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H. M. SRIVASTAVA
SHIGEYOSHI OWA
S. K. CHATTERJEA

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A Note on Certain Classes of Starlike Functions.

H. M. SRIVASTAVA - SHIGEYOSHI OWA - S. K. CHATTERJEA (*)

SUMMARY - A number of distortion theorems are proved for the classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ of analytic and univalent functions with negative coefficients, studied recently by S. K. Chatterjea [1]. We also present several interesting results for the convolution product of functions belonging to the classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$.

1. Introduction.

Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathcal{N} = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. Further, let $\mathcal{S}(n)$ be the subclass of $\mathcal{A}(n)$ consisting of analytic and univalent functions in the unit disk $\mathcal{U}$. Then a function $f(z) \in \mathcal{S}(n)$ is said to be

(*) Indirizzo degli AA.: H. M. SRIVASTAVA: Department of Mathematics, University of Victoria, Victoria, British Columbia V8W 2Y2, Canada; SHIGEYOSHI OWA: Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577, Japan; S. K. CHATTERJEA: Department of Pure Mathematics, Calcutta University, Calcutta 700 019, India.

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in the class $\mathcal{S}_\alpha(n)$ if and only if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha\,, \qquad (z \in \mathcal{U}), \tag{1.2}$$

for $0 \leqq \alpha < 1$. Also a function $f(z) \in \mathcal{S}(n)$ is said to be in the class $\mathcal{K}_\alpha(n)$ if and only if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha\,, \qquad (z \in \mathcal{U}), \tag{1.3}$$

for $0 \leqq \alpha < 1$.

We note that $f(z) \in \mathcal{K}_\alpha(n)$ if and only if $zf'(z) \in \mathcal{S}_\alpha(n)$, and that

$$\mathcal{S}_\alpha(n) \subseteq \mathcal{S}_0(n)\,, \qquad \mathcal{K}_\alpha(n) \subseteq \mathcal{K}_0(n)\,, \qquad \text{and} \qquad \mathcal{K}_\alpha(n) \subset \mathcal{S}_\alpha(n)$$

for $0 \leqq \alpha < 1$.

The class of functions of the form (1.1) was considered by Chen [2]. The classes $\mathcal{S}_\alpha(1)$ and $\mathcal{K}_\alpha(1)$ were first introduced by Robertson [9], and were studied subsequently by Schild [10], Pinchuk [8], Owa and Srivastava ([6], [7]), and others (see, for example, Duren [3] and Goodman [4]).

Let $\mathcal{T}(n)$ denote the subclass of $\mathcal{S}(n)$ consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \qquad (a_k \geqq 0)\,. \tag{1.4}$$

Denote by $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{S}_\alpha(n)$ and $\mathcal{K}_\alpha(n)$ with $\mathcal{T}(n)$, that is,

$$\mathcal{T}_\alpha(n) = \mathcal{S}_\alpha(n) \cap \mathcal{T}(n) \qquad (0 \leqq \alpha < 1;\; n \in \mathcal{N}) \tag{1.5}$$

and

$$\mathcal{C}_\alpha(n) = \mathcal{K}_\alpha(n) \cap \mathcal{T}(n) \qquad (0 \leqq \alpha < 1;\; n \in \mathcal{N})\,. \tag{1.6}$$

The classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ were studied recently by Chatterjea [1]. In particular, $\mathcal{T}_\alpha(1)$ and $\mathcal{C}_\alpha(1)$ are the classes $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively, which were introduced by Silverman [12].

In order to prove our results for functions belonging to the general classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$, we shall require the following lemmas given by Chatterjea [1]:

LEMMA 1. *Let the function* $f(z)$ *be defined by* (1.4). *Then* $f(z)$ *is in the class* $\mathcal{T}_\alpha(n)$ *if and only if*

$$\sum_{k=n+1}^{\infty}\left(\frac{k-\alpha}{1-\alpha}\right)a_k \leqq 1\,, \qquad (n \geqq 1). \tag{1.7}$$

LEMMA 2. *Let the function* $f(z)$ *be defined by* (1.4). *Then* $f(z)$ *is in the class* $\mathcal{C}_\alpha(n)$ *if and only if*

$$\sum_{k=n+1}^{\infty}\left(\frac{k(k-\alpha)}{1-\alpha}\right)a_k \leqq 1\,, \qquad (n \geqq 1). \tag{1.8}$$

REMARK. Lemma 1 follows immediately from a result due to Silverman [12, p. 110, Theorem 2] upon setting

$$a_k = 0\,, \qquad k = 2, 3, \ldots, n\,. \tag{1.9}$$

Lemma 2, on the other hand, is a similar consequence of another result due to Silverman [12, p. 111, Corollary 2].

