RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

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**On subfunctors of the identity**

*Rendiconti del Seminario Matematico della Università di Padova*, tome 76 (1986), p. 89-98

<http://www.numdam.org/item?id=RSMUP_1986__76__89_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 76 (1986)

# On Subfunctors of the Identity.

RADOSLAV DIMITRIĆ (*)

## 1. Introduction.

This note is concerned with the study of multiplicative and hereditary functors i.e. those subfunctors $F$ of the identity satisfying, for every $B \leqslant A$, $F(A/B) = (B + FA)/B$ and $FB = B \cap FA$ respectively. It also discusses some concepts of $F$-purity, connected with Fuchs' Problem 15 (in [2]) and the note is to be considered as an introduction to a more thorough treatment of this Problem. Some of the ideas here have been anticipated in [1].

Throughout we deal with a category of (unital) $R$-modules where $R$ is a commutative ring with unit. $F$ will denote a subfunctor of the identity in that category i.e. (an additive) functor such that, for every homomorphism $f: B \to A$, $Ff = f|FB$ (thus $f(FB) \leqslant FA$) and $FA \leqslant A$. This implies in particular that, for every $B \leqslant A$, $(B + FA)/B \leqslant F(A/B)$ and that $F$ commutes with direct sums.

By purity we mean $RD$-purity denoted by $B \leqslant_* A$, meaning that every equation $rx = b \in B$ $(r \in R)$ having solution $x \in A$ also has a solution in $B$.

## 2. Multiplicative functors.

DEFINITION 1. A subfunctor $F$ of the identity is called *multiplicative* if, for every submodule $B$ of an $R$-module $A$, $F(A/B) = (B + FA)/B$.

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The following proposition describes all multiplicative functors in the category of modules over PID's.

PROPOSITION 2. If $R$ is PID, then a subfunctor $F$ of the identity is multiplicative if and only if it is a multiplication by an $r \in R$.

PROOF. One way is trivial. Assume now that $F$ is multiplicative. Every $R$-module $A$ is a quotient of a free module: $A = (\oplus R)/B$. This gives $FA = (B + \oplus FR)/B = (B + r \oplus R)/B = rA$, for an $r \in R$.

A characterization of multiplicative functors is given in

THEOREM 3. The following are equivalent:

(1) $F$ is multiplicative;

(2) $F$ is epi-exact i.e. if $0 \to B \to A \xrightarrow{q} C \to 0$ is an exact sequence, then so is $FA \xrightarrow{q} FC \to 0$;

(3) $F$ is a radical i.e. $F(A/FA) = 0$, for every $A$ and $\mathcal{N} = \{A : FA = 0\}$ is closed under quotients and submodules.

PROOF. (1) $\Leftrightarrow$ (2): The relations $qFA = (B + FA)/B \leqslant F(A/B)$ prove this part immediately.

(1) $\Rightarrow$ (3): $F(A/FA) = (FA + FA)/FA = 0$. Now, if $A \in \mathcal{N}$, then $F(A/B) = (B + FA)/B = B/B = 0$, thus $\mathcal{N}$ is closed under quotients.

(3) $\Rightarrow$ (1): Since $A/FA \in \mathcal{N}$, the hypothesis that $\mathcal{N}$ is closed under quotients, gives $F\big(A/FA/(B + FA)/FA\big) = 0 = F\big(A/(B + FA)\big) = F\big(A/B/(B + FA)/B\big)$, so that

$$\big((B + FA)/B + F(A/B)\big)/(B + FA)/B = 0 .$$

This implies $F(A/B) \leqslant (B + FA)/B$, thus proving that $F$ is indeed multiplicative.

The theorem that follows enables us to work with classes of modules closed under submodules, quotients and direct products as well as with filter like families of ideals of $R$ closed under arbitrary intersections, instead of working with multiplicative functors. We give its detailed proof.

THEOREM 4. There is a bijective correspondence between:

(1) Multiplicative functors.

(2) Classes of modules closed under submodules, quotients and direct products.

