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Infinite Boundary Value Problems for Surfaces of Prescribed Mean Curvature.

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The Dirichlet problem for surfaces of prescribed mean curvature consists in determining a function $u \in C^2(\Omega)$ satisfying the equation

$$L(u) = (1 + |Du|^2)\Delta u - \sum_{i,j=1}^{n} D_i u D_j u D_{ij} u - nH(x)(1 + |Du|^2)^{\frac{3}{2}} = 0$$

in a bounded domain Ω and taking a given boundary value φ on $\partial\Omega$. We consider Dirichlet problem where infinite boundary values are admitted on subsets of the boundary. Jenkins and Serrin developed an existence and uniqueness theory for this problem in the case $H \equiv 0$ and $n = 2$ [9], while Spruck extended Jenkins-Serrin's results to the case H = constant and $n = 2$ [10].

In [1] Massari proved an existence and uniqueness theorem in the case H not constant, arbitrary n and the boundary value φ is finite in Γ_0, $+\infty$ in Γ_1, $-\infty$ in Γ_2, where Γ_0, Γ_1, Γ_2 are open, disjoint subsets of $\partial\Omega$ and $\Gamma_0 \neq \emptyset$. In this work we complete Massari's results studying the case $\Gamma_0 = \emptyset$.

In the first section we recall some basic results of U. Massari [1], [2] Giusti [3], Miranda [5], Emmer [4] which we shall use. In the second section we prove an existence and uniqueness theorem for the following Dirichlet problem

$$\begin{aligned} L(u) &= 0 && \text{in } \Omega \\ \lim_{y \to x} u(y) &= +\infty && \forall x \in \Gamma_1 \\ \lim_{y \to x} u(y) &= -\infty && \forall x \in \Gamma_2 \end{aligned}$$

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where Γ_1 and Γ_2 are open, not empty and disjoint subsets of $\partial\Omega$.
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1. *A*) Let Ω be a bounded domain (*i.e.* open and connected) in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$ and

$$\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup N$$

where Γ_0, Γ_1, Γ_2 are open, not empty, disjoint C^1 sets and N is a closed set such that $H_{n-1}(N) = 0$.

B) Let A_0, A_1, A_2 open disjoint sets with Lipschitz continuous boundary such that

$$H_{n-1}(\partial\Omega \cap \partial A_i) = 0 \qquad i = 0, 1, 2\ .$$

C) Let H a bounded Lipschitz continuous function in $\bar{\Omega}$ such that for every Caccioppoli set $B \subset \Omega$ meas $B \neq 0$, meas $B \neq$ meas Ω we have

$$\left| n \int_B H(x)\, dx \right| < \int_{\mathbb{R}^n} |D\varphi_B|$$

where $\int_{\mathbb{R}^n} |D\varphi_B|$ is the perimeter of B; *i.e.* the total variation of the vector valued measure $D\varphi_B$:

$$\int_{\mathbb{R}^n} |D\varphi_B| = \sup \left\{ \int \varphi_B \operatorname{div} g\, dx \colon g \in [C_0^1(\mathbb{R}^n)]^n\ |g| \leq 1 \right\}$$

D) Let $\Omega_0 = \Omega \cup A_0$ be a connected set and

$$\Lambda(x) \geq \frac{n|H(x)|}{n-1} \qquad \forall x \in \Gamma_0 \qquad \Gamma_0 = A_0 \cap \partial\Omega$$

$$\Lambda(x) = \frac{nH(x)}{n-1} \qquad \forall x \in \Gamma_1 \qquad \Gamma_1 = A_1 \cap \partial\Omega$$

$$\Lambda(x) = -\frac{nH(x)}{n-1} \qquad \forall x \in \Gamma_2 \qquad \Gamma_2 = A_2 \cap \partial\Omega$$

where $\Lambda(x)$ is the mean curvature of $\partial\Omega$ at x.

