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A Remark on Subgroup Lattices of Finite Soluble Groups.

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Introduction.

It is a long standing conjecture that every finite lattice is isomorphic to a section of the subgroup lattice of a suitable finite group. This note does not aim at proving or disproving this conjecture but to provide material for the fact that the conjecture is false if restricted to finite soluble groups. In particular, we want to improve knowledge about the structure of minimal nontrivial sections.

Subgroups of soluble groups are soluble. It suffices therefore to consider the set of all subgroups of G containing a given subgroup U; this section is nontrivial whenever U is not a maximal subgroup of G. We will see (Lemma 1) that a minimal nontrivial section includes a maximal chain of length two, and that there is at most one neighbour of the maximal element belonging to a longer chain. The number of neighbours of the maximal element is not arbitrary (Lemma 2) and restricts the form of the proper subsections. Also pairs of minimal sections having a member different from G in common can not be chosen arbitrarily (Theorem 2 and 3).

The results given here depend strongly on the famous theorem of 0. Ore on maximal subgroups of finite soluble groups. More attention to the way G operates on a chief factor is paid in the particular situation that $U \cap V$ is not the proper intersection of two subgroups of U (Theorem 1). Examples show that there are still many possibilities left.

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Conventions. If $A \leqslant B$, the smallest convex sublattice of the subgroup lattice of B containing A and B is denoted by $\{B; A\}$. We will also use the word « section » in this situation.

Let L be maximal in M wich is in turn maximal in G. If $T \supset L$ is a maximal subgroup of G and L is not maximal in T, we will say that T is *exceptional* in $\{G; L\}$. If furthermore L is contained in more than two maximal subgroups of G, we call $\{G, L\}$ *extensive.*

The linearly ordered lattice (« chain ») consisting of $k+1$ members will be denoted by $\mathcal{C}_k$; the modular lattice consisting of $n+2$ members among which are n pairwise incomparable by $\mathcal{M}_n$. This nomenclature is introduced for briefness only. For distinction, Z_n is the cyclic group of order n. Other group theoretic notation is standard; all groups considered finite and soluble.

1. We remind the reader here of the Theorem of Ore to have it at our disposal, and we will incidate first consequences.

Theorem of Ore. Let G be a finite soluble group and M a maximal subgroup of G and M_G the intersction of all conjugates of M. Then we have

(1) G/M_G possesses one and only one minimal normal subgroup,

(2) If S is another maximal subgroup of G and $S_G = M_G$, then S and M are conjugate in G.

For a proof see Ore [Theorem 11, 12, p. 451, 452]. We will denote the minimal normal subgroup mentioned in (1) by M^*/M_G; hence M^* is unique with respect to $M^*M = G$, $M^* \cap M = M_G$.

We make a first use of this in

Lemma 1. Assume that there are two different maximal subgroups of the group G conatining a given subgroup U. Then there are two different maximal subgroups R, S of G containing U such that $R \cap S$ is a maximal subgroup of R.

Corollary. Minimal sections with at least two different maximal objects possess at least one maximal chain of lenght two.

Proof. We consider two different maximal subgroups A and B of G which contain U. We distinguish at first two cases: $A_G = B_G$ or not.

(i) $A_G = B_G$.

By (2) of the theorem of Ore, A and B are conjugate, and $A^* = B^*$. Now $(A \cap B)A^*$ is a proper subgroup of G, since $(A/A_G)(A^*/A_G) = G/A_G$ is a semidirect product. There is a maximal subgroup C of G containing $(A \cap B)A^*$. Obviously $U \subset C$ and $C_G \supseteq A^* \supset A_G$. We have therefore reduced this case to the second one, namely

(ii) $A_G \neq B_G$.

Without loss of generality we may assume $A_G \not\subseteq B_G$.
Then

$$A_G B_G = A_G B^*,$$

and since

$$A^*/A_G \cong A^*B^*/A_G B^*$$

is a chief factor of G, so is

$$(A^*B^* \cap B)/A_G B^* \cap B)$$

a chief factor of B, and the corresponding subgroup of B avoiding this chief factor, namely $A \cap B$, is maximal in B.
The Lemma is proved for $A = S$ and $B = R$.

