-1}} u^2(\eta-\delta)\,|D\Phi|^2 dx\Big]^{1/2},$$

where S is a positive constant independent of δ. The last three inequalities yield that

$$|J| \leqslant \frac{\gamma}{4} \int_\delta^\infty \int_{R_{n-1}} |Du|^2(\eta-\delta)\Phi^2 dx + C_1\Big[\int_\delta^{\delta_0} \int_{R_{n-1}} u^2(\eta-x_n)x_n^{-\beta_1}\Phi^2 dx +$$
$$+ \int_\delta^\infty \int_{R_{n-1}} u^2\Phi^2 dx + \int_\delta^\infty \int_{R_{n-1}} u^2(\eta-\delta)\Phi^2 dx + \int_\delta^\infty \int_{R_{n-1}} u^2(\eta-\delta)\,|D\Phi|^2 dx\Big],$$

where C_1 is a positive constant dependent on n, γ, C, $\|D\eta\|_{L^\infty}$ and $\|c_2\|_{L^n}$. Consequently we derive from (5) the following inequality.

$$\text{(6)}\qquad \int_\delta^{\delta_0} \int_{R_{n-1}} |Du|^2(\eta-\delta)\Phi^2 dx \leqslant C_1\Big[\int_{R_{n-1}} u(x',\delta)^2\Phi^2 dx +$$
$$+ \int_\delta^\infty \int_{R_{n-1}} u^2|D\Phi|^2 dx + \int_\delta^\infty \int_{R_{n-1}} u^2\Phi^2 dx + \int_{\delta_0}^\infty \int_{R_{n-1}} |Du|^2\Phi^2 dx +$$
$$+ \int_\delta^{\delta_0} \int_{R_{n-1}} u^2 x_n^{-\beta_1}(\eta-\delta)\Phi^2 dx + \int_\delta^{\delta_0} \int_{R_{n-1}} u^2(x_n-\delta)^{2\alpha-1}\Phi^2 dx +$$

$$+\int_{\delta}^{\delta_0}\int_{R_{n-1}} u^2 x_n^{-\beta_2}(\eta-\delta)\,\Phi^2\,dx+\int_{\delta}^{\delta_0}\int_{R_{n-1}} u^2 x_n^{-}\;\Phi^2\,dx+$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}} u^2(\eta-\delta)^{2-\Theta}\Phi^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}} f^2(\eta-\delta)^{\Theta}\Phi^2\,dx\Bigg]$$

where a positive C_2 depends on the structure of the operator L. If the condition (I) holds, then by Lemma 1 for every $0\leqslant\mu<1$ the integral

$$\int_{\delta}^{\delta_0}\int_{R_{n-1}} u^2(x_n-\delta)^{-\mu}\,dx \text{ is bounded on } \left[0,\frac{\delta_0}{2}\right].$$

Now we put $\Phi=\Phi_\nu\Psi$, where Φ_ν is an increasing sequence of non-negative functions in $C_0^\infty(R_{n-1})$ converging to 1 as $\nu\to\infty$ with the gradient bounded independently of ν. Letting $\nu\to\infty$ in (11) it follows from Lemma 3 and the condition (Ψ) that

$$\int_{\delta}^{\delta_0}\int_{R_{n-1}} |Du|^2\,(x_n-\delta)\,\Psi^2\,dx\leqslant\tilde{C}_2\Bigg[\int_{R_{n-1}} u(x',\delta)^2\,\Psi(x'^2)\,dx+$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}} u^2\Psi^2\,dx+\int_{\delta_0/2}^{\infty}\int_{R_{n-1}} u^2\Psi^2\,dx+\int_{\delta}^{\delta_0}\int_{R_{n-1}} u^2 x_n^{-\beta_1}(\eta-\delta)\,\Psi^2\,dx+$$

$$+\int_{\delta}^{\delta_0}\int_{R_{n-1}} u^2 x_n^{-\beta_2}(\eta-\delta)\,\Psi^2\,dx+\int_{\delta}^{\delta_0}\int_{R_{n-1}} u^2 x_n^{-\beta}\,\Psi^2\,dx+$$

$$+\int_{\delta}^{\delta_0}\int_{R_{n-1}} u^2(x_n-\delta)^{2\alpha-1}\Psi^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}} u^2(\eta-\delta)^{2-\Theta}\Psi^2\,dx+$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}} f^2(\eta-\delta)^{\Theta}\Psi^2\,dx+\int_{\delta_0/2}^{\infty}\int_{R_{n-1}} f^2\Psi^2\,dx\Bigg]$$

and consequently the implication I $\Rightarrow$ II follows the Lebesgue Monotone Convergence Theorem.

