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Inequalities Involving Γ-Functionals and Semi Complete Lattice Homomorphisms.

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Summary - Sufficient conditions for the validity of inequalities involving Γ-functionals and lattice homomorphisms are provided. In particular, inequalities of De Giorgi and Franzoni and a theorem of Butazzo and Dal Maso on sequential Γ-limits are generalized.

In various problems of analysis one wants to deduce convergence properties of, say, sums, suprema or infima of sequences of functions from convergences of individual sequences. Analogous questions arise for convergence of sets.

Take, for example, the contingent epi-derivative of an extended-real-valued function f on a topological vector space X:

$$(D_{(-,-)}f)(x)h = \sup_{\substack{t>0 \\ Q\in\mathcal{N}(h)}} \inf_{\substack{0<t'<t \\ h'\in Q}} \frac{1}{t'}\big(f(x+t'h') - f(x)\big),$$

where $\mathcal{N}(h)$ denotes the neighborhood filter of h. Suppose now that we are given a real-valued function g on X and a subset A of X. If we need to estimate the contingent epi-derivative of $g+\psi_A$ (ψ_A being the indicator function of A: equal 0 on A and $+\infty$ elsewhere) in terms of derivatives of g and tangents of A, we are in front of a

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problem of the type described above. Let $\dot{+}$ stand for the upper extension of the addition.

In this case we have that [4]

$$D_{(-,-)}(g + \psi_A)(x) \geqslant (D_{(-,-)}g)(x) \dot{+} \psi_{T_{(-,-)}A(x)},$$

(where $T_{(-,-)}A(x)$ is the contingent cone of A at x, i.e., the set of those vectors h for every neighborhood Q of which and every $t > 0$ there is $0 < t' < t$ such that $(x + t'Q) \cap A \neq \emptyset$) and, for instance,

$$D_{(-,-)}(g + \psi_A)(x) \leqslant (D_{(+,-)}g)(x) \dot{+} \psi_{T_{(-,+)}A(x)}$$

(where the tangent epi-derivative $D_{(+,-)}g$ of g is

$$(D_{(+,-)}g)(x)\, h = \sup_{Q \in \mathcal{N}(h)} \inf_{t>0} \sup_{0<t'<t} \inf_{h' \in Q} \frac{1}{t'} \big(f(x + t'h') - f(x)\big)$$

and $T_{(-,+)}A(x)$ is the almost interior cone of A at x, i.e., the set of those h for which there is a neighborhood Q and a sequence (t_n) tending to 0 such that $x + t_n Q \subset A$).

The discussed derivatives and approximating cones are expressible in terms of Γ-functionals (introduced by De Giorgi [2] as a generalization of a concept due to De Giorgi and T. Franzoni [3]). Let, for $i = 1, 2, \ldots, n$, X_i be a nonempty set, $\mathcal{F}_i$ a filter on X_i and α_i either $+1$ or -1. Let f be an extended-real-valued function on $X_1 \times X_2 \times \ldots \times X_n$. The $\Gamma(\mathcal{F}_1^{\alpha_1}, \ldots, \mathcal{F}_n^{\alpha_n})$ functional calculated at f is, by definition,

$$\Gamma(\mathcal{F}_1^{\alpha_1}, \ldots, \mathcal{F}_n^{\alpha_n}) f = \operatorname*{ext}^{-\alpha_n}_{F_n \in \mathcal{F}_n} \ldots \operatorname*{ext}^{-\alpha_1}_{F_1 \in \mathcal{F}_1} \operatorname*{ext}^{\alpha_1}_{x_1 \in F_1} \ldots \operatorname*{ext}^{\alpha_n}_{x_n \in F_n} f(x_1, \ldots, x_n)\,, \tag{0.1}$$

where ext^{+1} stands for the supremum and ext^{-1} for the infimum (we shall abbreviate: ext^{+}, ext^{-}).

In the language of Γ-functionals, the convergence properties for sums, suprema, infima, etc., that we have announced at the very begining, amount to certain inequalities involving Γ-functionals.