Under these hypotheses there exist minima for the functionals

$$(1.1)\qquad I(v,\varphi)=\int_{\Omega}\sqrt{1+|Dv|^2}+n\int_{\Omega}H(x)v(x)\,dx+\int_{\partial\Omega}|v-\varphi|\,dH_{n-1}$$

$$(1.2)\qquad I'(v,\varphi)=\int_{\Omega}\sqrt{1+|Dv|^2}-n\int_{\Omega}H(x)v(x)\,dx+\int_{\partial\Omega}|v-\varphi|\,dH_{n-1}$$

for every $\varphi\in L^1(\partial\Omega)$. Such minima belong to $C^{2,\alpha}(\Omega)\cap BV(\Omega)$.

If φ is a bounded function then the minimum is bounded; if $\varphi\in C(\Gamma_0)$ then the minimum takes the boundary value φ in Γ_0. (see [6], [7]).

Let $\{\Omega_h\}_{h\in\mathbb{N}}$ be a sequence of smooth sets with

$$\Omega_1\subset\subset\Omega_2\subset\subset\ldots;\quad \bigcup_{h=1}^{\infty}\Omega_h=\Omega;\quad H_{n-1}(\partial\Omega)=\lim_{h\to\infty}H_{n-1}(\partial\Omega_h)\,.$$

We need to recall the following results:

THEOREM 1.1 [1]. Let $u\in C^2(\Omega)$ be a solution of equation $L(u)=0$ such that

$$\lim_{y\to x}u(y)=+\infty\qquad\forall x\in\Gamma_1\,.$$

Then for every open set A with Lipschits continuous boundary such that

$$H_{n-1}(\partial A\cap\partial\Omega)=0$$

we have

$$\lim_{h\to\infty}\int_{\partial\Omega_h\cap A}Tu\nu\,dH_{n-1}=H_{n-1}(\Gamma_1)\qquad\Gamma_1=A\cap\partial\Omega$$

where $Tu=Du/\sqrt{1+|Du|^2}$ and ν is the exterior normal to $\partial\Omega$.

THEOREM 1.2 [1]. Let $u\in C^2(\Omega)$ be a solution of $L(u)=0$ such that

$$\lim_{y\to x}u(y)=-\infty\qquad\forall x\in\Gamma_2\,.$$

Then for every open set A with Lipschitz continuous boundary such that

$$H_{n-1}(\partial A \cap \partial \Omega) = 0$$

we have

$$\lim_{h\to\infty} \int_{\partial\Omega_h \cap A} Tu\cdot\nu \, dH_{n-1} = -H_{n-1}(\Gamma_2) \qquad \Gamma_2 = A \cap \partial\Omega$$

THEOREM 1.3 [1]. If $\Gamma_2 = \emptyset$, $\Gamma_0 \neq \emptyset$, $\varphi: \Gamma_0 \to \mathbf{R}$ is a bounded continuous function in Γ_0 then the set $P = \{x \in \Omega: u(x) = +\infty\}$ minimizes the functional

$$(1.3) \qquad \mathcal{F}_1(B) = \int_\Omega |D\varphi_B| + n\int_\Omega H(x)\varphi_B(x)\,dx + \int_{\partial\Omega} |\varphi_B - \varphi_{\Gamma_1}|\,dH_{n-1}\,.$$

REMARK 1.1. If $\Gamma_1 = \emptyset$, an analogous result is valid for the set $N = \{x \in \Omega: u(x) = -\infty\}$. In particular N minimizes the functional

$$(1.4) \qquad \mathcal{F}_2(B) = \int_\Omega |D\varphi_B| - n\int_\Omega H(x)\varphi_B(x)\,dx + \int_{\partial\Omega} |\varphi_B - \varphi_{\Gamma_2}|\,dH_{n-1}\,.$$

THEOREM 1.4 [1]. Let φ be continuous in $x_0 \in \Gamma_0$, u minimizes the functional $I(v, \varphi)$ and $t > \varphi(x_0)$. Let $R > 0$ be such that

$$\varphi(x) < t - R \qquad \forall x \in B_R(x_0) \cap \partial\Omega$$

$$B_R(x_0) \cap \Omega = \{(x', x_n) \in (B' \times \mathbf{R}) \cap B_R(x_0): x_n > \psi(x')\}$$

where $B' \subset \mathbf{R}^{n-1}$ is an open set and $\psi: B' \to \mathbf{R}$ is a Lipschitz continuous function. Then the set

$$E = \{(x, y) \in \Omega \times \mathbf{R}: y < u(x)\}$$

minimizes the functional

$$\int_A |D\varphi_F| + n\int_A \varphi_F(x, y)\hat{H}(x)\,dx\,dy$$

where $A = B_R(x_0, t) \subset \mathbb{R}^{n+1}$ and $\hat{H}$ is the following function

$$\hat{H}(x', x_n) = \begin{cases} H(x', x_n) & \text{if } x_n \geqslant \psi(x') \\ H(x', \psi(x')) & \text{if } x_n < \psi(x') . \end{cases}$$