To show II $\Rightarrow$ I we note that

$$\frac{1}{2}\int_{E_{n-1}} a_{nn}(x',\delta)\,u(x',\delta)^2\,\Phi(x')^2\,dx' =$$

$$= -\frac{1}{2}\int_{\delta}^{\infty}\int_{R_{n-1}} a_{nn}(x',\delta)\,u(x)^2\,\Phi(x')^2\,D_n^2\eta(x_n)\,dx +$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}} [a_{nn}(x)-a_{nn}(x',\delta)]\,D_n u(x)\cdot u(x)\,D_n\eta(x_n)\,\Phi(x')^2\,dx +$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}} \sum_{i,j=1}^{n} a_{ij}D_i u\,D_j u(\eta-\delta)\,\Phi^2\,dx + \int_{\delta}^{\infty}\int_{R_{n-1}} \sum_{i=1}^{n-1} a_{in}D_i u\cdot uD_n\eta\Phi^2\,dx +$$

$$+2\int_{\delta}^{\infty}\int_{R_{n-1}} \sum_{i,j=1}^{n} a_{ij}D_i u\cdot u(\eta-\delta)\,\Phi\,D_i\Phi\,dx +$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}} \sum_{i=1}^{n} b_i\,D_i u\cdot u(\eta-\delta)\,\Phi^2\,dx +$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}} cu^2(\eta-\delta)\,\Phi^2\,dx - \int_{\delta}^{\infty}\int_{R_{n-1}} fu(\eta-\delta)\,\Phi^2\,dx\,.$$

According to Lemma 2 the condition (II) implies that for every $0\leqslant \leqslant\mu<1$ the integral

$$\int_{\delta}^{\infty}\int_{R_{n-1}} u^2(x_n-\delta)^{-\mu}\,\Psi^2\,\mathrm{d}x$$

s bounded independently of δ. Repeating the argument from the step I $\Rightarrow$ II the result follows.

REMARK 1. It follows from the proof of Theorem 1 that the condition (II) implies:

$$\text{for each } T>0 \sup_{0<x_n<T}\int_{R_{n-1}} u(x',x_n)^2\,\Psi(x')^2\,dx' < \infty\,.$$

THEOREM 2. *Let* $u \in W^{1,2}_{\Psi}(R^+_n)$ *be a solution of* (1) *in* R^+_n. *Suppose that one of the conditions* (I) *or* (II) *holds. Then there exists a function* $\varphi \in L^2_{\Psi}(R_{n-1})$ *such that*

$$\lim_{\delta\to 0} \int_{R_{n-1}} u(x', \delta) g(x') \Psi(x')^2 dx' = \int_{R_{n-1}} \varphi(x') g(x') \Psi(x')^2 dx'$$

for every $g \in L^2_{\Psi}(R_{n-1})$.

The proof of this theorem is an obvious modification of the proof of Theorem 2 in [3].

Our next objective is to establish the L^2_{Ψ}-convergence of $u(\cdot, \delta)$ to φ as $\delta \to 0$. To do this we first show the norm of $u(\cdot, \delta)$ converges to the norm of φ. The result then follows by the uniform convexity of the space L^2_{Ψ}.

THEOREM 3. *Let* $u \in W^{1,2}_{\Psi}(R^+_n)$ *be a solution of* (1) *in* R^+_n. *Suppose that one of the conditions* (I) *or* (II) *of Theorem* 1 *holds. Then there exists a function* $\varphi \in L^2_{\Psi}(R_{n-1})$ *such that*

$$\lim_{\delta\to 0} \int_{R_{n-1}} [u(x', \delta) - \varphi(x')]^2 \Psi(x')^2 dx' = 0 .$$

The proof is similar to that of theorem 3 in [3] and therefore is omitted.

3. The energy estimate.

The result of Section 2 suggest the following definition of the Dirichlet problem (1), (2).

A weak solution $u \in W^{1,2}_{\Psi}(R^+_n)$ of (1) is a solution of the Dirichlet problem (1), (2) if

$$\lim_{\delta\to 0} \int_{R_{n-1}} [u(x', \delta) - \varphi(x')]^2 \Psi(x')^2 dx' = 0 . \tag{7}$$

To solve the Dirichlet problem (1), (2) we first consider the problem

$$Lu + \lambda u = f \quad \text{in } R_{n-1}\times(\delta, \infty) , \tag{1_δ}$$

$$u(x', \delta) = 0 \quad \text{on } R_{n-1}. \tag{2_δ}$$

Here the boundary condition (2_δ) is understood in the sense that $u \in \overset{\circ}{W}{}^{1,2}(R_{n-1}\times(\delta,\infty))$.

We begin by establishing an energy estimate for a solution u_δ of (1_δ), (2_δ) (see Theorem 5 below). Thus a solution of (1), (2) (with $\varphi \equiv \equiv 0$) is obtained as a limit of u_δ. The existence of a solution of the problem (1_δ), (2_δ) will be proved in Section 4.