2. We will now take stock of the different minimal top sections that are possible.

LEMMA 2. Assume that $\{G; U\}$ is a section of $L(G)$ containing more than one maximal subgroup of G and that $\{G; U\}$ is minimal among all sections $\{G; V\}$ in this respect. Then there is a prime p such that

(1) the number of maximal subgroups containing U is $1 + q$, where q is a power of p,

(2) there is at most one exceptional maximal subgroup in the section $\{G; U\}$, K, say, and $\{K; U\}$ is modular with minimal non-distributive subsections isomorphic to $\mathcal{M}_{1+p^t}$, where t is variable,

(3) if $q \neq 1$ and there is an exceptional maximal subgroup K, then there is a subsection of $\{K; U\}$ which is isomorphic to the subgroup lattice of the elementary abelian group of order q.

Proof. We take two maximal subgroups A, B of G which contain U and are not conjugate, By Lemma 1 this is possible. Minimality of the section yields $U = A \cap B$. Again we distinguish two cases: A_G and B_G are incomparable or comparable.

Case 1. A_G and B_G are incomparable. In this case

$$G/A_G \cap B_G$$

possesses two minimal normal subgroups $A^* \cap B_G/A_G \cap B_G$ and $A_G \cap B^*/A_G \cap B_G$ such that any minimal normal subgroup is contained in the product of these two. If $R/A_G \cap B_G$ is such a minimal normal subgroup, $R(A \cap B)$ is a maximal subgroup of G, and all maximal subgroups of G containing $A \cap B = U$ are obtained in this manner. It is well known that such further minimal normal subgroups exist if and only if $A^* \cap B_G/A_G \cap B_G$ and $A_G \cap B^*/A_G \cap B_G$ are operator isomorphic in $G/A_G \cap B_G$, and that the number of minimal normal subgroups is in this case $1 + q$, where q is the number of minimal normal of central G-endomorphisms in the factor, a divisor of the order of the factor. Since chief factors are p-groups, q is a power of a prime. If the two factors are not operator isomorphic, $q = 1$. The section is modular and no lower neighbour of G is exceptional.

Case 2. $A_G \supset B_G$. In this case $B \cap A$ is maximal in B and $A \supseteq A_G \supseteq$ $\supseteq B^*$. If $x \in N(A \cap B) \cap A$, then $x^{-1} Bx \cap A = B \cap A$, and if x is not contained in B, the two maximal subgroups B and $x^{-1} Bx$ are different since B is selfnormalizing. To find the number of these conjugates of B, we need the index $|A \cap N(A \cap B) : A \cap B|$. Now

$$A = (A \cap B)B^* = (A \cap N(A \cap B))B^* \subseteq A$$

and

$$A \cap N(A \cap B)/(A \cap B) = B^* \cap N(A \cap B)/B^* \cap (A \cap B) =$$
$$= B^* \cap N(A \cap B)/B_G.$$

Since B^*/B_G is an elementary abelian p-group, there are some power of p many conjugates of A containing $A \cap B$. On the other hand $A = (A \cap B)B^*$, and there are no conjugates of A different from A containing $A \cap B$. Since maximal subgroups containing $A \cap B$ avoid one of the two chief factors A^*/A_G or B^*/B_G, we find $1 + p^k$ maximal subgroups containing $A \cap B$ (where again p^k may be 1).

We have found that $U = A \cap B$ is a maximal subgroup of B and all its conjugates. The only exceptional lower neighbour of G in the section $\{G; U\}$ is therefore A if there is one at all. If $A \cap B \subset T \subset A$, then by

$$A = A \cap BB^* = (A \cap B)B^*,$$

$$T = T \cap A = T \cap (A \cap B)B^* = (A \cap B)\,(T \cap B^*),$$

and $T \cap B^*$ is a subgroup of A which contains B_G. So the section $\{U; A\}$ is isomorphic to the lattice of all A-invariant subgroups contained in B^* and containing B_G. This is a modular lattice, and since B^*/B_G is an elementary abelian p group, minimal nondistributive subsections possess $1 + p^m$ lower neighbours of the top element, where m varies. This follows from case 1 of this proof. The prime occurring in (1) and (2) is the one dividing the order of B^*/B_G, so both primes are the same.

If there is an exceptional maximal subgroup, the number q is the index $|A \cap N(A \cap B): A \cap B|$.

Since $A = (A \cap B)B^*$ we have

$$A \cap N(A \cap B) = (B^* \cap N(A \cap B))\,(A \cap B)$$

and

$$(B^* \cap N(A \cap B)) \cap (A \cap B) = B_G.$$

So $A \cap N(A \cap B)/B_G$ is the direct product of $(B^* \cap N(A \cap B))/B_G$ and

$$(A \cap B)/B_G,\ (B^* \cap N(A \cap B))/B_G = B^*/B_G \cap Z(A/B_G)$$

in particular

$$\{A \cap N(A \cap B);\, A \cap B\} = \{B^* \cap N(A \cap B);\, B_G\}.$$

This yields (3).