THEOREM 1.5 [1]. If $\Gamma_2 = \emptyset$, $\Gamma_0 \neq \emptyset$, $\varphi\colon \Gamma_0 \to \mathbb{R}$ is a below bounded continuous function and $\emptyset$ is the unique minimum of $\mathcal{F}_1(B)$, then there exists $u \in C^2(\Omega)$ such that

$$L(u) = 0 \quad \text{in } \Omega$$

$$u = \varphi \quad \text{in } \Gamma_0$$

$$\lim_{y \to x} u(y) = +\infty \quad \forall x \in \Gamma_1$$

THEOREM 1.6 [1]. If $\Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, $\varphi\colon \Gamma_0 \to \mathbb{R}$ is an upper bounded continuous function and if $\emptyset$ is the unique minimum of $\mathcal{F}_2(B)$, then there exists $u \in C^2(\Omega)$ such that

$$L(u) = 0 \quad \text{in } \Omega$$

$$u = \varphi \quad \text{in } \Gamma_0$$

$$\lim_{y \to x} u(y) = -\infty \quad \forall x \in \Gamma_2 .$$

THEOREM 1.7 [1]. Let $\varphi\colon \Gamma_0 \to \mathbb{R}$ be a continuous function, $\Gamma_0 \neq \emptyset$. Necessary and sufficient condition for the existence of a solution $u \in C^2(\Omega)$ to Dirichlet problem

$$L(u) = 0 \quad \text{in } \Omega$$

$$u = \varphi \quad \text{in } \Gamma_0$$

$$\lim_{y \to x} u(y) = +\infty \quad \forall x \in \Gamma_1$$

$$\lim_{y \to x} u(y) = -\infty \quad \forall x \in \Gamma_2$$

is that the unique minimum of the functionals $\mathcal{F}_1(B)$ and $\mathcal{F}_2(B)$ is the empty set.

THEOREM 1.8 [1]. Let u_1, $u_2 \in C^2(\Omega)$ be two solutions of the problem

$$\begin{aligned} L(u) &= 0 \quad \text{in } \Omega \\ u &= \varphi \quad \text{in } \Gamma_0 \\ \lim_{y\to x} u(y) &= +\infty \quad \forall x \in \Gamma_1 \\ \lim_{y\to x} u(y) &= -\infty \quad \forall x \in \Gamma_2 . \end{aligned}$$

Then

i) if $\Gamma_0 \neq \emptyset$ then $u_1 = u_2$ in Ω;

ii) if $\Gamma_0 = \emptyset$ then $u_1 = u_2 +$ constant in Ω.

THEOREM 1.9 [2]. Let E a set minimizing the functional

$$\int_K |D\varphi_E| + \int_K \varphi_E(x) A(x)\, dx$$

in an open set $\Omega \subset \mathbf{R}^n$ with $n \geqslant 2$ and $|A(x)| \leqslant A$. If $x \in \partial E$ and $\overline{B_\varrho(x)} \subset \Omega$, $\varrho > 0$, then we have

$$\varrho^{1-n} \int_{B_\varrho(x)} |D\varphi_E| + (n-1) A \omega_n \varrho \geqslant \omega_{n-1} \tag{1.5}$$

THEOREM 1.10 [4]. If the set E is a local minimum for the functiona

$$\mathfrak{L}(E) = \int \varphi_E(x) H(x)\, dx + \int |D\varphi_E|$$

in the open set $\Omega \subset \mathbf{R}^n$ with obstacle L and if $\partial L \cap \Omega$ is of class C^1, then there exists an open set $\Omega_0 \subset \Omega$ with $\partial L \cap \Omega \subset \Omega_0$ such that $\partial E \cap \Omega_0$ is of class C^1.