THEOREM 4. *Let u_δ be a solution in $W^{1,2}(R_{n-1}\times(\delta,\infty))$ of the problem (1_δ), (2_δ). Then there exist positive constants λ_0, δ_1, τ and C_1 such that*

$$(8)\qquad \int_{R_n^+}|Du_\delta(x)|^2\big(\eta(x_n)-\delta\big)\Psi(x')^2\,dx+\lambda\int_{R_n^+}u_\delta(x)^2\big(\eta(x_n)-\delta\big)\Psi(x')^2\,dx+$$
$$+\sup_{\delta<s<\delta+\tau}\int_{R_{n-1}}u_\delta(x',s)^2\Psi(x')^2\,dx'\leqslant C_1\int_{R_n^+}f(x)^2[\min(1,x_n)]^{\Theta}\Psi(x')^2\,dx$$

for all $\lambda\geqslant\lambda_0$ and $0<\delta<\delta_1$.

PROOF. Let $0<\delta<\delta_0/2$ and put $v(x)=u_\delta(x)\big(\eta(x_n)-\delta\big)\Phi(x)^2$ for $x\in R_{n-1}\times(\delta,\infty)$, where Φ is a non-negative function in $C_0^\infty(R_{n-1})$. Taking v as a test function in the integral identity defining a solution u_δ we easily arrive at the estimate

$$(9)\qquad \int_\delta^{\delta_0}\int_{R_{n-1}}|Du_\delta|^2(\eta-\delta)\Psi^2\,dx+\lambda\int_\delta^{\infty}\int_{R_{n-1}}u_\delta^2(\eta-\delta)\Psi^2\,dx\leqslant$$
$$\leqslant C_2\Big[\int_\delta^\infty\int_{R_{n-1}}u_\delta^2\Psi^2\,dx+\int_{\delta_0}^\infty\int_{R_{n-1}}|Du|^2\Psi^2\,dx+\int_\delta^{\delta_0}\int_{R_{n-1}}u_\delta^2(\eta-\delta)x_n^{-\beta_1}\Psi^2\,dx+$$
$$+\int_\delta^{\delta_0}\int_{R_{n-1}}u_\delta^2(\eta-\delta)x_{x_n}^{-\beta_2}\Psi^2\,dx+\int_\delta^{\delta_0}\int_{R_{n-1}}u_\delta^2(x_n-\delta)^{2\alpha-1}\Psi^2\,dx+$$
$$+\int_\delta^{\delta_0}\int_{R_{n-1}}u_\delta^2x_n^{-\beta}\Psi^2\,dx+\int_\delta^{\infty}\int_{R_{n-1}}u_\delta^2(\eta-\delta)^{2-\Theta}\Psi^2\,dx+$$
$$+\int_{R_n^+}f^2[\min(1,x)]^{\Theta}\Psi^2\,dx\Big],$$

for all $\delta \leqslant \delta_0/2$, provided λ is sufficiently large, say $\lambda \geqslant \lambda_1$, where a positive constant C_2 depends on the structure of L. By Lemma 2 we can write the estimate (9) in the form

$$(10)\qquad \int_{\delta}^{\delta_0}\int_{R_{n-1}} |Du_\delta|^2 (\eta-\delta)\Psi^2\,dx + \lambda \int_{\delta}^{\infty}\int_{R_{n-1}} u_\delta^2(\eta-\delta)\Psi^2\,dx \leqslant$$
$$\leqslant C_3\Big[\int_{R_n^+} f^2[\min(1,x_n)]^\Theta \Psi^2\,dx + \int_{R_{n-1}} u(x',\delta_0)^2\Psi(x')^2\,dx' +$$
$$+\int_{\delta_0}^{\infty}\int_{R_{n-1}} |Du_\delta|^2\Psi^2\,dx + \int_{\delta_0}^{\infty}\int_{R_{n-1}} u_\delta^2\Psi^2\,dx\Big]$$

for all $\delta \leqslant \delta_0/2$ and $\lambda \geqslant \lambda_2$, where λ_2 is a sufficiently large and a positive constant C_3 depends on the structure of L. Now note that by an obvious modification of inequality (2) in [10] (p. 179) we have

$$(11)\qquad \int_{R_{n-1}} u_\delta(x',\delta_0)^2\Psi(x')^2\,dx' \leqslant S\int_{\delta_0}^{\infty}\int_{R_{n-1}} |Du_\delta(x)|^2\Psi(x')^2\,dx,$$

where a positive S is independent of u. Using (11) and Lemma 3 with $r=\delta_0/2$ we can write the estimate (10) in the form

$$(12)\qquad \int_{\delta}^{\infty}\int_{R_{n-1}} |Du_\delta|^2 (\eta-\delta)\Psi^2\,dx + \lambda \int_{\delta}^{\infty}\int_{R_{n-1}} u_\delta^2(\eta-\delta)\Psi^2\,dx \leqslant$$
$$\leqslant C_4\Big[\int_{R_n^+} f^2[\min(1,x_n)]^\Theta \Psi^2\,dx + \int_{\delta_0/2}\int_{R_{n-1}} u_\delta^2\Psi^2\,dx\Big]$$

for all $\delta \leqslant \delta_0/4$ and $\lambda \geqslant \lambda_2$, with a modified λ_2 if necessary. On the other hand note that

$$\eta(x_n) \geqslant \frac{\delta_0}{8} + \delta \text{ for all } \delta \leqslant \frac{\delta_0}{8} \text{ and } x_n \geqslant \frac{\delta_0}{4}$$

and consequently

$$\int_{\delta_0/2}^{\infty}\int_{R_{n-1}} u_\delta^2\Psi^2\,dx \leqslant \frac{8}{\delta_0}\int_{\delta}^{\infty}\int_{R_{n-1}} u_\delta^2\Psi^2(\eta-\delta)\,dx\,.$$