De Giorgi and Franzoni give four such inequalities [3, Prop. 1.19], which amount to the following. Consider a filter $\mathcal{F}$ on a set X and the usual cofinite filter $\mathcal{N}$ on the set $\mathbf{N}$ of natural numbers; consider, as well, extended-real-valued functions f, g on $\mathbf{N} \times X$ and an increasing continuous (in each variable) function φ: $\overline{\mathbf{R}} \times \overline{\mathbf{R}} \to \overline{\mathbf{R}}$.

Then

$$\varphi\big(\Gamma(\mathcal{N}^-, \mathcal{F}^-)f, \Gamma(\mathcal{N}^-, \mathcal{F}^-)g\big) \leqslant \Gamma(\mathcal{N}^-, \mathcal{F}^-)\varphi(f, g) ;$$

$$\varphi\big(\Gamma(\mathcal{N}^+, \mathcal{F}^-)f, \Gamma(\mathcal{N}^-, \mathcal{F}^-)g\big) \leqslant \Gamma(\mathcal{N}^+, \mathcal{F}^-)\varphi(f, g) ;$$

$$\varphi\big(\Gamma(\mathcal{N}^-, \mathcal{F}^-)f, \Gamma(\mathcal{N}^+, \mathcal{F}^+)g\big) \geqslant \Gamma(\mathcal{N}^-, \mathcal{F}^-)\varphi(f, g) ;$$

$$\varphi\big(\Gamma(\mathcal{N}^+, \mathcal{F}^-)f, \Gamma(\mathcal{N}^+, \mathcal{F}^+)g \geqslant \Gamma(\mathcal{N}^+, \mathcal{F}^-)\varphi(f, g) .$$

What matters for such functions φ is to be complete lattice homomorphisms separately in each variable.

The above inequalities specialize to the suprema $\vee$ and infima $\wedge$ on $\overline{\mathbb{R}}$ (and on every completely distributive lattice). However, the extended upper addition $\dot{+} : \overline{\mathbb{R}}\times\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is only a meet semi complete lattice homomorphism while the lower addition $\underset{\cdot}{+} : \overline{\mathbb{R}}\times\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a joint semi complete lattice homomorphism. Therefore, the inequalities of De Giorgi and Franzoni may be used for $\dot{+}$ and $\underset{\cdot}{+}$ only if f and g are bounded (from below or from above).

G. Butazzo and G. Dal Maso considered such inequalities on the occasion of a study of convergences of optimal control problems [1]. Their results concern the so-called sequential Γ-functionals.

Consider a sequence $F = (f_k)$ of extended-real-valued functions on $X_1 \times \ldots \times X_n$. As before, let $\mathcal{F}_i$ be a filter on X_i. By $\varepsilon\mathcal{F}$ we denote the set of all the sequences which generate filters finer than $\mathcal{F}$. Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be either $+1$ or -1. By definition,

$$\Gamma^{\alpha_0}_{\mathrm{seq}}(\mathcal{F}_1^{\alpha_1}, \ldots, \mathcal{F}_n^{\alpha_n})F = \operatorname*{ext}^{\alpha_1}_{(x_{1k})\in\varepsilon\mathcal{F}_1} \ldots \operatorname*{ext}^{\alpha_n}_{(x_{nk})\in\varepsilon\mathcal{F}_n} \operatorname*{ext}^{-\alpha_0}_{k_0\in\mathbb{N}} \operatorname*{ext}^{\alpha_0}_{k\geqslant k_0} f_k(x_{1k}, \ldots, x_{nk}) . \tag{0.2}$$

It is known [6] that if the filters $\mathcal{F}_1, \ldots, \mathcal{F}_n$ admit countable bases, then

$$\Gamma(\mathcal{N}^{\alpha_0}, \mathcal{F}_1^{\alpha_1}, \mathcal{F}_2^{\alpha_2}) = \Gamma^{\alpha_0}_{\mathrm{seq}}(\mathcal{F}_1^{\alpha_1}, \mathcal{F}_2^{\alpha_2}) ,$$

but the equality fails, in general, for $n \geqslant 3$. Let $\varphi : \overline{\mathbb{R}}\times\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be continuous and increasing in each variable.