(3) Families $\mathcal{F} = \{I \lhd R\}$ satisfying

a) $I \in \mathcal{F},\ J \supset I \Rightarrow J \in \mathcal{F}$;

b) $\{I_k\}_{k\in K} \subset \mathcal{F} \Rightarrow \bigcap_{k\in K} I_k \in \mathcal{F}$ .

PROOF. (1) $\to$ (2): For given multiplicative functor $F$ define $\mathcal{C}_F = \{A: FA = 0\}$. $\mathcal{C}_F$ is clearly closed under submodules and quotients (Theorem 3). If $A_i \in \mathcal{C}_F$ $(i \in I)$, then for every $j \in I$, $0 = FA_j = F\Big(\prod_{i\in I} A_i / \prod_{i\neq j} A_i\Big) = \Big(\prod_{i\neq j} A_i + F\Big(\prod_{i\in I} A_i\Big)\Big) / \prod_{i\neq j} A_i$. Therefore, for every $j \in I$, $F\Big(\prod_{i\in I} A_i\Big) \leqslant \prod_{i\neq j} A_i$, thus $F\Big(\prod_{i\in I} A_i\Big) = 0$ i.e. $\mathcal{C}_F$ is also closed under arbitrary products.

(2) $\to$ (1): For given class $\mathcal{C}$ of modules closed under submodules, quotients and direct products, define $F_\mathcal{C}$ by $F_\mathcal{C} A =$ the smallest submodule $B$ of $A$ such that $A/B \in \mathcal{C}$. We show that $F_\mathcal{C}$ satisfies (3) in Theorem 3: By the way we defined $F_\mathcal{C}$, $F_\mathcal{C}(A/F_\mathcal{C} A) = 0$. If $F_\mathcal{C} A = 0$ i.e. $A \in \mathcal{C}$ it gives $A/B \in \mathcal{C}$ and $B \in \mathcal{C}$, for every $B \leqslant A$. Hence $F_\mathcal{C}(B) = F_\mathcal{C}(A/B) = 0$, and this proves that $\mathcal{N}$ is closed under submodules and quotients.

The correspondence (1) $\leftrightarrow$ (2) is bijective:

$$\mathcal{C} \to F_\mathcal{C} \to \mathcal{C}_{F_\mathcal{C}}\colon\ A \in \mathcal{C}_{F_\mathcal{C}} \Leftrightarrow F_\mathcal{C} A = 0 \Leftrightarrow A \in \mathcal{C}\,.$$

$F \to \mathcal{C}_F \to F_{\mathcal{C}_F}$: $A/F_{\mathcal{C}_F}A \in \mathcal{C}_F$ or equivalently $F(A/F_{\mathcal{C}_F}A) = 0$, hence $(F_{\mathcal{C}_F}A + FA)/F_{\mathcal{C}_F}A = 0$ i.e. $FA \leqslant F_{\mathcal{C}_F}A$. By Theorem 3, $F(A/FA) = 0$, thus, by the definition of $F_{\mathcal{C}_F}$ we get $F = F_{\mathcal{C}_F}$.

(2) $\to$ (3): $\mathcal{C} \to \mathcal{F}_\mathcal{C} = \{I \lhd R\colon R/I \in \mathcal{C}\}$

a) $I \in \mathcal{F}_\mathcal{C} \Leftrightarrow R/I \in \mathcal{C}$; $J \supset I$ implies $R/J \in \mathcal{C}$ so $J \in \mathcal{F}_\mathcal{C}$;

b) If, for every $k \in K$, $I_k \in \mathcal{F}_\mathcal{C}$ i.e. $R/I_k \in \mathcal{C}$, we get $\prod_{k\in K} R/I_k \in \mathcal{C}$ and, by the known fact $R/\bigcap_{k\in K} I_k \leqslant \prod_{k\in K} R/I_k$, we obtain $\bigcap I_k \in \mathcal{F}_\mathcal{C}$.