Therefore there exist positive constants C_5 and λ_2, both depending on the structure of L, such that

$$(13)\qquad \int_{\delta}^{\infty}\int_{R_{n-1}} |Du_\delta|^2(\eta-\delta)\Psi^2\,dx+\lambda\int_{\delta}^{\infty}\int_{R_{n-1}} u_\delta^2(\eta-\delta)\Psi^2\,dx\leqslant$$
$$\leqslant C_4\int_{R_n^+} f^2[\min(1,x_n)]^{\Theta}\Psi^2\,dx$$

for all $\delta\leqslant\delta_0/8$ and $\lambda\geqslant\lambda_2$.

Let $0<\tau_0\leqslant\delta_0/2$, as in Theorem 1 we derive the following inequality

$$(14)\qquad \int_{R_{n-1}} u_\delta(x',\delta+\tau)^2\Psi(x')^2\,dx'\leqslant C_6\Big[\int_{R_n^+} f[\min(1,x_n)]^{\Theta}\Psi^2\,dx+$$
$$+\int_{\delta+\tau}^{\infty}\int_{R_{n-1}}|Du|^2(\eta-\delta-\tau)\Psi^2\,dx+\lambda\int_{\delta+\tau}^{\infty}\int_{R_{n-1}}u_\delta^2(\eta-\delta-\tau)\Psi^2\,dx+$$
$$+\int_{\tau+\delta}^{\infty}\int_{R_{n-1}}u_\delta^2\Psi^2\,dx+\int_{\tau+\delta_0}^{\infty}\int_{R_{n-1}}|Du|^2\Psi^2\,dx+$$
$$+\int_{\tau+\delta}^{\tau+\delta_0}\int_{R_{n-1}}u_\delta^2(x_n-\delta)^{2\alpha-1}\Psi^2\,dx+\int_{\delta+\tau}^{\tau+\delta_0}\int_{R_{n-1}}u_\delta^2x_n^{-\beta_2}(\eta-\delta-\tau)\Psi^2\,dx+$$
$$+\int_{\tau+\delta}^{\tau+\delta_0}\int_{R_{n-1}}u_\delta^2x_n^{-\beta}\Psi^2\,dx+\int_{\tau+\delta}^{\tau+\delta_0}\int_{R_{n-1}}u_\delta^2(\eta-\delta-\tau)^{2-\Theta}\Psi^2\,dx+$$
$$+\int_{\tau+\delta}^{\tau+\delta_0}\int_{R_{n-1}}u_\delta^2x_n^{-\beta_1}(\eta-\delta-\tau)\Psi^2\,dx\Big]$$

for all $\delta\leqslant\delta_0/2$ and $0<\tau\leqslant\tau_0/2$. Note that if $0<\tau\leqslant\delta_0/2$ and $0<\tau<\delta_0/2$, then for every $0\leqslant\mu<1$ we have

$$(15)\qquad \int_{\tau+\delta}^{\tau+\delta_0}\int_{R_{n-1}}u_\delta^2(\eta-\delta-\tau)^{-\mu}\Psi^2\,dx\leqslant\int_{\tau+\delta}^{\delta+\tau_0}\int_{R_{n-1}}u_\delta^2(\eta-\delta-\tau)^{-\mu}\Psi^2\,dx+$$

$$+\int_{\delta+\tau_0}^{\infty}\int_{R_{n-1}} u_\delta^2(\eta-\delta-\tau)^{-\mu}\Psi^2\,dx \leqslant \frac{(\tau_0)^{1-\mu}}{1-\mu}\sup_{\delta<s<\delta+\tau_0}\int_{R_{n-1}} u(x',s)^2\Psi(x')^2\,dx' + \\ + \left(\frac{2}{\tau_0}\right)^{\mu}\int_{\delta+\tau_0}^{\infty}\int_{R_{n-1}} u_\delta^2\Psi^2\,dx\,.$$

Since $\eta(x_n)\geqslant\tau_0/2+\delta$ for all $\delta\leqslant\tau_0/2$ and $x_n\geqslant\tau_0$ we get

$$(16)\qquad \int_{\delta+\tau_0}^{\infty}\int_{R_{n-1}} u_\delta^2\Psi^2\,dx \leqslant \frac{2}{\tau_0}\int_{\delta}^{\infty}\int_{R_{n-1}} u_\delta^2(\eta-\delta)\,\Psi^2\,dx\,.$$