2. Distortion theorems.

We first prove

THEOREM 1. *Let the function* $f(z)$ *defined by* (1.4) *be in the class* $\mathcal{T}_\alpha(n)$. *Then*

$$|z| - \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^{n+1} \leqq |f(z)| \leqq |z| + \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^{n+1} \tag{2.1}$$

and

$$1 - \frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n \leqq |f'(z)| \leqq 1 + \frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n \tag{2.2}$$

for $z \in \mathcal{U}$. *The results* (2.1) *and* (2.2) *are sharp.*

PROOF. By virtue of Lemma 1, we have

$$\left(\frac{n+1-\alpha}{1-\alpha}\right)\sum_{k=n+1}^{\infty} a_k \leqq \sum_{k=n+1}^{\infty}\left(\frac{k-\alpha}{1-\alpha}\right) a_k \leqq 1\,, \tag{2.3}$$

or

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{1-\alpha}{n+1-\alpha}\,. \tag{2.4}$$

It follows from (2.4) that

$$|f(z)| \geqq |z| - |z|^{n+1}\sum_{k=n+1}^{\infty} a_k \geqq |z| - \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^{n+1} \tag{2.5}$$

and

$$|f(z)| \leqq |z| + |z|^{n+1}\sum_{k=n+1}^{\infty} a_k \leqq |z| + \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^{n+1}\,. \tag{2.6}$$

Note that

$$\sum_{k=n+1}^{\infty} k a_k \leqq 1-\alpha + \frac{\alpha(1-\alpha)}{n+1-\alpha} = \frac{(n+1)(1-\alpha)}{n+1-\alpha}\,. \tag{2.7}$$

Hence

$$|f'(z)| \geqq 1 - |z|^n \sum_{k=n+1}^{\infty} k a_k \geqq 1 - \frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n \tag{2.8}$$

and

$$|f'(z)| \leqq 1 + |z|^n \sum_{k=n+1}^{\infty} k a_k \leqq 1 + \frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n\,. \tag{2.9}$$

Further, by taking the function given by

$$f(z) = z - \frac{1-\alpha}{n+1-\alpha} z^{n+1}\,, \tag{2.10}$$

we can show that the results (2.1) and (2.2) are sharp.

COROLLARY 1. *Let the function $f(z)$ defined by* (1.4) *be in the class $\mathcal{C}_\alpha(n)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| < n/(n+1-\alpha)$.*

Next we state

THEOREM 2. *Let the function* $f(z)$ *defined by* (1.4) *be in the class* $\mathcal{C}_\alpha(n)$. *Then*

$$|z| - \frac{1-\alpha}{(n+1)(n+1-\alpha)}|z|^{n+1} \leqq |f(z)| \leqq \leqq |z| + \frac{1-\alpha}{(n+1)(n+1-\alpha)}|z|^{n+1} \tag{2.11}$$

and

$$1 - \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^n \leqq |f'(z)| \leqq 1 + \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^n \tag{2.12}$$

for $z \in \mathcal{U}$. *The results* (2.11) *and* (2.12) *are sharp.*

PROOF. In view of Lemma 2, we have

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{1-\alpha}{(n+1)(n+1-\alpha)} \tag{2.13}$$

and

$$\sum_{k=n+1}^{\infty} k a_k \leqq \frac{1-\alpha}{n+1-\alpha}, \tag{2.14}$$

and the assertions (2.11) and (2.12) of Theorem 2 follow from (2.13) and (2.14), respectively.

Noting that the results (2.11) and (2.12) are sharp for the function given by

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1}, \tag{2.15}$$

the proof of Theorem 2 is completed.

COROLLARY 2. *Let the function* $f(z)$ *defined by* (1.4) *be in the class* $\mathcal{C}_\alpha(n)$. *Then the unit disk* $\mathcal{U}$ *is mapped onto a domain that contains the disk* $|w| < r_0$, *where* r_0 *is given by*

$$r_0 = 1 - \frac{1-\alpha}{(n+1)(n+1-\alpha)}. \tag{2.16}$$

3. Convolution product.

Let $f_i(z)$ $(i = 1, 2)$ be defined by

$$f_i(z) = z - \sum_{k=n+1}^{\infty} a_{i,k} z^k \qquad (a_{i,k} \geqq 0) . \tag{3.1}$$

We denote by $f_1 * f_2(z)$ the convolution product of the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_1 * f_2(z) = z - \sum_{k=n+1}^{\infty} a_{1,k} a_{2,k} z^k , \tag{3.2}$$

and we prove

THEOREM 3. *Let the functions $f_i(z)$ $(i = 1, 2)$ be in the class $\mathcal{C}_\alpha(n)$. Then $f_1 * f_2(z)$ belongs to the class $\mathcal{C}_\beta(n)$, where*

$$\beta = \frac{n + 1 - \alpha^2}{n + 2 - 2\alpha} . \tag{3.3}$$

The result is sharp.