(3) $\to$ (2): $\mathcal{F} \to \mathcal{C}_\mathcal{F} = \{A \mid$ for every $x \in A$, $\operatorname{ann} x \in \mathcal{F}\}$. It is clear that $\mathcal{C}_\mathcal{F}$ is closed under submodules. That it is closed under quotients we see from the fact that if $\bar{x} \in A/B$ then $\operatorname{ann} \bar{x} \geqslant \operatorname{ann} x \in \mathcal{F}$

and $a$) applies. If $A_i \in \mathcal{C}_{\mathcal{F}}$ $(i \in I)$, then, for every $x_i \in A_i$, $\operatorname{ann} x_i \in \mathcal{F}$. For arbitrary $x \in \prod_{i \in I} A_i$, $\operatorname{ann} x = \bigcap_{i \in I} \operatorname{ann} x_i$, thus $b$) gives $\prod A_i \in \mathcal{C}_{\mathcal{F}}$.

The correspondence $(2) \leftrightarrow (3)$ is bijective:

$\mathcal{C} \to \mathcal{F}_{\mathcal{C}} \to \mathcal{C}_{\mathcal{F}_{\mathcal{C}}}$: $A \in \mathcal{C}_{\mathcal{F}_{\mathcal{C}}} \Leftrightarrow$ for every $x$ in $A$, $\operatorname{ann} x \in \mathcal{F}_{\mathcal{C}} \Leftrightarrow$ for every $x$ in $A$, $R/\operatorname{ann} x \in \mathcal{C} \Leftrightarrow$ for every $x$ in $A$, $Rx \in \mathcal{C} \Leftrightarrow \bigoplus_{x \in A} Rx \in \mathcal{C}$. This implies that $A \in \mathcal{C}$ since $A$ is the quotient of $\bigoplus_{x \in A} Rx$.

$\mathcal{C} \to \mathcal{C}_{\mathcal{F}} \to \mathcal{F}_{\mathcal{C}_{\mathcal{F}}}$: $I \in \mathcal{F}_{\mathcal{C}_{\mathcal{F}}} \Leftrightarrow R/I \in \mathcal{C}_{\mathcal{F}} \Leftrightarrow$ for every $\bar{r} \in R/I$, $\operatorname{ann} \bar{r} \in \mathcal{F} \Leftrightarrow I = \bigcap_{r \in R} \operatorname{ann} \bar{r} \in \mathcal{F}$ (the last equivalence is valid since $R$ is assumed to be having a unit).

The correspondence $(1) \leftrightarrow (3)$ is realized by

$$F \to \mathcal{F}_F = \{I \lhd R\colon F(R/I) = 0\}$$

and $\mathcal{F} \to F_{\mathcal{F}} A =$ the smallest submodule $B$ of $A$ such that for every $\bar{x} \in A/B$, $\operatorname{ann} \bar{x} \in \mathcal{F}$.

Notice that if $\mathcal{F} \neq \emptyset$ in Theorem 4, then $R \in \mathcal{F}$.

Also $0 \in \mathcal{F} \Leftrightarrow F$ is zero functor $\Leftrightarrow \mathcal{C} =$ the category of $R$-modules.

If $R$ is PID, then by Proposition 2, the only multiplicative functors are multiplications by $r \in R$, thus $\mathcal{C}$ consists of all $R$-modules bounded by a fixed $r \in R$, while $\mathcal{F} = \{sR\colon s|r\}$, for an $r \in R$.

## 3. Hereditary functors.

A very important class of functors, in a sense dual to multiplicative functors, is introduced in

Definition 5. Call a subfunctor $F$ of the identity ***hereditary*** if for every submodule $B$ of an $R$-module $A$, $FB = B \cap FA$.

As a corollary to this definition, we derive that for a hereditary functor $F$ and arbitrary $R$-modules $C$, $D$, $FC \cap FD = F(C \cap D)$.

Combining Proposition 1.4 and Exercise 1.1 in [4] we state

Theorem 6. The following are equivalent:

(1) $F$ is hereditary;

(2) $F$ is left exact i.e. if $0 \to B \xrightarrow{\beta} A \xrightarrow{\alpha} C$ is an exact sequence, then so is $0 \to FB \xrightarrow{\beta} FA \xrightarrow{\alpha} FC$;

(3) $F^2 = F$ and $\mathcal{C} = \{A: FA = A\}$ is closed under submodules.