LEMMA 3. Assume that U and V are maximal subgroups of G and $[G: U \cap V]$ is a nondistributive modular section. Then

either

(i) U^*/U_G and V^*/V_G are operator isomorphic and $U^* = V^*$, the number of maximal subgroups containing $U \cap V$ is $1 + q$, where q is a prime power,

or

(ii) the number of maximal subgroups containing $U \cap V$ is $1 + p$, where p is a prime, and U and V can be chosen such that

$$U = U_G = V^*,$$

$$|V^*/V_G| = p,$$

$$|G/U| \text{ is a prime dividing } p-1.$$

Proof. We distinguish two cases: (i) no two maximal subgroups containing $U \cap V$ are conjugate, (ii) there is a conjugate of, say, U containing $U \cap V$.

In case (i), we have at least three maximal subgroups U, V, W containing $U \cap V$, and $U_G \cap V_G = V_G \cap W_G = W_G \cap U_G = (U \cap V)_G$ (since no two of them are conjugate).

Now U^* contains $U_G V_G W_G$ and U^*/U_G is the unique minimal normal subgroup of G/U_G. Since all three U_G, V_G, and W_G are incomparable, we find

$$U^* = U_G V_G W_G$$

and by symmetry

$$V^* = U_G V_G W_G,$$

so $U^* = V^* (= W^*)$,

and the three quotients $U_G/(U \cap V)_G$, $V_G/(U \cap V)_G$ and $W_G/(U \cap V)_G$ are operator isomorphic. On the other hand, if $S/(U \cap V)_G$ is some minimal normal subgroup of $G/(U \cap V)_G$ contained in $U^*/(U \cap V)_G$, then $(U \cap V)S$ is another maximal subgroup of G containing $U \cap V$. The number of maximal subgroups containing $U \cap V$ is therefore the number of such normal subgroups, which in turn is $1 + q$, where q is the number of operator isomorphisms of $U_G/(U \cap V)_G$ into $V_G/(U \cap V)_G$, a power of p if the quotients are p-groups.

In case (ii) and if U is not conjugate to V, we have

$$V_G = U^* .$$

Since $[G: U \cap V]$ is modular, $U \cap V$ is maximal in U, and so

$$U \cap \mathcal{N}(U \cap V) \neq U \cap V$$

is not a proper subgroup of U, therefore $U \cap V$ is a normal subgroup of U and $U/(U \cap V)$ is a cyclic group of prime order. Since there are conjugates of U containing $U \cap V$, which obviously satisfy the same conditions as U, and since U is maximal in G, we find that $U \cap V$ is a normal subgroup of G, also U^*, as subgroup of prime index, is a maximal subgroup of G and so identical with V. Since there are no proper subgroups of $V/(U \cap V)$, this quotient group is also of order a (different) prime, and this order coincides with the number of conjugates of the sylow subgroup $U/(U \cap V)$ of $G/(U \cap V)$. This concludes the proof: the cases (i) and (ii) are exactly those of the lemma.

3. We consider here the case that $U \cap V$ has only one upper neighbour in $\{U; U \cap V\}$ and find restrictions on $\{G; U \cap V\}$.

THEOREM 1. Let U and V be two maximal subgroups of the soluble group G such that

(i) $U \cap V$ is maximal in V,

(ii) there is only one upper neighbour K of $U \cap V$ in $\{U; U \cap V\}$,

(iii) there are $r > 2$ lower neighbours of G in $\{G; U \cap V\}$.

Then

(*a*) $p = r - 1$ is a prime,

(*b*) $F_{r-2} \supseteq K$, where $F_0 = U$ and for all i, F_{i+1} is the intersection of all lower neighbours of F_i in $\{U; U \cap V\}$,

(*c*) if furthermore $F_{r-2} = K$, then $\{U; U \cap V\} \simeq \mathfrak{C}_{r-1}$.

PROOF. By (iii), $V_G = U^*$ and $N(U \cap V) \neq U \cap V$. We may assume $U_G = 1$. Since $N(U \cap V) = (U \cap V)\,(C(U \cap V) \cap U^*)$, the section $\{N(U \cap V);\ U \cap V\}$ is isomorphic to the subgroup lattice of some elementary abelian p-group. By (ii), $C(U \cap V) \cap U^* = K$, which is therefore of order p. So the number $r-1$ of conjugates of V containing $U \cap V$ is this prime p, and (a) is shown.