THEOREM (Butazzo, Dal Maso). *If for* $i = 0, 1, \ldots, n$,

$$\alpha_i + \beta_i \leqslant \gamma_i - 1 , \tag{0.3}$$

then

$$(0.4)\qquad \varphi(\Gamma^{\alpha_0}_{\text{seq}}(\mathcal{F}_1^{\alpha_1}, \ldots, \mathcal{F}_n^{\alpha_n})f,\ \Gamma^{\beta_0}_{\text{seq}}(\mathcal{F}_1^{\beta_1}, \ldots, \mathcal{F}_n^{\beta_n})g) \leqslant$$
$$\leqslant \Gamma^{\gamma_0}_{\text{seq}}(\mathcal{F}_1^{\gamma_1}, \ldots, \mathcal{F}_n^{\gamma_n})\varphi(f, g)$$

If, on the other hand,

$$(0.5)\qquad \alpha_i + \beta_i \geqslant \gamma_i + 1\,,$$

then the opposite inequality holds.

In (0.4), $\varphi(f, g)$ stands for a function on $X = X_1 \times X_2 \times \ldots \times X_n$ and is an abbreviation of $\varphi(f, g) \circ (1_X \times 1_X)$, where 1_X is the identity of X.

The important theorem above admits simple proofs. This is probably the reason why the authors called it a «proposition» and provided only an outline of.

In [8, Prop. 16] Volle establishes the last of the inequalities of De Giorgi and Franzoni in the case of φ being $+$. This indicates that some of Γ-inequalities hold for semi complete lattive homomorphisms.

In this paper we establish general inequalities involving Γ-functionals and semi complete lattice homomorphisms and derive from them the Butazzo-Dal Maso theorem.

Demonstrations of our results rely on the representation theory for Γ-functionals developed by G. H. Greco in [5], [6].

1. Results.

Clearly, formulae (0.1), (0.2) are still meaningful, if f is supposed to be valued in an arbitrary complete lattice L. Throughout this paper we shall assume that the lattices are completely distributive. Recall that every complete chain (in particular, the extended real line) and each product of complete chains is completely distributive. A mapping φ: $L^k \to L$ is called a *joint* (resp. *meet*) *semi complete lattice homomorphism* (separately in each variable) if it commutes with suprema (resp. infima) of nonempty subsets of L. If φ satisfies both the properties, then it is said a *complete lattice homomorphism.* Consider an $n \times k$-matrix $[\alpha_{ij}]$ composed of $+1$ and -1 and a vector $[\gamma_i]$, $1 \leqslant i \leqslant n$.

1.1. THEOREM. *Let φ be a joint semi complete lattice homomorphism. If, for each $i \leqslant n$,*

$$\sum_{j=1}^{k} \alpha_{ij} \leqslant \gamma_i - k + 1 , \tag{1.1}$$

then

$$\varphi\big(\Gamma(\mathcal{F}_1^{\alpha_{11}}, \ldots, \mathcal{F}_n^{\alpha_{n1}}) f_1, \ldots, \Gamma(\mathcal{F}_1^{\alpha_{1k}}, \ldots, \mathcal{F}_n^{\alpha_{nk}}) f_k\big) \leqslant \leqslant \Gamma(\mathcal{F}_1^{\gamma_1}, \ldots, \mathcal{F}_n^{\gamma_n}) \varphi(f_1, \ldots, f_k), \tag{1.2}$$

for each collection of functions $f_1, \ldots, f_k$ from $X_1 \times \ldots \times X_n$ to L.

1.2. THEOREM. *Let φ be a meet semi complete lattice homomorphism. If, for each $i \leqslant n$,*

$$\sum_{j=1}^{k} \alpha_{ij} \geqslant \gamma_i + k - 1 , \tag{1.3}$$

then the opposite inequality holds in (1.2).