On the other hand (11) we have

$$(17)\qquad \sup_{\delta<s<\delta+\tau_0}\int_{R_{n-1}} u_\delta(x',s)^2\Psi(x')^2\,dx' \leqslant \sup_{\delta<s<\delta+\tau_0/2}\int_{R_{n-1}} u_\delta(x',s)^2\Psi(x')\,dx' + \\ + \sup_{\delta+\tau_0/2<s<\delta+\tau_0}\int_{R_{n-1}} u_\delta(x',s)^2\Psi(x')^2\,dx' \leqslant \sup_{\delta<s<\delta+\tau_0/2}\int_{R_{n-1}} u_\delta(x',s)^2\Psi(x')^2\,dx' + \\ + S\int_{\tau_0/2}^{\infty}\int_{R_{n-1}} |Du_\delta|^2\Psi^2\,dx\,.$$

Consequently combining (13), (14), (15), (16) and (17) we obtain

$$(18)\qquad \sup_{\delta<s<\delta+\tau_0/2}\int_{R_{n-1}} u_\delta(x',s)^2\Psi(x')^2\,dx' \leqslant \\ C_7\left[\int_{R_n^+} f^2[\min(1,x_n)]^{\theta}\Psi^2\,dx + \int_{\tau_0/2}^{\infty}\int_{R_{n-1}} |Du_\delta|^2\Psi^2\,dx\right]$$

for all $\delta<\min(\delta_0/8,\tau_0/2)$ and $\lambda\geqslant\lambda_2$, provided λ_2 is sufficiently large and τ_0 is sufficiently samll and C_7 is a positive constant depending on the structure of L. Applying Lemma 3 with $r=\tau_0/4$ we deduce from the last estimate that

$$(19)\qquad \sup_{\delta<s<\delta+\tau_0/2}\int_{R_{n-1}} u_\delta(x',s)^2\Psi(x')^2\,dx' \leqslant \tilde{C}_7\left[\int_{R_n^+} f^2[\min(1,x_n)]^2\Psi^2\,dx + \\ + \int_{\tau_0/4}^{\infty}\int_{R_{n-1}} u_\delta^2\Psi^2\,dx + \int_{\tau_0/2}^{\infty}\int_{R_{n-1}} f^2\Psi^2\,dx\right]$$

for all $\delta < \min(\delta_0/8, \tau_0/8)$. Since $\eta(x_n) \geqslant \tau_0/8 + \delta$ for all $\delta < \tau_0/8$ and $x_n \geqslant \tau_0/4$ we have

$$(20) \qquad \int_{\tau_0/4}^{\infty} \int_{R_{n-1}} u_\delta^2 \Psi^2 \, dx \leqslant \frac{8}{\tau_0} \int_{R_{n-1}}^{\infty} u_\delta^2 \Psi^2 (\eta - \delta) \, dx$$

for all $\delta \leqslant \min(\delta_0/8, \tau_0/8)$. Combining the estimates (13), (19) and (20) we easily deduce (8) with $\tau = \tau_0/2$ and $\delta_1 = \min(\delta_0/8, \tau_0/8)$ provided λ_0 is sufficiently large.

4. The existence of a solution to the Dirichlet problem.

We begin by proving the existence of a solution in $W^{1,2}_{\Psi}(R_{n-1} \times \times (\delta, \infty))$ of the problem (1_δ), (2_δ).

We need the following result due to G. Bottaro and M. E. Marina [1].

THEOREM 5. *Suppose that* $f \in L^2(R_n^+)$, $b_i \in L^n(R_n^+) \cap L^\infty(R_n^+)$ $(i = 1, \ldots, n)$, $c \in L^n(R_n^+) + L^\infty(R_n^+)$ *and that* $c(x) \geqslant \text{Const} > 0$ *on* R_n^+. *Then the Dirichlet problem*

$$Lu = f(x) \quad \text{in } R_n^+; \; u(x', 0) = \quad \text{on } R_{n-1}$$

has unique solution in $\mathring{W}^{1,2}(R_n^+)$.

THEOREM 6. *Assume that*

$$b_i \in L^n(R_n^+) \cap L^\infty(R_n^+) \; (i = 1, \ldots, n) \qquad c \in L^n(R_n^+) + L^\infty(R_n^+)$$

and that $c(x) \geqslant 0$ *on* R_n^+. *Then there exists* $\lambda_0 > 0$ *such that for every* $\varphi \in L^2_\Psi(R_{n-1})$ *and all* $\lambda \geqslant \lambda_0$ *there exists a unique solution* $u \in W^{1,2}_\Psi(R_n^+)$ *of the Dirichlet problem* (1), (2).