PROOF. Employing the technique used earlier by Schild and Silverman [11], we need to find the largest β such that

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\beta}{1-\beta}\right) a_{1,k} a_{2,k} \leqq 1 . \tag{3.4}$$

Since

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha}\right) a_{1,k} \leqq 1 \tag{3.5}$$

and

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha}\right) a_{2,k} \leqq 1 , \tag{3.6}$$

by the Cauchy-Schwarz inequality we have

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha}\right) \sqrt{a_{1,k} a_{2,k}} \leqq 1 . \tag{3.7}$$

Thus it is sufficient to show that

$$\left(\frac{k-\beta}{1-\beta}\right) a_{1,k} a_{2,k} \leqq \left(\frac{k-\alpha}{1-\alpha}\right) \sqrt{a_{1,k} a_{2,k}}\,, \qquad (k \geqq n+1), \tag{3.8}$$

that is, that

$$\sqrt{a_{1,k} a_{2,k}} \leqq \frac{(k-\alpha)(1-\beta)}{(1-\alpha)(k-\beta)}\,. \tag{3.9}$$

Note that

$$\sqrt{a_{1,k} a_{2,k}} \leqq \frac{1-\alpha}{k-\alpha}\,, \qquad (k \geqq n+1)\,. \tag{3.10}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{k-\alpha} \leqq \frac{(k-\alpha)(1-\beta)}{(1-\alpha)(k-\beta)}\,, \qquad (k \geqq n+1), \tag{3.11}$$

or, equivalently, that

$$\beta \leqq \frac{(k-\alpha)^2 - k(1-\alpha)^2}{(k-\alpha)^2 - (1-\alpha)^2}\,, \qquad (k \geqq n+1)\,. \tag{3.12}$$

Since

$$\Phi(k) = \frac{(k-\alpha)^2 - k(1-\alpha)^2}{(k-\alpha)^2 - (1-\alpha)^2} \tag{3.13}$$

is an increasing function of k, letting $k = n+1$ in (**3**.12) we obtain

$$\beta \leqq \Phi(n+1) = \frac{n+1-\alpha^2}{n+2-2\alpha}\,, \tag{3.14}$$

which completes the proof of the theorem.

Finally, by taking the functions given by

$$f_i(z) = z - \frac{1-\alpha}{n+1-\alpha} z^{n+1}\,, \qquad (i = 1, 2), \tag{3.15}$$

we can see that the result is sharp.

Similarly, we have

THEOREM 4. *Let the functions* $f_i(z)$ $(i = 1, 2)$ *defined by* (3.1) *be in the class* $\mathcal{C}_\alpha(n)$. *Then* $f_1 * f_2(z)$ *belongs to the class* $\mathcal{C}_\beta(n)$, *where*

$$\beta = \frac{n(n+1)(n+2-2\alpha)}{(n+1)^3 - 2n(n+2)\alpha + n\alpha^2 - 1}. \tag{3.16}$$

The result is sharp for the functions given by

$$f_i(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1}, \quad (i = 1, 2). \tag{3.17}$$

Finally, we prove

THEOREM 5. *Let the functions* $f_i(z)$ $(i = 1, 2)$ *defined by* (3.1) *be in the class* $\mathcal{C}_\alpha(n)$. *Then the function*

$$h(z) = z - \sum_{k=n+1}^{\infty} (a_{1,k}^2 + a_{2,k}^2) z^k \tag{3.18}$$

belongs to the class $\mathcal{C}_\beta(n)$, *where*

$$\beta = \frac{n^2 + 2n\alpha(1-\alpha) - (1-\alpha)^2}{n^2 + 2n(1-\alpha) - (1-a)^2}. \tag{3.19}$$

The result is sharp for the functions $f_i(z)$ $(i = 1, 2)$ *defined by* (3.15).

PROOF. By virtue of Lemma 1, we obtain

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha}\right)^2 a_{1,k}^2 \leqq \left[\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha}\right) a_{1,k}\right]^2 \leqq 1 \tag{3.20}$$

and

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha}\right)^2 a_{2,k}^2 \leqq \left[\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha}\right) a_{2,k}\right]^2 \leqq 1. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha}\right)^2 (a_{1,k}^2 + a_{2,k}^2) \leqq 2. \tag{3.22}$$

Therefore, we need to find the largest β such that

$$\frac{k-\beta}{1-\beta} \leqq \frac{1}{2}\left(\frac{k-\alpha}{1-\alpha}\right)^2, \quad (k \geqq n+1), \tag{3.23}$$

that is,

$$\beta \leqq \frac{(k-\alpha)^2 - 2k(1-\alpha)^2}{(k-\alpha)^2 - 2(1-\alpha)^2}, \quad (k \geqq n+1). \tag{3.24}$$

Since

$$\Psi(k) = \frac{(k-\alpha)^2 - 2k(1-\alpha)^2}{(k-\alpha)^2 - 2(1-\alpha)^2} \tag{3.25}$$

is an increasing function of k, we readily have

$$\beta \leqq \Psi(n+1) = \frac{n^2 + 2n\alpha(1-\alpha) - (1-\alpha)^2}{n^2 + 2n(1-\alpha) - (1-\alpha)^2}, \tag{3.26}$$

and Theorem 5 follows at once.

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