We recall the definition of generalized solution, introduced by M. Miranda (see [5]).

DEFINITION 1.1. A function $u: \Omega \to \overline{\mathbf{R}}$ is called a generalized solution of equation

$$\operatorname{div} Tu = nH(x)$$

if the set $E = \{(x, y) \in \Omega\times\mathbf{R}: y < u(x)\}$ minimizes the functional

$$\int |D\varphi_E| + n\int H(x)\varphi_E(x, y)\,dx\,dy \tag{1.6}$$

in $\Omega\times\mathbf{R}$.

That means that for every set $V \subset \Omega\times\mathbf{R}$, coinciding with E outside some compact set $K \subset \Omega\times\mathbf{R}$ we have

$$\int_K |D\varphi_E| + n\int_K H(x)\varphi_E(x, y)\,dx\,dy \leqslant \int_K |D\varphi_V| + n\int_K H(x)\varphi_V(x, y)\,dx\,dy\,.$$

We note that the function $u(x)$ can take the values $\pm\infty$.

It follows from [8] theorem 2.3 that every classical solution of $\operatorname{div} Tu = nH(x)$ is a generalized solution and reciprocally, every local bounded generalized solution is a classical solution of $\operatorname{div} Tu = nH(x)$. We introduced the sets:

$$P = \{x \in \Omega: u(x) = +\infty\}\,; \qquad N = \{x \in \Omega: u(x) = -\infty\}\,;$$

$$G = \Omega - (P \cup N) - \partial P \cap \partial N\,.$$

We have the following results

(1.7) the function $u(x)$ is regular in G and is a classical solution of $\operatorname{div} Tu = nH(x)$.

(1.8) Let $\{u_k\}$ be a sequence of generalized solution of $\operatorname{div} Tu = nH(x)$ in Ω and let E_k be the corresponding domains (1.6). Then a subsequence of E_k will converge in $L^1_{\text{loc}}(\Omega\times\mathbf{R})$ to a set $E = \{(x, y) \in \Omega\times\mathbf{R}: y < u(x)\}$ and $u(x)$ is a generalized solution of $\operatorname{div} Tu = nH(x)$. We say in this case that a subsequence of $\{u_k\}$ converges locally to $u(x)$.

THEOREM 1.11 [3]. *Let u, v be two C^2-functions in Ω such that $\operatorname{div} Tu \leqslant \operatorname{div} Tv$ in Ω. Suppose that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with Γ_1 open set in $\partial\Omega$ and that $u, v \in C(\Omega \cup \Gamma_1)$, $u(x) \geqslant v(x)$ in Γ_1 and*

$$\lim_{t\to 0^+} \int_{\partial\Omega_t - A} (1 - Tu\cdot\nu)\,dH_{n-1} = 0$$

for every open set $A \supset \Gamma_1$. Then

a) if $\Gamma_1 \neq \emptyset$ then $u \geqslant v$ in Ω;

b) if $\Gamma_1 = \emptyset$ then $u = v +$ constant in Ω.

2. THEOREM 2.1. We suppose that $\Gamma_0 = \emptyset$, $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$. If $\emptyset$ and Ω are the unique minima for the functionals

$$\mathcal{F}_1(B) = \int_\Omega |D\varphi_B| + n\int_\Omega H(x)\varphi_B(x)\,dx + \int_{\partial\Omega} |\varphi_B - \varphi_{\Gamma_1}|\,dH_{n-1}$$

$$\mathcal{F}_2(B) = \int_\Omega |D\varphi_B| - n\int_\Omega H(x)\varphi_B(x)\,dx + \int_{\partial\Omega} |\varphi_B - \varphi_{\Gamma_2}|\,dH_{n-1}$$

then there exists a solution to the Dirichlet problem

$$\begin{aligned} &\operatorname{div} Tu = nH(x) && \text{in } \Omega \\ &\lim_{y\to x} u(y) = +\infty && \forall x \in \Gamma_1 \\ &\lim_{y\to x} u(y) = -\infty && \forall x \in \Gamma_2 . \end{aligned}$$

REMARK 2.1. It follows from the hypothesis on the functionals $\mathcal{F}_1(B)$ and $\mathcal{F}_2(B)$ that

$$n\int_\Omega H(x)\,dx + H_{n-1}(\Gamma_2) = H_{n-1}(\Gamma_1) \tag{2.1}$$

REMARK 2.2. It follows from theorem 1.8 that the solution of the problem is unique up to an additive constant.