PROOF. (1) $\Rightarrow$ (2): Exactness at $FB$ is straightforward since $F$ is a subfunctor of the identity. The exactness at $FA$ is established from the chain of equalities:

$$\ker(F\alpha) = FA \cap \ker\alpha = FA \cap \operatorname{im}\beta = FA \cap \beta B = F(\beta B) = \\ = \beta(FB) = \operatorname{im} F\beta;$$

(2) $\Rightarrow$ (3): The exact sequence $0 \to FA \to A \to A/FA$ induces another exact sequence $0 \to FFA \to FA \to F(A/FA)$ that gives $F^2 = F$. If $FA = A$ and $B \leqslant A$, then exactness of $0 \to B \to A \to A/B$ induces exactness of $0 \to FB \to A \to F(A/B)$ thus $FB = B$; this shows that $\mathcal{C}$ is closed under submodules;

(3) $\Rightarrow$ (1): Let $B \leqslant A$. $FA \in \mathcal{C}$ so $B \cap FA \in \mathcal{C}$ or equivalently $FB \leqslant B \cap FA = F(B \cap FA) \leqslant FB$ therefore $F$ is hereditary.

One of several important characteristics of hereditary functors is to be found in

PROPOSITION 7. A subfunctor of the identity commutes with inverse limits if and only if it is hereditary and commutes with direct products.

PROOF. By a known fact, $F$ commutes with inverse limits if and only if it commutes with direct products and equalizers. The last condition is equivalent to heredity of $F$.

Here, for the sake of completeness and usefulness, we state Proposition 3.3 from [4].

THEOREM 8. There is a bijective correspondence between:

(1) Hereditary functors.

(2) Classes $\mathcal{C}$ of modules closed under submodules, quotients and direct sums.

(3) Families $\mathcal{F}$ of ideals of $R$ (called *pretopologies* on $R$) satisfying

*a*) $I \in \mathcal{F},\ a \in R \Rightarrow (I:a) \in \mathcal{F}$;

*b*) $J \in \mathcal{F},\ I \supset J \Rightarrow I \in \mathcal{F}$;

*c*) $I, J \in \mathcal{F} \Rightarrow I \cap J \in \mathcal{F}$.

PROOF. The correspondences are as follows:

$$\begin{aligned}
\mathcal{F} &\to \mathcal{C} &&= \{A\colon \text{for every } x \in A,\ \operatorname{ann} x \in \mathcal{F}\}\,,\\
\mathcal{C} &\to \mathcal{F} &&= \{I \lhd R\colon R/I \in \mathcal{C}\}\,,\\
\mathcal{C} &\to FA &&= \text{the largest submodule } B \text{ of } A \text{ belonging to } \mathcal{C}\,,\\
F &\to \mathcal{C} &&= \mathcal{T} = \{A\colon FA = A\}\,,\\
\mathcal{F} &\to FA &&= \{x \in A\colon \operatorname{ann} x \in \mathcal{F}\}\,,\\
F &\to \mathcal{F} &&= \{I \lhd R\colon F(R/I) = R/I\}\,.
\end{aligned}$$

At the end of this section we examine subfunctors of the identity that are at the same time hereditary and multiplicative.

PROPOSITION 9. For a subfunctor $F$ of the identity, the following are equivalent:

(1) $F$ commutes with direct limits;

(2) for every homomorphism $f\colon B \to A$, $F(A/fB) \cong FA/fFB$ canonically;

(3) for every subgroup $B \leqslant A$ there is a canonical isomorphism $FA/FB \cong F(A/B)$;

(4) $F$ is hereditary and multiplicative;

(5) $F$ is exact.