PROOF. The proof is based on the following energy estimate: there exist positive contants d, λ_0 and C_1 depending on the structure of L such that if $u \in W^{1,2}_\Psi(R_n^+)$ is a solution of (1), (2) for $\lambda \geqslant \lambda_0$, then

$$(21) \qquad \int_{R_n^+} |Du|^2 \min(1, x_n) \Psi^2 \, dx + \lambda \int_{R_n^+} u^2 \min(1, x_n) \Psi^2 \, dx +$$

$$+ \sup_{0<\delta<d} \int_{R_{n-1}} u(x')^2 \Psi(x')^2 dx' \leqslant C_1 \Big[\int_{R_n^+} f^2[\min(1, x_n)]^\Theta \Psi^2 dx + \\ + \int_{R_{n-1}} \varphi(x')^2 \Psi(x')^2 dx' \Big],$$

the proof of which is now a routine. Let $\{\varphi_m\}$ be sequence of functions in $C_0^1(R_{n-1})$ converging in $L^2_\Psi(R_{n-1})$ to φ. Put

$$f_m(x) = \begin{cases} f(x) & \text{in } (|x'| < m) \times \left(\frac{1}{m}, \infty\right), \\ 0 & \text{elsewhere}, \end{cases}$$

$m = 1, 2, \ldots$. It follows from Theorem 6 that for every m and $\lambda > 0$ there exists a unique solution u_m in $W^{1,2}(R_n^+)$ of the Dirichlet problem

$$Lu + \lambda u = f_m \text{ in } R_n^+,$$

$$u(x', 0) = \varphi_m(x') \text{ on } R_{n-1}.$$

According to (21)

$$\int_{R_n^+} |Du_q - Du_p|^2 \min(1, x_n) \Psi^2 dx + \lambda \int_{R_n^+} (u_p - u_q)^2 \min(1, x_n) \Psi^2 dx + \\ + \sup_{0<\delta<d} \int_{R_{n-1}} [u_p(x', \delta) - u_q(x', \delta)]^2 \Psi(x')^2 dx' \leqslant \\ \leqslant C_1 \Big[\int_{R_n^+} (f_p - f_q)^2 \min(1, x_n) \Psi^2 dx + \int_{R_{n-1}} (\varphi_p - \varphi_q)^2 \Psi^2 dx' \Big]$$

for all p and q, provided $\lambda \geqslant \lambda_0$. Hence $\{u_m\}$ is the Cauchy sequence in the norm

$$\Big[\int_{R_n^+} |Du|^2 \min(1, x_n) \Psi^2 dx + \sup_{0<\delta<d} \int_{R_{n-1}} u(x', \delta)^2 \Psi^2 dx + \\ + \int_{R_n^+} u^2 \min(1, x_n) \Psi^2 dx \Big]^{1/2}$$

and the result follows.

In the sequel we adopt the assumptions of Section 1 with (B iii) replace by

(B' iii) for every $\delta > 0$

$$b_i \in L^n(R_{n-1}\times(\delta, \infty)) \cap L^\infty(R_{n-1}\times(\delta, \infty)) \quad (i = 1, \dots, n)$$

and

$$c \in L^n(R_{n-1}\times(\delta, \infty)) + L^\infty(R_{n-1}\times(\delta, \infty))$$

and moreover there exist positive constants b, C, B, β_1 and β_2 with $0 \leqslant \beta_1 < 1$, $0 \leqslant \beta_2 < 2$ such that

$$0 \leqslant c(x) \leqslant C x_n^{-\beta_2} \qquad \text{on } R_{n-1}\times(0, b)$$

and

$$|b_i(x)| \leqslant B x_n^{-\beta_1} \ (i = 1, \dots, n) \qquad \text{on } R_{n-1}\times(0, b).$$

THEOREM 7. *Let* $\varphi \in L^2_\Psi(R_{n-1})$. *Then there exists* $\lambda_1 > 0$ *such that for every* $\lambda \geqslant \lambda_1$ *the Dirichlet problem* (1), (2) *admits a unique solution in* $W^{1,2}(R_n^+)$.

PROOF. We first assume that $\varphi = 0$ on R_{n-1}. Since for every $\delta > 0$ the coefficients b_i $(i = 1, \dots, n)$ and c satisfy the assumptions of Theorem 6 in $R_{n-1}\times(\delta, \infty)$, there exists $\lambda_\delta > 0$ such that for every $\lambda \geqslant \lambda_\delta$ the problem (1_δ), (2_δ) admits a unique solution u_δ in $W^{1,2}_\Psi(R_{n-1}\times \times(\delta, \infty))$. In view of Theorem 7 we may assume that λ_δ can be chosen independently of δ in a small interval $(0, \delta_1]$. Consequently by Theorem 5 there exist positive constants τ, δ_1, λ_1 and C_1 such that

$$(22) \qquad \int_{R_n^+} |Du_\delta|^2(\eta - \delta)\Psi^2 dx + \lambda \int_{R_n^+} u_\delta^2(\eta - \delta)\Psi^2 dx + \\ + \sup_{\delta < s < \delta + \tau} \int_{R_n^+} u_\delta(x', s)^2 \Psi(x')^2 dx' \leqslant C_1 \int_{R_n^+} f^2[\min(1, x_n)]^\Theta \Psi^2 dx$$

for all $\lambda \geqslant \lambda_1$ and $0 < \delta \leqslant \delta_1$. The estimate (29) yields the existence of a sequence δ_n and a function $u \in W^{1,2}_{\mathrm{loc}}(R_n^+)$ such that:

for every compact set $K \subset R_n^+$

$$(23) \qquad \lim_{n\to\infty} u_{\delta_n} = u \text{ weakly in } W^{1,2}(K).$$