We prove theorem 2.1 in two steps.

1*st step*. Let u_h be the solution of the problem

$$\begin{aligned} &\operatorname{div} Tu_h = nH(x) && \text{in } \Omega \\ &u_h(x) = h && \text{in } \Gamma_1 \\ &u_h(x) = 0 && \text{in } \Gamma_2 . \end{aligned}$$

For every $h \in \mathcal{A}$ we can find a constant c_h with $0 < c_h < h$ such that

$$(2.2)\qquad \begin{cases} \operatorname{meas}\left(\{x \in \Omega\colon u_h(x) \geqslant c_h\}\right) \geqslant \dfrac{|\Omega|}{4} \\[2ex] \operatorname{meas}\left(\{x \in \Omega\colon u_h(x) \leqslant c_h\}\right) \geqslant \dfrac{|\Omega|}{4}\,. \end{cases}$$

We set $v_h = u_h - c_h$ then v_h is a generalized solution of

$$\begin{aligned} &\operatorname{div} Tv_h = nH(x) && \text{in } \Omega\,, \\ &v_h = h - c_h && \text{in } \Gamma_1\,, \\ &v_h = -c_h && \text{in } \Gamma_2\,. \end{aligned}$$

It follows from (1.8) that a subsequence of $\{v_h\}$ will converge locally to a generalized solution $v(x)$ of

$$(2.3)\qquad \operatorname{div} Tv = nH(x)\,.$$

We prove that the sets P_v and N_v are empty and hence the set G is Ω. Then v is a locally bounded function in Ω and it is a classical solution of (2.3). First we prove that

a) $\lim_{h\to\infty} c_h = +\infty$

b) $\lim_{h\to\infty} (h - c_h) = +\infty$.

a) If $\lim_{h\to\infty} c_h = c_0$ with $c_0 \in \mathbf{R}$, then passing possibly to a subsequence we can suppose that $v_h \to u$, solution of the problem

$$(2.4)\qquad \begin{cases} \operatorname{div} Tu = nH(x) & \text{in } \Omega \\ \lim\limits_{y\to x} u(y) = +\infty & \forall x \in \Gamma_1 \\ u = -c_0 & \text{in } \Gamma_2\,. \end{cases}$$

Let $\{\Omega_h\}$ be a sequence of smooth open sets with

$$\Omega_1 \subset\subset \Omega_2 \subset\subset \ldots; \quad \Omega = \bigcup_{h=1}^{\infty} \Omega_h; \quad H_{n-1}(\partial\Omega) = \lim_{h\to\infty} H_{n-1}(\partial\Omega_h)\,.$$

If we integrate (2.4) in Ω_h we get

$$(2.5)\qquad n\int_{\Omega_h} H(x)\,dx = \int_{\partial\Omega_h\cap A_1} Tu\cdot\nu\, dH_{n-1} + \int_{\partial\Omega_h\cap A_2} Tu\cdot\nu\, dH_{n-1}\,.$$

This is possible because u is solution of problem (2.4) and from theorem 1.3 the set P_u minimizes the functional $\mathcal{F}_1(B)$, hence $P_u = \emptyset$ or $P_u = \Omega$ But from (2.2)

$$\begin{cases} \text{meas}\left(\{x\in\Omega\colon v_h(x)\geqslant 0\}\right)\geqslant \dfrac{|\Omega|}{4} \\ \text{meas}\left(\{x\in\Omega\colon v_h(x)\leqslant 0\}\right)\geqslant \dfrac{|\Omega|}{4} \end{cases}$$

so we get $P_u \neq \Omega$ that is $P_u = \emptyset$.