By (ii), U^* is a noncyclic p-group; it is the direct product of suitably chosen minimal normal subgroups of V^*. Forming the products L_i of those minimal normal subgroups of V^* which are operator isomorphic we have with

$$U^* = \sum_{i-1}^{t} L_i$$

a description of U^* as a direct product with factors that are permuted by G. Since U^* is a minimal normal subgroup of G, we have

$$K^G = U^*,$$

and for $H_j = \sum_{\substack{i=l \\ i \neq j}}^{t} L_i$ we find

$$K \not\subseteq H_j \qquad \text{and} \qquad |L_j \cap KH_j| = p\,.$$

Now $(L_j \cap KH_j)^{V^*}$ is a minimal mornal subgroup of V^*, and

$$(L_j \cap KH_j)^{V^*(U \cap V)} = (L_j \cap KH_j)^G = U^*\,.$$

So

$$(L_j \cap KH_j)^{V^*} = L_j$$

and any two different minimal normal subgroups of V^* are not operator isorphic.

Choose a minimal normal subgroup Q of $U \cap V$. If Q is not a p-group, we have

$$[Q, U^*] \times (C(Q) \cap U^*) = U^*,$$

and since the right factor is different from 1 by (iii) and different from U^* by uniqueness of U^* as minimal normal subgroup, we find that the direct product is nontrivial, a contradiction to the existence of K as described in (ii). So Q is a p-group. Assume now the existence of some element y in Q which does not leave all L_i invariant by conjugation. Then y will permute p different of the L_i's into each other, let L_j be one of them. So $[L_j, y] \neq 1$, and if

$$[L_j, y] = [L_{j,1}\, y]\,,\ [[L_{j,k}y], y] = [L_{j,k+1}y]\,,$$

then

$$1 = [L_{j,p}y] \neq [L_{j,p-1}y]\,.$$

On the other hand, if all y in Q leave all L_i invariant, then if y is different from 1 it is not contained in $C(V^*/U^*)$ and so y operates as a nontrivial power automorphism on some $V^*/C(L_j) \cap V^*$. Now y will permute a suitable basis of L_j and again

$$[L_{j,p-1}y] \neq 1\,.$$

Now obviously

$$[U^*,_{p-1}Q] \neq 1\,,$$

on the other hand

$$[F_i, Q] \subseteq F_{i+1}$$

since all minimal factors are centralized by Q. We derive

$$F_{r-2} = F_{p-1} \supseteq [U^*,_{p-1}Q] \supseteq K$$

showing (*b*).

If furthermore $F_{p-1} = K$, we have two possibilities:

Case A: U^* is a minimal normal subgroup of V^*.

Then V^*/U^* is cyclic of order some prime q and $G/V^* \cong U \cap V$ is cyclic of order dividing $q-1$ and possesses no minimal normal subgroups of order prime to p. We deduce that G/V^* is of order p exactly and q divides $p^p - 1$ (but not $p-1$).

Case B: U^* is not minimal normal in V^*.

Then the minimal normal cubgroups of V^* have order p and V^*/U^* is a q-group where q divides $p-1$. The element $y \neq 1$ of Q wiol permute exactly p of the L_i's and

$$[U^*,_{p-1}Q] = [U^*,_{p-1}y] = K\,.$$

So U^* is the direct product of p factors L_i, and G/V^* is a (nontrivial) normal subgroup of $\mathrm{Hol}(Z_p)$. Now (*c*) follows easily.

REMARKS. Condition (iii) is indispensable for (*b*) as is shown by $\mathrm{Hol}(Z_3 \times Z_3)$ and its maximal subgroups of index 4 and 9. Also the equality $F_{r-2} = K$ is indispensable in (*c*): For this consider the maximal subgroups of index 2^8 and 9 of the extension of the elementary abelian group of order 2^8 by $(\mathrm{Hol}(Z_3 \times Z_3))^2$.

4. *Examples.* We will use a simple construction principle to provide us with a wider range of examples. We begin with a well known general statement.

LEMMA 4. Let A and B be two groups, M/N a chief factor of A. We consider the factor M^B/N^B of the wreath product A wr B.

(i) If M/N is not central, then M^B/N^B is a non-central chief factor of A wr B.