It is obvious that u satisfies (1). We extend u_δ and $D_i u\sqrt{\eta-\delta}$ ($i = 1, \dots, n$) by 0 outside $R_{n-1}\times(\delta, \infty)$. By a weak compacteness of bounded sets in $L^2_\Psi(R_{n-1})$ we may assume that there exist functions $v_i (i = 1, \dots, n)$ and v in $L^2_\Psi(R^+_n)$ such that

$$\lim_{n\to\infty} D_i u_{\delta_n}\sqrt{\eta-\delta_n} = v_i (i = 1, \dots, n) \quad \text{and} \quad \lim_{n\to\infty} u_{\delta_n} = v \tag{24}$$

weakly in $L^2_\Psi(R^+_u)$. We claim that

$$v_i = D_i u\sqrt{\eta} \;\; (i = 1, \dots, n) \text{ and } v = u \tag{25}$$

a.e. in R^+_n. We only show the first relation (32).

Let $g \in L^2_\Psi(R^+_n)$ be an arbitrary function with a compact support. Then

$$\Big|\int_{R^+_n}(v_i - D_i u\sqrt{\eta})g\Psi^2 dx\Big| \leqslant \Big|\int_{R^+_n}(v_i - D_i u_{\delta_n}\sqrt{\eta-\delta_n})g\Psi^2 dx\Big| + \\ + \Big|\int_{\operatorname{supp} g}(D_i u_{\delta_n}\sqrt{\eta-\delta_n} - D_i u_{\delta_n}\sqrt{\eta})g\Psi^2 dx\Big| + \\ + \Big|\int_{\operatorname{supp} g}(D_i u_{\delta_n} - D_i u)\sqrt{\eta}\, g\Psi^2 dx\Big| + R_1 + R_2 + R_3 .$$

By (23) $\lim_{n\to\infty} R_3 = 0$. From (24) we obtain $\lim_{n\to\infty} R_1 = 0$. The Hölder inequality and boundedness of $\int_{\operatorname{supp} g} |Du_{\delta_n}|^2\, \Psi^2\, dx$ imply that $\lim_{n\to\infty} R_2 = 0$. Therefore

$$\int_{R^+_n} D_i u\sqrt{\eta}\, g\, \Psi^2\, dx = \int_{R^+_n} v_i\, g\, \Psi^2\, dx$$

for every $g \in L^2_\Psi(R^+_n)$ with compact support and the first relation easily follows. It is also clear that

$$\int_{R^+_n} |Du|^2 \min(1, x_n)\, \Psi^2\, dx + \int_{R^+_n} u^2\, \Psi^2\, dx \leqslant \int_{R^+_n} f^2[\min(1, x_n)]^\Theta\, \Psi^2\, dx .$$

Theorem 4 implies the existence of a function $h \in L^2_\Psi(R_{n-1})$ such that

$$\lim_{\delta\to 0} \int_{R_{n-1}} [u(x', \delta) - h(x')]^2 \Psi(x')^2 dx = 0 .$$

Hence we have to show that $h = 0$ a.e. on R_{n-1}. Let $\Phi \in C_0^1(R_{n-1})$. Then

$$\int_{R_{n-1}} a_{nn}(x', 0)\Phi(x')h(x')\Psi(x')^2 dx' =$$

$$= \int_{R_n^+} [a_{nn}(x) - a_{nn}(x', 0)] D_n u(x) \Phi(x') D_n\eta(x_n) \Psi(x')^2 dx -$$

$$- \int_{R_n^+} D_n\big(a_{nn}(x', 0) D_n\eta(x_n)\big) \Phi(x')\Psi(x')^2 dx +$$

$$+ \int_{R_n^+} \sum_{i=1}^{n-1} a_{in} D_i u\, D_n\eta\, \Phi\Psi^2 dx + \int_{R_n^+} \sum_{i,j=1}^{n} a_{ij} D_i u\, D_j(\Phi\Psi^2)\eta\, dx +$$

$$+ \int_{R_n^+} \sum_{i=1}^{n} b_i D_i u\, \Phi\Psi^2 \eta\, dx + \int_{R_n^+} (c + \lambda)\Phi\Psi^2\eta\, dx - \int_{R_n^+} f\Phi\Psi^2\eta\, dx = K ,$$

and

$$0 = \int_\delta^\infty \int_{R_{n-1}} [a_{nn}(x) - a_{nn}(x', \delta)] D_n u_\delta D_n\eta\, \Phi\Psi^2 dx -$$

$$- \int_\delta^\infty \int_{R_{n-1}} D_n\big(a_{nn}(x', \delta) D_n\eta\big) u_\delta \Phi\Psi^2 dx + \int_\delta^\infty \int_{R_{n-1}} \sum_{i=1}^{n-1} a_{in} D_i u_\delta D_n\eta\, \Phi\Psi^2 dx +$$

$$+ \int_\delta^\infty \int_{R_{n-1}} \sum_{i,j=1}^{n-1} a_{ij} D_i u_\delta D_j(\Phi\Psi^2)(\eta - \delta)\, dx + \int_\delta^\infty \int_{R_{n-1}} \sum_{i=1}^{n} b_i D_i u_\delta \Phi\Psi^2(\eta - \delta)\, dx +$$

$$+ \int_\delta^\infty \int_{R_{n-1}} (c + \lambda) u_\delta \Phi\Psi^2(\eta - \delta)\, dx - \int_\delta^\infty \int_{R_{n-1}} f\Phi\Psi^2(\eta - \delta)\, dx = K_\delta .$$