Moreover $N_u = \emptyset$ because in problem (2.4) $\Gamma_2 = \emptyset$. We have

$$\lim_{y\to x} u(y) = +\infty \qquad \forall x\in\Gamma_1$$

$$\partial\Omega\cap A_1 = \Gamma_1\,; \qquad H_{n-1}(\partial\Omega\cap\partial A_1) = 0$$

and from theorem 1.1 we get

$$\lim_{h\to\infty} \int_{\partial\Omega_h\cap A_1} Tu\cdot\nu\, dH_{n-1} = H_{n-1}(\Gamma_1)$$

On the other hand

$$\min\lim_{h\to\infty} \int_{\partial\Omega_h\cap A_2} Tu\cdot\nu\, dH_{n-1} > -H_{n-1}(\Gamma_2)\,.$$

In fact if

$$\min\lim_{h\to\infty} \int_{\partial\Omega_h\cap A_2} Tu\cdot\nu\, dH_{n-1} = -H_{n-1}(\Gamma_2)\,.$$

it follows from theorem 1.11 that every solution w of equation

$$\operatorname{div} Tw = -nH(x) \quad \text{ in } \Omega$$

with

$$w \leqslant -u \quad \text{in } \partial\Omega \setminus \Gamma_2$$

must be

$$w \leqslant -u \quad \text{in } \Omega .$$

This is a contradiction because for every boundary value $\varphi \in C(\Gamma_2)$ a minimum of $I'(v, \varphi)$ takes it. Passing to the limit as $h \to \infty$ we have

$$n\int_{\Omega} H(x)\,dx \geqslant \lim_{h\to\infty} \int_{\partial\Omega_h A_1} Tu\cdot\nu\, dH_{n-1} + \operatorname{minlim}_{h\to\infty} \int_{\partial\Omega_h \cap A_2} Tu\cdot\nu\, dH_{n-1} >$$
$$> H_{n-1}(\Gamma_1) - H_{n-1}(\Gamma_2) .$$

This contradicts (2.1).

b) We prove that $\lim\limits_{h\to\infty} (h - c_h) = +\infty$

If $\lim\limits_{h\to\infty} (h - c_h) = \gamma_0$ with $\gamma_0 \in \mathbf{R}$, then passing possibly to a subsequence we can suppose that $v_h \to u$, solution of the problem

$$\text{(2.7)} \qquad \begin{cases} \operatorname{div} Tu = nH(x) & \text{in } \Omega \\ u = \gamma_0 & \text{in } \Gamma_1 \\ \lim\limits_{y\to x} u(y) = -\infty & \forall x \in \Gamma_2 . \end{cases}$$

Arguing the same way of (*a*) we get

$$n\int_{\Omega} H(x)\,dx < H_{n-1}(\Gamma_1) - H_{n-1}(\Gamma_2)$$

contradicting (2.1).

It follows that a subsequence of $\{v_h\}$ will converge locally to a generalized solution v of problem

$$\begin{aligned} &\operatorname{div} Tv = nH(x) && \text{in } \Omega \\ &\lim_{y\to x} v(y) = +\infty && \forall x \in \Gamma_1 \\ &\lim_{y\to x} v(y) = -\infty && \forall x \in \Gamma_2 . \end{aligned}$$

From theorem 1.3 the set $P = \{x \in \Omega: v(x) = +\infty\}$ minimizes the functional $\mathcal{F}_1(B)$ nad the set $N = \{x \in \Omega: v(x) = -\infty\}$ the functional $\mathcal{F}_2(B)$. Therefore P and N are $\emptyset$ or Ω, but

$$\text{meas}\left(\{x \in \Omega: v_h(x) \geqslant 0\}\right) \geqslant \frac{|\Omega|}{4}$$

$$\text{meas}\left(\{x \in \Omega: v_h(x) \leqslant 0\}\right) \geqslant \frac{|\Omega|}{4}$$

and hence $P = N = \emptyset$.