(ii) If M/N is a central chief factor of A, the lattice of normal subgroups of A wr B lying between N^B and M^B is isomorphic to the lattice of B-invariant subgroups of $(M/N)^B$ in (M/N) wr B.

The proof follows easily from the definition of the standard wreath product.

COROLLARY 1. Consider two groups A and B with the following properties:

(1) A possesses only one minimal normal subgroup, denote it by K, further K is a p-group and complemented by T.

(2) A/K possesses only one minimal normal subgroup L/K which is complemented by S/K.

(3) $[A, K] = K$, $[A, L] = L$.

(4) $[S, K] \cap Z(S) = 1$.

Denote by $\mathfrak{L}$ the lattice of all B-invariant subgroups of $(Z(S) \cap K)$ *wr* R which are contained in $(Z(S) \cap K)^B$. Then we find for the wreath product A wr B:

L^B/K^B and $K^B/1 = K^B$ are chief factors of A wr B with

complements $T^B B$ and $S^B B$, and

$$\mathfrak{S} = \{(T^B B \cap S^B B)\, K^B;\ (T^B B \cap S^B B)\} = \mathfrak{L} \times \{(S \cap T)\,[S, K];\ (S \cap T)\},$$

furthermore $|Z(S) \cap K| = |Z(S^B B) \cap K^B|$.

PROOF. By (4), there is no central chief factor of S in $\{[S, K]; 1\}$, so this section is transferred directly into the wreath product according to Lemma 6 (i). For $Z(S) \cap K$ we apply Lemma 6 (ii), obtaining also the order of $Z(S^B B) \cap K^B$.

For the examples formed according to the Corollary we will only state A and B and the section $\mathfrak{S} = \{(T^B R \cap S^B R) K^B; (T^B R \cap S^B R)\}$, the subgroups S, T and K being clear from the context.

(I) Take a prime p different from 2 and 3 and define

$$A = (D_{2p} \operatorname{wr} Z_3)^2 \quad \text{and} \quad R = Z_{p^t q}.$$

If $p \equiv 1 \bmod 3$, we obtain $\mathfrak{S} = \mathfrak{C}_1 \times \mathfrak{C}_1 \times (\underbrace{\mathfrak{C}_{p^t} \times \ldots \times \mathfrak{C}_{p^t}}_{k \text{ times}})$, if $p \equiv -1$ mod 3 one direct factor $\mathfrak{C}_1$ is missing. The number k is the number of irreducible factors of $x^q - 1$ in $Z_p[x]$.

(II) Choose $A = (D_6 \operatorname{wr} Z_5)^2$ and $B = Z_{3^t q}$ to obtain the analogous result (with one factor $\mathfrak{C}_1$) for $\mathfrak{S}$, with 3 substituted for p.

(III) Choose $A = \operatorname{Aut}(F)$, where F is the group of all one-to-one transformations $y \to ay + b$ of the field of order 8 onto itself, and $B = Z_{2^t q}$ to obtain the analogous result (with one factor $\mathfrak{C}_1$) for $\mathfrak{S}$, with 2 substituted for p.

(IV) Choose two different odd primes p, r. Construct A such that $|K| = p^q$, $|L/K| = 2^m$, $|A/L| = r$ and all these factors are chief factors of A, also K and L/K are non central. If $B = Z_{p^t q}$, we have again that $\mathfrak{S}$ is a direct product of chains $\mathfrak{C}_1$ and $\mathfrak{C}_{p^t}$, where the number of factors $\mathfrak{C}_1$ is the number of prime factors of $x^r - 1$ in $Z_p[x]$.

(V) In this example we will not use a wreath product: choose a prime p and a prime divisor q of $p^{p^n} - 1$ but not of $p^{p^{n-1}} - 1$ (this q exists since $\gcd(p^{p^n} - 1, (p^{p^{n-1}} - 1)^2) = \gcd(p^{p^n} - 1, p^{p^{n-1}} - 1)$). The noncentral extension N of the elementary abelian p-group M of rank p^n by the cyclic group of order q admits an automorphism of order p^n. Consider as G the extension of M by this automorphism; choose as maximal subgroups a p-sylow subgroup U and the normalizer V of a q-sylow subgroup. Now $\{U; U \cap V\}$ is a chain $\mathfrak{C}_{p^n}$.