PROOF. (1) $\Leftrightarrow$ (2): By the known fact, $F$ (being a subfunctor of the identity) commutes with direct limits if and only if it commutes with coequalizers or, equivalently, with coqernels. This is exactly (2);

(2) $\Rightarrow$ (3) is obvious;

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5): If $0 \to B \to A \to C \to 0$ is exact, then by Theorems 3 and 6, $0 \to FB \to FA \to FC \to 0$ is exact if and only if $F$ is multiplicative and hereditary. The latter sequence is exact if and only if $FA/FB \cong F(A/B)$ canonically;

(3), (4) $\Rightarrow$ (1): If $\{A_i\ (i \in I),\ \pi_i^j\colon A_i \to A_j\}$ is a direct spectar, then $\varinjlim A_i = (\bigoplus A_i)/B$, where $B = \{a_i - \pi_i^j a_i\colon i \leqslant j\} \leqslant \bigoplus A_i$. $F$-induced spectar is

$$\{FA_i\ (i \in I),\ \pi_i^j|\colon FA_i \to FA_j\} \quad \text{and} \quad \varinjlim FA_i = (\bigoplus FA_i)/B_F\,,$$

where $B_F = \{b_i - \pi_i^j | b_i : i \leqslant j\} \leqslant \bigoplus FA_i$. Since $B_F = B \cap (\bigoplus FA_i) = B \cap F(\bigoplus A_i) = FB$ we have

$$F \varinjlim A_i = F\big((\bigoplus A_i)/B\big) \simeq (\bigoplus FA_i)/FB = \varinjlim FA_i ,$$

thus $F$ commutes with direct limits.

PROPOSITION 10. If $F$ is a subfunctor of the identity in the category of modules over a PID, then $F$ is multiplicative and hereditary if and only if $F$ is the Zero-functor or the identity itself.

PROOF. By Proposition 2, $F$ is multiplication by an $r \in R$. Since $F$ is hereditary, we have $rR = R \cap rQ$ ($Q$-the quotient field of $R$); this is possible only if $r$ is invertible so the only choices are either $r = 1$ or $r = 0$.

## 4. $F$-purity.

Following Fuchs (we only change the name a little; see [2], Problem 15) we have

DEFINITION 11. For a subfunctor $F$ of the identity, call the exact sequence

$$E\colon 0 \to B \to A \to C \to 0$$

*$F$-exact* if the induced sequence

$$FE\colon 0 \to FB \to FA \to FC \to 0 \quad \text{is exact}.$$

There are several concepts of $F$-purity introduced in the literature. For instance, Nunke in [3] calls the exact sequence $E$, $F$-pure if it is an element of $F\,\mathrm{Ext}(A/B, B)$.

If $n = k \cdot m$, the exact sequence $0 \to Z(k) \to Z(n) \to Z(m) \to 0$ gives an example of $n$-exact sequence in the category of Abelian groups; this sequence is not $n$-pure in Nunke's sense. Conversely $0 \to \bigoplus Z \to \prod Z \to \prod Z / \bigoplus Z \to 0$ is Nunke $D$-pure but is not $D$-exact sequence ($D$ being the divisible part of a group).

THEOREM 12. If $F$ is a subfunctor of the identity and $E$ a given exact sequence, then the following are equivalent:

(1) $E$ is $F$-exact;

(2) $FB = B \cap FA$ and $F(A/B) = (B + FA)/B$;

(3) $FA/FB \cong F(A/B)$ canonically.

PROOF. The same reasoning applies as in proofs of Theorems 3 and 6.

EXAMPLES. 1. By Theorem 29.1 in [2], $E$ is pure exact if and only if it is $r$-exact, for every $r \in R$.

2. If $E$ is pure exact, then it is $S$-exact (here $S$ denotes the socle subfunctor). By Theorem 12 we need to show only that $S(A/B) = (B + SA)/B$. The last follows from the known fact (see e.g. Theorem 28.1 in [2]) that $B$ is pure submodule of $A$ if and only if for every $\bar{c} \in A/B$ there is an $a \in \bar{c}$ of the same order as $\bar{c}$.

3. If $E$ is splitting, then it is $F$-exact for every subfunctor $F$ of the identity. Indeed if $A = B \oplus C$, then $B \cap FA = B \cap (FB \oplus FC) = FB$ and $F(A/B) = FC = (B \oplus FA)/B$, thus $F$ satisfies both conditions in Theorem 12 (2).

PROPOSITION 13. Assume that $E$ is an exact sequence, where $A = A_0 \oplus A_1$, with $A_0$ torsion free and $A_1$ torsion. Then $E$ is $T$-exact ($T$-the torsion functor) if and only if $T((A_0 + B)/B) = 0$ or, equivalently, $A_0 \cap B \leqslant_* A_0$.