We get $G = \Omega$ and v is a classical solution of the equation

$$\operatorname{div} Tv = nH(x) \quad \text{in } \Omega .$$

2nd step. We prove that the function v takes on the required boundary value, more precisely we prove

i) $\lim_{y \to x} v(y) = +\infty \ \forall x \in \Gamma_1$

ii) $\lim_{y \to x} v(y) = -\infty \ \forall x \in \Gamma_2$

i) Let $x_0 \in \Gamma_1$ and let $\{x_h\}$ be a sequence of points in Ω such that $x_h \xrightarrow[h \to \infty]{} x_0$. We suppose that $v(x_h) \xrightarrow[h \to \infty]{} t \in \mathbb{R}$. The function $v_h = u_h - c_h$ minimizes $I(v, \varphi_h)$ where

$$\varphi_h(x) = \begin{cases} h - c_h & \text{if } x \in \Gamma_1 \\ -c_h & \text{if } x \in \Gamma_2 . \end{cases}$$

Then $-v_h$ minimizes the functional $I'(v, -\varphi_h)$. Let $r > 0$ such that

$$h - c_h > t + r .$$

It follows from theorem 1.4 that the set

$$E_h = \{(x, y) \in \Omega \times \mathbb{R}: y < -v_h(x)\}$$

minimizes the functional

$$\int_A |D\varphi_F| - n \int_A \varphi_F(x, y) H(x)\, dx\, dy \tag{2.8}$$

in $A = B_r(x_0', -t)$ in the class $\{F \subset \mathbf{R}^{n+1}: F \subset \Omega\times\mathbf{R}\ F \Delta E_h \subset\subset A\}$. The same minimal property is true for limit set

$$E = \{(x, y) \in \Omega\times\mathbf{R}: y < -v(x)\}\,.$$

For r small enough $(\partial\Omega\times\mathbf{R}) \cap A$ is of class C^1, it follows from theorem 1.10 that E has boundary of class C^1 in a neighborhood of $(\partial\Omega\times\mathbf{R}) \cap A$ and ∂E and $(\partial\Omega\times\mathbf{R})$ have the same normal in the contact points. Making smaller the open set we can suppose

$$(\Omega\times\mathbf{R}) \cap A = \{(x', x_n) \in (B'\times\mathbf{R}) \cap A: x_n > \psi(x')\}$$
$$E \cap A = \{(x', x_n) \in (B'\times\mathbf{R}) \cap A: x_n > g(x')\}$$

where $B' \subset \mathbf{R}^n$ in an open set, $x' = (x_1, \ldots, x_{n-1})$, g, $\psi \in C^1(B')$. In every point of Γ_1 mean curvature is

$$\Lambda(x) = \frac{nH(x)}{n-1}\,. \tag{2.9}$$

The weak form of (2.9) is that for every $\chi \in C_0^1(B')$

$$\int_{B'} \frac{D\psi\cdot D\chi}{\sqrt{1+|D\psi|^2}}\,dx' + n\int_{B'} H(x', \psi(x'))\,\chi(x')\,dx' = 0 \tag{2.10}$$

On the other hand we have for every $\chi \in C_0^1(B')$, $\chi \geqslant 0$

$$\frac{d}{dt}\Big[\int_{B'}\sqrt{1+|D(g+t\chi)|^2}\,dx' - n\int_{B'} dx' \int_{g+t\chi} H(x', x_n)\,dx_n\Big]_{t=0} \geqslant 0\,. \tag{2.11}$$

Hence

$$\int_{B'} \frac{Dg\cdot D\chi}{\sqrt{1+|Dg|^2}}\,dx' + n\int_{B'} H(x', g(x'))\,\chi(x')\,dx' \geqslant 0\,. \tag{2.12}$$

Subtracting (2.12) and (2.10) we get

$$\int_{B'} D\chi\Big(\frac{Dg}{\sqrt{1+|Dg|^2}} - \frac{D\psi}{\sqrt{1+|D\psi|^2}}\Big)\,dx' + n\int_{B'} \chi(x')[H(x', g(x')) - $$
$$- H((x', \psi(x'))]\,dx' \geqslant 0$$

for every $\chi \in C_0^1(B')$, $\chi \geqslant 0$.

Therefore $g - \psi$ is a supersolution of an elliptic equation and $g - \psi \geqslant 0$ in B'; $g - \psi = 0$ in the contact points.

It follows from maximum principle that $g - \psi = 0$ in B' and $\partial E = \partial\Omega \times \mathbf{R}$: a contradiction because

$$\big(x_h, -v(x_h)\big) \in \partial E \cap (\Omega \times \mathbf{R}) .$$

We suppose now that $v(x_h) \xrightarrow[h\to\infty]{} -\infty$.