COROLLARY 2. Consider two groups A and B with the notation and properties as in Corollary 1 except for (4); instead we impose

$$(4^*) \qquad 1 = R_0 \subset R_1 \subset \ldots\ldots\ldots \subset R_t = K$$

such that R_{i+1}/R_i are chief factors of S and $[R_i, S] = R_{i-1}$.

Then $\mathfrak{S} \simeq \mathfrak{L}^t$, the successive multiple extension of $\mathfrak{L}$ by itself, t times in all.

PROOF. By construction, all R_{i+1}/R_i are cyclic, and

$$\{(T^B B \cap S^B B)\, R_{i+1}B;\ (T^B B \cap S^B B)\, R_i^B\} = \mathfrak{L}$$

It is therefore sufficient to prove the following statement:

(*) If X is a $(T^B B \cap S^B B)$ invariant subgroup of K which is not contained in R_i^B, then X contains R_i^B.

For the moment we fix one direct factor H of the product A^B in A wr R for our consideration. We may suppose without loss of generality that X is contained in R_{i+1}^B. If t is an element of X which is not contained in R_i^B, there is at least one component of t, consisered ad element of the direct product A^B, which is not contained in R_i^B. Conjugating by an element of B, if necessary, we obtain a new element t^* with a component y in Y which is not contained in R_i^B. Forming commutation of y with elements of $T^B \cap {}^B S \cap Y$ we conclude that $R_i^B \cap Y$ is contained in X. Now application of conjugation by elements of B yields that X contains R_i^B. This shows (*) and Corollary 2.

(VI) Choose $A = S_4$, and $R = Z_3$. If V and U are the complements of the lowest 2- and 3-chief factors respectively, $\{G;\ U \cap V\}$ possesses three neighbours of G, among which U is exceptional and $\{U;\ U \cap V\}$ is the extension of the Boole lattice $\mathcal{M}_2$ by itself.

(VII) Taking $A = S_4$ wr Z_3 and $B = Z_3$ yelds $\{U;\ U \cap V\}$ isomorphic to the extension of the Boole lattice with three atoms by itself; by induction the extension of any finite Boole lattice by itself can be acchieved.

5. Considering more than one top section at the same time leads to further relations. This is shown by the following two statements.

THEOREM 2. Assume that U, V and W are maximal subgroups of G and $\{G; (U \cap V)\}$ and $\{G\colon (U \cap W)\}$ are modular and nondistributive. If furthermore one of the sections is isomoprhic to $\mathcal{M}_{1+q}$ where q is not a prime, then $\{G; (U \cap V)\}$ and $\{G; (U \cap W)\}$ are isomorphic.

PROOF. Assume first that $\{G; (U \cap V)\}$ is siomorphic to $\mathcal{M}_{1+q}$, where q is not a prime but a prime power. Then U^*/U_G is not of prime order, so case (ii) of our lemma can not apply. Therefore U^*/U_G is operator isomorphic to V^*/V_G and to W^*/W_G. The number of maximal subgroups is one more than the number of operaror isomorphisms, which is the same in both these cases. So the sections are isomorphis.

REMARK. In the dihedral group G of order 30, choose $|U| = 15$, $|V| = 10$, $|W| = 6$. Then $\{G; (U \cap V)\}$ and $\{G; (U \cap W)\}$ are modular, but the sections are nonisomorphic. This shows the necessity of the condition on the number of elements in the theorem.

THEOREM 3.

(i) If U, V_1, and V_2 are maximal in G and $U \cap V_i$ is maximal in V_i, then the sections $\{U: U \cap V_i\}$ are modular with minimal nondistributive sections $\mathcal{M}_{1+p^t}$ (for the same prime p).

(ii) If in addition to the hypotheses of (i) there are more than two maximal subgroups containing $U \cap V_i$ (for $i = 1, 2$) and none of the $U \cap V_i$ is maximal in U, then $\{U; U \cap V_1\} = \{U; U \cap V_2\}$.

PROOF. Under the hypotheses of (i), $\{G; U \cap V_i\}$ is a minimal nontrivial top section, and $\{U; U \cap V_i\}$ is isomorphic to the lattice of all $(U \cap V_i)$-invariant subgroups of U^*/U_G, which is a p-group. This shows (i).

To see (ii), we find $(V_1)_G = U^* = (V_2)_G$ and so V_1 and V_2 are conjugate in G. Since $UV_1 = G$, we have that $U \cap V_1$ and $U \cap V_2$ are conjugate and the sections $\{G; U \cap V_1\}$ and $\{G; U \cap V_2\}$ are isomorphic with U mapped onto itself.

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