PROOF. By Theorem 12, $E$ is $T$-pure if and only if $T((A_0 + A_1)/B) = (B + A_1)/B$ which is equivalent to $T((A_0 + B)/B) = 0$ or $A_0/(A_0 \cap B)$ being torsion free. This is same as $A_0 \cap B \leqslant_* A_0$.

We also introduce

DEFINITION 14. For a subfunctor $F$ of the identity, call the exact sequence

$$E\colon\ 0 \to B \to A \xrightarrow{q} C \to 0$$

*pure-F-exact* if the induced sequence

$$FE\colon\ 0 \to FB \to FA \xrightarrow{q} FC \to 0 \quad \text{is pure exact}.$$

PROPOSITION 15. *$E$ is pure exact if and only if it is pure-$T$-exact.*

PROOF. $\Rightarrow$: The proof that $T(A/B) = (B + TA)/B$ is by the same argument as in Example 2. That $TB \leqslant_* TA$ follows straightforwardly from $B \leqslant_* A$.

$\Leftarrow$: We need to prove that $B \leqslant_* A$. Let $ra = b$, $a \in A$, $b \in B$, $r \in R$. Then $rqa = qra = qb = 0$ so $qa \in TC$. By $T$-exactness there is an $a' \in TA$ with $qa' = qa$ i.e. $a - a' \in B$ and $qra' = 0$ thus there is a $b' \in TB$ such that $ra' = b'$. Since $TB \leqslant_* TA$, there is a $b'' \in TB$ with $ra' = rb'' = b'$ therefore $r(a' - b'') = 0$ hence $r\big(a - (a' - b'')\big) = ra = b$.

One can show in a similar fashion that $E$ is pure exact if and only if it is pure-$r$-exact for every $r \in R$ (see Example 1 in [1] and Exercise 1, § 29 in [2]).

Following Stenström (see [4], Exercise 1, § 6) call the exact sequence $E$ $\mathcal{F}$-pure ($\mathcal{F}$ is a pretopology on $R$) if for every $x \in C$ with $\operatorname{ann} x \in \mathcal{F}$, there exists $y \in A$ mapped upon $x$, such that $\operatorname{ann} y = \operatorname{ann} x$.

THEOREM 16. *Let $F$ be hereditary functor and $\mathcal{F}$ the pretopology corresponding to it in the correspondence given by Theorem 8. Then the exact sequence $E$ is pure-$F$-exact if and only if it is $\mathcal{F}$-pure.*

PROOF. $\Rightarrow$: Assume that $E$ is pure-$F$-exact sequence. We prove that it is $\mathcal{F}$-pure. To this end, let $x \in C$ with $\operatorname{ann} x \in \mathcal{F}$. By Theorem 8, this means that $x \in FC$, therefore there is a $y \in FA$ (i.e. $\operatorname{ann} y \in \mathcal{F}$) such that $qy = x$. We find a $y_0 \in A$ such that $qy_0 = x$ and $\operatorname{ann} x \leqslant \operatorname{ann} y_0$: if $rx = 0$, then $ry \in FB \leqslant_* FA$ so there is a $y_1 \in FB$ such that $r(y - y_1) = 0$; it is enough now to take $y_0 = y - y_1$.

$\Leftarrow$: We assume that $E$ is $\mathcal{F}$-pure and prove that $FE$ is pure exact. If $x \in FC$ i.e. $\operatorname{ann} x \in \mathcal{F}$, then there is a $y \in A$ with $qy = x$ and $\operatorname{ann} y = \operatorname{ann} x$ so that $y \in FA$. This shows that $E$ is $F$-exact. It is also pure exact: if $ry = b \in FB$, $y \in FA$, by $\mathcal{F}$-purity there exists $y_1 \in A$ such that $qy_1 = qy$ i.e. $y - y_1 \in FB$ and $\operatorname{ann} y_1 = \operatorname{ann} qy$. Since $rqy = 0$ we have $ry_1 = 0$ thus giving $r(y - y_1) = ry = b$.

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Manoscritto pervenuto in redazione il 9 febbraio 1985.