Let $r > 0$ be such that

$$v(x_h) < (k - c_k) - r$$

from theorem 1.4 the set $E_k = \{(x, y) \in \Omega \times \mathbf{R} : y < -v_k(x)\}$ minimizes the functional

$$\int_A |D\varphi_F| - n\int_A H(x)\varphi_F(x, y)\,dx\,dy \tag{2.13}$$

in $A = B_r\big(x_0, -v(x_h)\big)$.

Set

$$E_{h,k} = \{(x, y) \in \Omega \times \mathbf{R} : y < -v_k(x) + v(x_h)\}$$

we get

$$\int_{B_r(x_0,-v(x_h))} |D\varphi_{E_k}| - n\int_{B_r(x_0,-v(x_h))} H(x)\varphi_{E_k}(x, y)\,dy\,dx = \int_{B_r(x_0,0)} |D\varphi_{E_{h,k}}| - n\int_{B_r(x_0,0)} H(x)\varphi_{E_{h,k}}(x, y)\,dx\,dy$$

and $E_{h,k}$ minimizes (2.15) in $B_r(x_0, 0)$.

Passing to the limit as $k \to +\infty$, the set $E_h = \{(x, y) \in \Omega \times \mathbf{R}: y < -v(x) + v(x_h)\}$ minimizes (2.15) in $B_r(x_0, 0)$.

Because $x_h \xrightarrow[h\to\infty]{} x_0$, choose $\sigma \in (0, r/2)$ there exists $h_0 > 0$ such that for every $h \geqslant h_0$

$$|x_h - x_0| < \sigma$$

and for these h we get

$$\operatorname{meas}\big(B_\sigma(x_h, 0) \cap E_h\big) \leqslant \operatorname{meas}\big(B_r(x_0, 0) \cap E_h\big) .$$

On the other hand from the theorem 1.9 we have for every $0 < t < r$

$$t^{-n}\int_{B_t(x_h,0)}|D\varphi_{E_h}| + nt\|H\|_{L^\infty(B_t(x_h,0))} \geqslant \omega_n . \tag{2.14}$$

From the minimal property of E_h with $F = E_h - B_t(x_h, 0)$ we get

$$\int_{B_t(x_h,0)}|D\varphi_{E_h}| - n\int_{B_t(x_h,0)} H(x)\varphi_{E_h}(x, y)\,dx\,dy \leqslant \int_{\partial B_t(x_h,0)}|D\varphi_F| - \int_{\partial B_t(x_h,0)}|D\varphi_{E_h}| \leqslant \int_{\partial B_t(x_h,0)}\varphi_{E_h}\,dH_n$$

from (2.14)

$$\int_{\partial B_t(x_h,0)}\varphi_{E_h}\,dH_n \geqslant \left(\omega_n - nt\|H\|_{L^\infty(B_t(x_h,0))}\right)t^n - n\|H\|_{L^\infty(B_t(x_h,0))}\text{meas}\left(B_t(x_h, 0)\right) \geqslant$$

$$\geqslant \left(\omega_n - c\left(n, \|H\|_{L_\infty(B_t(x_h,0))}\right)t\right)t^n .$$

Integrating between 0 and σ we get

$$\text{meas}\,(B_\sigma(x_h, 0) \cap E_h) = \int_0^\sigma dt \int_{\partial B_t(x_h,0)}\varphi_{E_h}(x, y)\,dH_n \geqslant$$

$$\geqslant \left(\frac{\omega_n}{n+1} - c\left(n, \|H\|_{L^\infty}\right)\sigma\right)\sigma^{n+1} > 0$$

while it is

$$\varphi_{E_h}(x, y) \to 0 \qquad \text{a.e. } (x, y) \in \mathbb{R}^{n+1} .$$

Hence

$$v(x_h) \xrightarrow[h\to\infty]{} +\infty .$$

ii) Let $x_0 \in \Gamma_2$ and let $\{x_h\}$ be a sequence of points in Ω such that $x_h \xrightarrow[h\to\infty]{} x_0$. Arguing the same way of (i) we can prove that

$$\lim_{h\to\infty} v(x_h) = -\infty .$$

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