Let $[\alpha_0^j]$ be an k-vector and γ_0 a scalar of $+1$ and -1. With the same assumptions on φ, we have

1.3. COROLLARY. *If* (1.1) (*respectively,* (1.3)) *holds for each $i \leqslant n$, then*

$$\varphi\big(\Gamma_{\text{seq}}^{\alpha_{01}}(\mathcal{F}_1^{\alpha_{11}}, \ldots, \mathcal{F}_n^{\alpha_{n1}}) f_1, \ldots, \Gamma_{\text{seq}}^{\alpha_{0k}}(\mathcal{F}_1^{\alpha_{1k}}, \ldots, \mathcal{F}_n^{\alpha_{nk}}) f_k\big) \leqslant \leqslant (\textit{respectively}, \geqslant) \Gamma_{\text{seq}}^{\gamma_0}(\mathcal{F}_1^{\gamma_1}, \ldots, \mathcal{F}_n^{\gamma_n}) \varphi(f_1, \ldots, f_k) . \tag{1.4}$$

As before, the functions $\varphi(f_1, \ldots, f_k)$ are defined on $X = X_1 \times \ldots \times X_n$ and should be properly written as $\varphi(f_1, \ldots, f_k) \circ 1_X^k$.

We observe first of all that Theorems 1.1, 1.2 and Corollary 1.3 may be derived, by induction, from their special cases of $k = 2$. Then, (1.1) becomes (0.3) and (1.3) becomes (0.5)—the original conditions of Butazzo and Dal Maso.

Notice that the assumptions on φ are, in some sense, also necessary. Rather than to give a formal statement in this sense we consider two examples. Let $\varphi: L \to L$ be not a joint semicomplete lattice homomorphism: there exists a subset A of L such that $\sup \varphi(A) \not\geqslant \not\geqslant \varphi(\sup A)$. By taking the principal filter $\mathcal{N}_\iota(A)$ of A, we have $\Gamma(\mathcal{N}_\iota(A)^+) \varphi(1_L) \not\geqslant \varphi\big(\Gamma(\mathcal{N}_\iota(A)^+) 1_L\big)$. This is a counterexample to (1.2) in the case where $n = 1$ or, in other words, where φ is constant for all variables but one. In the particular case of $L = \overline{\mathbb{R}}$, φ which is not a joint semi complete lattice homomorphism satisfies $\sup \varphi(A) < \varphi(\sup A)$ for some A, and we may suppose that $A = [-\infty, a_0)$. By taking the filter $\mathcal{F}$ generated by $(r, a_0)_{r<a_0}$ we see that $\Gamma(\mathcal{F}^-)\varphi(1_{\overline{\mathbb{R}}}) = a_0$ while $\varphi(\Gamma(\mathcal{F}^-)1_{\overline{\mathbb{R}}}) = \varphi(a_0) > a_0$.

2. Semifilters and limitoids.

A non empty family $\mathcal{A}$ of subsets of a set X is called a *semifilter* on X, if $\phi \notin \mathcal{A}$ and if $A \in \mathcal{A}$ and $A \subset B$ implies that $B \in \mathcal{A}$. The *grill* $\mathcal{A}^{\#}$ of a semifilter $\mathcal{A}$ on X is the family of all the subsets of X that meet every element of $\mathcal{A}$. Semifilters $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ *meet*, in symbols,

$$A_1 \# A_2 \# \ldots \# A_n ,$$

if for each $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2, \ldots, A_n \in \mathcal{A}_n$, $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$.

Let L be a (completely distributive, complete) lattice. Following Greco [5], we call an *L-limitoid in X* every mapping $T: L^X \to L$ which is isotone, commutes with each complete lattice homomorphism $\sigma: L \to L$ ($T(\sigma \circ g) = \sigma T(g)$, for each $g \in L^X$) and such that $T(g)$ lies, for each g, in the complete lattice generated by $g(X)$.

Let $\mathcal{A}$ be a semifilter on X. The functional $\Gamma(\mathcal{A}): L^X \to L$ defined by

$$\Gamma(\mathcal{A})f = \sup_{A \in \mathcal{A}} \inf_{x \in A} f(x) , \quad f \in L^X, \tag{2.1}$$

is an L-limitoid and, by the Greco representation theorem [5], every L-limitoid in X is of the form (2.1); $\mathcal{A}$ is called the *carrier* of that limitoid.

Moreover, (2.1) establishes a complete lattice isomorphism between the lattice of semifilters (ordered by inclusion) and the lattice of L-limitoids [5].