By (24) and (25) it is clear that $\lim_{n\to\infty} (K - K_{\delta_n}) = 0$ and consequently

$$\int_{R_{n-1}} h\Phi\,\Psi^2\,\mathrm{d}x' = 0 \qquad \text{for every } \Phi \in C_0^1(R_{n-1}),$$

therefore $h = 0$ a.e. on R_{n-1}.

Now consider the general situation $\varphi \in L^2_\Psi(R_{n-1})$ and $\varphi \not\equiv 0$. As in Theorem 4 we derive an energy estimate for solutions in $W^{1,2}_\Psi$ of the problem (1), (2), namely, let $u \in W^{1,2}_\Psi(R_n^+)$ be a solution of the problem (1), (2), then there exist positive constants λ_0, τ and C depending on the structure of the operator L such that

$$\text{(26)} \qquad \int_{R_n^+} |Du|^2 \min(1, x_n)\Psi^2 dx + \lambda \int_{R_n^+} u^2 \min(1, x_n)\Psi^2 dx +$$
$$+ \sup_{0<s<\tau} \int_{R_{n-1}} u(x', s)^2 \Psi(x')^2 dx' \leqslant C_1 \Big[\int_{R_n^+} f^2[\min(1, x_n)]^\Theta \Psi^2 dx +$$
$$+ \int_{R_n} \varphi^2 \Psi^2 dx' \Big]$$

for all $\lambda \geqslant \lambda_0$. To solve the problem (1), (2) we take a sequence $\{\varphi_m\}$ in $C_0^1(R_{n-1})$ such that $\lim_{n\to\infty} \varphi_m = \varphi$ in $L^2_\Psi(R_{n-1})$ and consider the Dirichlet problem (1), (2) with $\varphi = \Phi_m$. Let Φ_m be a function in $C^1(\overline{R_n^+})$ such that $\Phi_m(x', 0) = \varphi(x')$ on R_{n-1} and $\Phi(x', x_n) = 0$ for $x_n \geqslant \delta_0$. By means of the transformation $v = u - \varphi_m$ this problem can be reduced to the Dirichlet problem with zero boundary data. Therefore by the previous step for every m there exist a solution u_n in $W^{1,2}_\Psi$ provided λ is sufficiently large. Now it is obvious that u_m converges to a solution of (1), (2) in the norm defined by the left hand side of the inequality (26) and this completes the proof.

REFERENCES

[1] G. Bottaro - M. E. Marina, *Problemi di Dirichlet per equazioni ellittiche di tipo variazionale su insiemi non limitati*, Boll. Un. Mat. Ital., (4) **8** (1983), pp. 46-56.

[2] J. H. CHABROWSKI - H. B. THOMPSON, *On the boundary values of solutions of linear elliptic equations*, Bull. Austral. Mat. Soc., **27** (1983), pp. 1-30.

[3] J. H. CHABROWSKI, *Dirichlet problem for a linear elliptic equation in unbounded domains with L^2-boundary data*, Rend. Sem. Mat. Univ. Padova, **71** (1983), pp. 287-328.

[4] J. H. CHABROWSKI, *On the Dirichlet problem for an elliptic equation with unbounded coefficients*, Boll. Un. Mat. Ital., (6), **5**-B (1986), pp. 71-91.

[5] E. B. FABES - D. S. JERISON - C. E. KENIG, *Necessary and sufficient conditions for absolute continuity elliptic-harmonic measure*, Ann. of Math., (2), **119** (1984), pp. 121-141.

[6] A. K. GUŠČIN - V. P. MIKHAILOV, *On the boundary values in L_p $(p > 1)$ of solutions of elliptic equations*, Mat. Sb., **108** (150) (1979), pp. 1-19.

[7] O. A. LADYZHENSKAJA - N. N. URAL'TSEVA, *Linear and quasilinear elliptic equations*, Academic Press, New York-London, 1968.

[8] V. P. MIKHAILOV, *On the boundary values of the solutions of elliptic equations*, Appl. Mat. Optim., **6** (1980), pp. 193-199.

[9] YU. A. MIKHAILOV, *Boundary values in L_p, $p > 1$, of solutions of second order elliptic equations*, Differentsial'nye Uravnenija, **19** (2), pp. 318-337.

[10] S. MIZOHATA, *The theory of partial differential equations*, Cambridge University Press, Cambridge, 1973.

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