By allowing $\mathcal{F}_1, \ldots, \mathcal{F}_n$ in (0.1) to be arbitrary semifilters, Greco considered an extension of Γ-limits. They are L-limitoids and the carrier of $\Gamma(\mathcal{A}_1^{\alpha_1}, \ldots, \mathcal{A}_n^{\alpha_n})$ is denoted by $(\mathcal{A}_1^{\alpha_1}, \ldots, \mathcal{A}_n^{\alpha_n})$. Recall that ([5], [6]), $(\mathcal{A}_1^{\alpha_1}, \ldots, \mathrm{A}_n^{\alpha_n})^{\#} = (\mathcal{A}_1^{-\alpha_1}, \ldots, \mathcal{A}_n^{-\alpha_n})$ and that

$$(\mathcal{A}^-, \mathcal{B}^-) = \mathcal{A} \times \mathcal{B} , \quad (\mathcal{A}^-, \mathcal{B}^+) = (\mathcal{A}^{\#} \times \mathcal{B})^{\#} ,$$

where $\mathcal{A}\times\mathcal{B}$ stands for the semifilter generated by $\{A\times B: A\in\mathcal{A}, B\in\mathcal{B}\}$.

We shall use in the sequel the following consequence of the above facts

$$\sup_{H\in\mathcal{A}^\#}\inf_{x\in H} f(x) = \inf_{A\in\mathcal{A}}\sup_{x\in A} f(x)\,. \tag{2.2}$$

3. Proofs.

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be semifilters on X, $\varphi\colon L^2\to L$ a mapping and let $f, g\in L^X$. Denote by 1_X the identity on X.

3.1. Lemma. *If φ is a joint semi complete lattice homomorphism then*

$$\varphi\big(\Gamma(\mathcal{A})f, \Gamma(\mathcal{B})g\big)\leqslant\Gamma(\mathcal{C})\varphi(f, g)\circ(1_X\cdot 1_X) \tag{3.1}$$

provided that, for each $A\in\mathcal{A}$ and $B\in\mathcal{B}$, $A\cap B\in\mathcal{C}$.

Proof. Since φ is a joint semi complete lattice homomorphism, we have

$$\varphi\big(\Gamma(\mathcal{A})f, \Gamma(\mathcal{B})g\big)\leqslant\sup_{\substack{A\in\mathcal{A}\\ B\in\mathcal{B}}}\inf_{\substack{x\in A\\ y\in B}}\varphi\big(f(x), g(y)\big) = \Gamma(\mathcal{A}\times\mathcal{B})\varphi(f, g)\,.$$

On the other hand,

$$\Gamma(\mathcal{C})\,[\varphi(f, g)\circ(1_X\cdot 1_X)] = \Gamma\big((1_X\cdot 1_X)\,\mathcal{C}\big)\varphi(f, g)\,.$$

As $\mathcal{A}\times\mathcal{B}\subset(1_X\times 1_X)\mathcal{C}$ if and only if, for each $A\in\mathcal{A}$ and $B\in\mathcal{B}$, $A\cap B\in\mathcal{C}$, we conclude that (3.1) holds, because of the isomorphism between limitoids and their carriers. ■

Proof of Theorem 1.1. As noticed, it is enough to consider the case $k=2$. In order to apply Lemma 3.1 we shall show that if, for each $i=1,\ldots,n$,

$$\alpha_i+\beta_i\leqslant\gamma_i-1\,, \tag{0.3}$$

then

$$A\in(\mathcal{F}_1^{\alpha_1},\ldots,\mathcal{F}_n^{\alpha_n})\,,\qquad B\in(\mathcal{F}_1^{\beta_1},\ldots,\mathcal{F}_n^{\beta_n}) \tag{3.2}$$

implies $A\cap B\in(\mathcal{F}_1^{\gamma_1},\ldots,\mathcal{F}_n^{\gamma_n})$.

Let $n = 1$. We have two cases: either $\alpha_1 = \beta_1 = \gamma_1 = -1$ or $\gamma_1 = +1$ and one of α_1, β_1 is -1, say, $\alpha_1 = -1$, $\beta_1 = +1$. In the first case, (3.2) follows from the fact that $\mathcal{F}_1$ is a filter; in the second case, we have that $A \in \mathcal{F}_1$, $B \in \mathcal{F}_1^{\#}$ implies $A \cap B \in \mathcal{F}_1^{\#}$, again because $\mathcal{F}_1$ is a filter.

Suppose that (3.2) holds for some n. Denote, for the sake of brevity,

$$\mathcal{A} = (\mathcal{F}_1^{\alpha_1}, \dots, \mathcal{F}_n^{\alpha_n}), \quad \mathcal{B} = (\mathcal{F}_1^{\beta_1}, \dots, \mathcal{F}_n^{\beta_n}), \quad \mathcal{C} = (\mathcal{F}_1^{\gamma_1}, \dots, \mathcal{F}_n^{\gamma_n}).$$

Consider the case of $\alpha_{n+1} = \beta_{n+1} = \gamma_{n+1} = -1$. Take $A \in \mathcal{A}$, $B \in \mathcal{B}$, $F, F' \in \mathcal{F}_{n+1}$. Since $F \cap F' \in \mathcal{F}_{n+1}$ and, by the induction hypothesis, $A \cap B \in \mathcal{C}$, we have that $(A \times F) \cap (B \times F') \in \mathcal{C} \times \mathcal{F}_{n+1}$.

Consider now the case $\alpha_{n+1} = -1$, $\beta_{n+1} = \gamma_{n+1} = +1$. We need prove that the intersection of arbitrary sets A from $\mathcal{A} \times \mathcal{F}_{n+1}$ and B from $(\mathcal{B}^{\#} \times \mathcal{F}_{n+1})^{\#}$ belongs to $(\mathcal{C}^{\#} \times \mathcal{F}_{n+1})^{\#}$. Equivalently, we need show that

$$[\mathcal{A} \times \mathcal{F}_{n+1}] \,\#\, [(\mathcal{B}^{\#} \times \mathcal{F}_{n+1})^{\#}] \,\#\, [\mathcal{C}^{\#} \times \mathcal{F}_{n+1}]. \tag{3.3}$$

By the induction hypothesis, $\mathcal{A} \# \mathcal{B} \# \mathcal{C}^{\#}$, hence, for each $A \in \mathcal{A}$ and $H \in \mathcal{C}^{\#}$, $A \cap H \in \mathcal{B}^{\#}$. If now $F, F' \in \mathcal{F}_{n+1}$, then

$$(A \times F) \cap (H \times F') = (A \cap H) \times (F \cap F') \in \mathcal{B}^{\#} \times \mathcal{F}_{n+1}.$$

As, for an arbitrary semifilter $\mathcal{D}, \mathcal{D}^{\#\#} = \mathcal{D}$, the above fact implies (3.3). ■

3.2. LEMMA. *Theorems* 1.1 *and* 1.2 *are equivalent.*

PROOF. Clearly, it is enough to assume that $k = 2$. Suppose that Theorem 1.1 is valid. We need deduce from that (for $k = 2$) the inverse of (1.2) holds if $\alpha_i + \beta_i \geqslant \gamma_i + 1$ for $i = 1, \dots, n$.

Consider now the dual order on L. Equipped with it, L is completely distributive and complete, joints in the original order become now meets, so that Theorem 1.1 holds for that order.

As $(-\alpha_i) + (-\beta_i) \leqslant (-\gamma_i) - 1$, we may apply it to the functionals

$$\Gamma(\mathcal{F}_1^{-\alpha_1}, \dots, \mathcal{F}_n^{-\alpha_n}),\ \Gamma(\mathcal{F}_1^{-\beta_1}, \dots, \mathcal{F}_n^{-\beta_n}) \text{ and } \Gamma(\mathcal{F}_1^{-\gamma_1}, \dots, \mathcal{F}_n^{-\gamma_n}).$$

Set $\mathcal{A} = (\mathcal{F}_1^{\alpha_1}, \dots, \mathcal{F}_n^{\alpha_n})^{\#}$, $\mathcal{B} = (\mathcal{F}_1^{\beta_1}, \dots, \mathcal{F}_n^{\beta_n})^{\#}$ and $\mathcal{C} = (\mathcal{F}_1^{\gamma_1}, \dots, \mathcal{F}_n^{\gamma_n})^{\#}$.

Then (1.2) becomes (3.1) and, returning to the original order, (3.1) becomes

$$(3.4) \qquad \varphi(\Gamma(\mathcal{A}^{\#})f, \Gamma(\mathcal{B}^{\#})g) \geqslant \Gamma(\mathcal{C}^{\#})\varphi(f, g)\circ(1_X \cdot 1_X),$$

which amounts to the inverse of (1.2).

In an analogous way, one derives Theorem 1.1 from Theorem 1.2. ■

Proof of Corollary 1.3. The left-hand side of (1.4), for $k = 2$ is equal to

$$\operatorname*{ext}^{\alpha_1}_{(x_{1l})\in\varepsilon\mathcal{F}_1} \operatorname*{ext}^{\beta_1}_{(y_{1l})\in\varepsilon\mathcal{F}_1} \operatorname*{ext}^{\alpha_n}_{(x_{nl})\in\varepsilon\mathcal{F}_n} \operatorname*{ext}^{\beta_n}_{(y_{nl})\in\varepsilon\mathcal{F}_n} \varphi[\Gamma(\mathcal{N}^{\alpha_0})f(l, x_{1l}, \ldots, x_{nl}), \Gamma(\mathcal{N}^{\beta_0})g(l, y_{1l}, \ldots, y_{nl}).$$

If (0.3) holds, the above is less or equal to

$$\operatorname*{ext}^{\gamma_1}_{(x_{1l})\in\varepsilon\mathcal{F}_1} \ldots \operatorname*{ext}^{\gamma_n}_{(x_{nl})\in\varepsilon\mathcal{F}_n} \varphi[\Gamma(\mathcal{N}^{\alpha_0})f(l, x_{1l}, \ldots, x_{nl}), \Gamma(\mathcal{N}^{\beta_0})g(l, x_{1l}, \ldots, x_{nl})]$$

and we complete the proof by applying Theorem 1.1 to the above formula. ■

4. An example of applications.

We shall show how to apply our theorems to the type of problems considered at the beginning of the paper. For a subset A of X define the function ν_A by

$$\nu_A(x) = \begin{cases} -\infty, & \text{if } x \in A, \\ +\infty, & \text{otherwise}. \end{cases}$$

The upper restriction g_A of an arbitrary extended-real-valued function g on X to the set A may be represented in two ways

$$g_A = g \dot{+} \psi_A = g \vee \nu_A$$

(g_A is equal to g on A and to $+\infty$ elsewhere). Hence, by applying our theorems in the case of $\varphi =$ supremum, we may obtain, for instance, 4.1. THEOREM. *Let g be differentiable. The contingent epi-derivative of the restriction of g to a set A is equal to the restriction of the derivative of g to the contingent cone of A.*

PROOF. Since g is differentiable, we have that the derivative Dg is equal $D_{(-,-)}g = D_{(+,+)}g$, where

$$(D_{(+,+)}g)(x)h = \inf_{\substack{t>0 \\ Q\in\mathcal{N}(h)}} \sup_{\substack{0<t'<t \\ h'\in Q}} \frac{1}{t'}\big(g(x+t'h')-g(x)\big).$$

On one hand, by applying Theorem 1.1 with $\alpha=\beta=\gamma=-1$ to the functions $(1/t')[g(x+t'h')-g(x)]$ and $(1/t')\big(\nu_A(x+t'h') \dotplus -\nu_A(x)\big)$ and for the homomorphism $\vee$ (the supremum of these difference quotients is equal to the difference quotient of g_A), we get

$$(D_{(-,-)}g_A)(x) \geqslant Dg(x)\vee\nu_{T_{(-,-)}A}(x) = \big(Dg(x)\big)_{T_{(-,-)}A(x)}.$$

On the other, by applying Theorem 1.2 with

$$\alpha_1=\alpha_2=+1\,, \quad \beta_1=\beta_2=-1 \quad \text{and} \quad \gamma_1=\gamma_2=-1$$

we get $Dg(x)\vee\nu_{T_{(-,-)}A}(x) \geqslant (D_{(-,-)}g_A)(x)$. ■

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